Journal of Mathematical Extension
Vol. 16, No. 11, (2022) (8)1-19
URL: https://doi.org/10.30495/JME.2022.2049
ISSN: 1735-8299
Original Research Paper

Some Results on the Comaximal Colon Ideal Graph

S. Rajaee
Payame Noor University (PNU), Tehran, Iran

A. Abbasi*
University of Guilan

Abstract. In this paper, R is a commutative ring with a non-zero identity and M is a unital R-module. We introduce the comaximal colon ideal graph $C^*(R)$ and colon submodule graph $C^*(M)$; and study the interplay between the graph-theoretic properties and the corresponding algebraic structures. $C^*(R)$ is a simple connected supergraph of the comaximal ideal graph $C(R)$ with $\mathrm{diam}(C^*(R)) \leq 2$. Moreover, we prove that if $|\mathrm{V}(C^*(R)| \geq 3$, then $\mathrm{gr}(C^*(R)) = 3$. We prove that if $|\mathrm{Max}(R)| = n$, then $C^*(R)$ containing a complete n-partite subgraph. Also if M is a finitely generated multiplication module, then $C^*(R) \cong C^*(M)$. Moreover, for $\mathbb{Z}$-module $\mathbb{Z}_n$ which n is not a prime, $C^*(\mathbb{Z}_n) \cong K_{d(n)}$, where $d(n)$ is the number of all divisors of the positive integer n other than 1 and n.

AMS Subject Classification:13A15; 13F05; 05C75.
Keywords and Phrases: Colon ideal, Comaximal ideals, Graphs on commutative rings, Graphs of submodules.

Received: May 2021; Accepted: May 2022.
*Corresponding Author
The first author is partialy supported by the Grant of Payame Noor University of Iran.

1 Introduction

Throughout this paper R is a commutative with non-zero identity and M is an unitary R-module. Recently, there has been considerable attention in the work to associating graphs with algebtaic structures. There are several associated graphs to R and M. The most well-known are zero-divisor graph [7], total graph [1, 4], unit graph [11], the annihilator ideal graph [2] and trace graph [13].

In [12], Sharma and Bhatwadekar defined a graph on R, $G(R)$, with vertices as elements of R where, two distinct vertices a, and b are adjacent, if and only if $Ra + Rb = R$. In [15], Ye and Wu introduced the comaximal ideal graph of a ring R, denoted by $C(R)$ such that vertices are the proper ideals of R which are not contained in the Jacobson radical of R, and two vertices I_1 and I_2 are adjacent if and only if $I_1+I_2 = R$. They studied the diameter, girth and bipartiteness of $C(R)$ and showed that $C(R)$ is a simple, connected graph with diameter not more than three, and both the clique number and the chromatic number of the graph are equal to the number of maximal ideals of the ring R. Dorbidi and Manaviyat, classify all comaximal ideal graphs with finite independence number and present a formula to calculate this number, see [9]. Also, the domination number of $C(R)$ for a ring R is determined.

Let $\mathbb{P}$ denote the set of prime integers. Let S be a subset of R and I be an ideal. We define the subset $(I\colon S) = \{a \in R\colon aS \subset I\}$. Clearly, $(I\colon S)$ is an ideal of R which is called the *ideal quotient* or *colon ideal*. Let $\mathbb{I}^*(R)$ denote the set of all proper non-trivial ideals of R and $\mathbb{S}^*(M)$ denote the set of all proper non-trivial submodules of M. The Krull dimension of a ring R is the supremum of the lengths of all chains of prime ideals which is denoted by $\dim(R)$.

Recall that an R-module M is a *multiplication module* if every submodule N of M has the form $N = IM$ for some ideal I of R. Equivalently, $N = (N : M)M$, see [6, 8, 14]. A multiplication module M with $\mathrm{Ann}(M) = 0$ is called a *faithful multiplication module*. A *local module* is a module with exactly one maximal submodule. A non-zero module M over a ring R whose only submodules are the module itself and the zero module, is called a *simple* module. A co-semisimple module is an R-module M in which every proper submodule is an intersection of maximal submodules. We will say M is a cancellation module if for all ideals

I and J of R, $IM = JM$ implies that $I = J$. An R-module M is called *prime* if for every non-zero submodule K of M, $\mathrm{Ann}(K) = \mathrm{Ann}(M)$. For any unexplained notions or terminology we refer the reader to [5].

According to [10], a non-zero module M is define to be *hollow* if every submodule N of M is small, abreviated $N \ll M$, in case for any submodule K of M; the equality $N + K = M$ implies that $K = M$. An R-module M is called a *uniserial module* if for all non-zero elements m and n in M, either $Rm \subseteq Rn$ or $Rn \subseteq Rm$. Equivalently, for all submodules K and N of M, either $K \subseteq N$ or $N \subseteq K$. A ring R is said to be *uniserial* (*hollow*) if R is uniserial (hollow) as an R-module.

For a graph G, by $\mathrm{V}(G)$ and $\mathrm{E}(G)$, we mean the set of all vertices and edges of G, respectively. We recall that a graph is *connected* if there exists a path connecting any two of it's distinct vertices. We say that a graph is *empty* or *totally disconnected* if $\mathrm{E}(G) = \emptyset$. The distance between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of a shortest path connecting them (if such a path does not exist, then $\mathrm{d}(a, b) = \infty$; we also defne $\mathrm{d}(a, a) = 0$). The *diameter* of a graph G, denoted by $\mathrm{diam}(G)$, is equal to $\sup\{\mathrm{d}(a, b) \colon a; b \in \mathrm{V}(G)\}$. A graph is *complete* if it is connected with diameter less than or equal to one. The *girth* of a graph G, denoted $\mathrm{gr}(G)$, is the length of a shortest cycle in G, provided G contains a cycle; otherwise; $\mathrm{gr}(G) = \infty$. We denote the complete graph on n vertices by K_n and the complete bipartite graph on m and n vertices by $K_{m,n}$. We allow m and n to be infinite cardinals. We will sometimes call a $K_{1,m}$ a *star* graph. A *universal vertex* is a vertex of an undirected graph that is adjacent to all other vertices of the graph. A graph that contains a universal vertex may be called a *cone*. The *chromatic number* of a graph G, denoted $\chi(G)$, is defined to be the minimum number of colors which can be assigned to the vertices of G in such away that every two adjacent vertices have different colors. A subset C of the vertex set of a graph G is called a *clique* if any two distinct vertices of C are adjacent. The *clique number* of G, denoted by $\omega(G)$ is the size of the largest clique of G, and clearly $\chi(G) \geq \omega(G)$. A *regular graph* is a graph whose vertices all have equal degree. A *n-regular graph* is a regular graph whose common degree is n. An *independent vertex set* of a graph G is a subset of the vertices such that no two vertices in the subset represent an edge of G. The *independence number* is the cardinality of the largest independent vertex set which is denoted by $\alpha(G)$. For a loopless graph G, it's line graph $\mathrm{L}(G)$ is constructed in such way that the vertex set of $\mathrm{L}(G)$ is in $1-1$ correspondence with the edge set of G and two vertices of $\mathrm{L}(G)$ are joined by an edge if and only if the correspondence edges of G are adjacent in G. A graph G is called planar if there exists a drawing of G in the plane in which no two edges intersect in a point other than a vertex of G. By Kuratowski's theorem, a graph is planar if and only if it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

2 The Comaximal Colon Ideal Graph of a Commutative Ring

In this paper, all considered graphs are non-null. Let I and J be two elements of $\mathbb{I}^*(R)$ and denote $C(I,J)$ as the sum of two colon ideals $(I:J)+(J:I)$. First, we introduce the comaximal colon ideal graph $C^*(R)$ as follows.

Definition 2.1. Let $C^*(R)$ be the simple graph of vertex set $\mathbb{I}^*(R)$ and two vertices I, J of $\mathbb{I}^*(R)$ are adjacent if $C(I,J)=R$.

Proposition 2.2. *Suppose that n is a positive integer and that $R = F_1 \times \cdots \times F_n$ where F_k is a field for all $k = 1, \cdots, n$. Then $C^*(R)$ is a complete graph with $2(2^{n-1}-1)$ vertices.*

Proof. Note that every non-trivial ideal of R has the form $I_1 \times \cdots \times I_n$ where at least one component I_k is equal to zero ideal of F_k and at least one component I_s is equal to F_s. Let I and J be two non-trivial ideals of R. We denote $(I\colon J)_k$ for the kth component of $(I\colon J)$. Then $(I\colon J)_k = (I_k\colon_{F_k} J_k)$ for all $1 \leq k \leq n$. If $I_k = J_k = 0$ or $I_k = J_k = F_k$, then $(I\colon J)_k = (J\colon I)_k = F_k$. Let $I_k = 0$ and $J_k = F_k$ for some $1 \leq k \leq n$. Then $(I\colon J)_k = 0$ and $(J\colon I)_k = F_k$. Hence, $(I\colon J)_k + (J\colon I)_k = F_k$ for all $k = 1, \cdots, n$. So, $(I\colon J) + (J\colon I) = R$ as we desired. □

As noted $C(R)$ is a simple graph in which $\mathrm{V}(C(R)) \subseteq \mathrm{V}(C^*(R))$ and two disjoint vertices are adjacent if and only if they are comaximal ideals. It is clear that any adjacent pair in $C(R)$ is also adjacent in $C^*(R)$. But the converse is not true. For example, in the $R=\mathbb{Z}_8$, $\langle 2\rangle$

and $\langle 4\rangle$ are non-adjacent in $C(R)$, but they are adjacent in $C^*(R)$. It is clear that a comparable pair of vertices I, J of $C^*(R)$ forms an edge. Whence, a chain of non-trivial ideals forms a complete subgraph.

Remark 2.3. Let $(R, \mathfrak{m})$ is a local ring. Then $C^*(R)$ is a complete graph if and only if R is a uniserial ring. In fact, non-comparable ideals of a local ring are not adjacent. Because, $C(I, J) \subseteq \mathfrak{m}$ for all proper ideals I, J.

Observation 2.4. *Let I and J be disjoint vertices of the graph. If $I + J = R$, $I \subseteq J$ or $J \subseteq I$, then $C(I, J) = R$ and hence $I — J$ is an edge. Otherwise, $I — I + J — J$ is a path. This implies that $C^*(R)$ is a connected graph and* $\operatorname{diam}(C^*(R)) \leq 2$.

Theorem 2.5. *Let $|\operatorname{Max}(R)| = n$ and I be an ideal properly contained in $\mathrm{J}(R)$, then $\deg(I) \geq 2^n - 1$. In particular, $\deg(\mathrm{J}(R)) \geq 2^n - 2$.*

Proof. As we have $I \subset \mathrm{J}(R)$, it is evident that $I \subset \bigcap_{\lambda \in B} \mathfrak{m}_\lambda$ for all nonempty sets B of Λ. The number of non-empty subsets B of Λ equals to $2^n - 1$. Thus, $\deg(I) \geq 2^n - 1$. The second part is clear. □

Theorem 2.6. *Let $\mathfrak{m} \in \operatorname{Max}(R)$ and I be a vertex of $C^*(R)$ disjoint from $\mathfrak{m}$. For all positive integers n where $\sqrt{(I : \mathfrak{m}^n)} \nsubseteq \mathfrak{m}$, I is adjacent to $\mathfrak{m}^n$. Moreover, if $\alpha(C^*(R)) \geq 3$, then $C^*(R)$ is not a line graph.*

Proof. Fix a positive integer n. Since, $\sqrt{(I : \mathfrak{m}^n)} \nsubseteq \mathfrak{m}$. Hence, $\sqrt{(I : \mathfrak{m}^n)} + \sqrt{\mathfrak{m}^n} = R$. Therefore, $(I : \mathfrak{m}^n) + \mathfrak{m}^n = R$. This shows that

$$(I : \mathfrak{m}^n) + (\mathfrak{m}^n : I) \supseteq (I : \mathfrak{m}^n) + \mathfrak{m}^n = R.$$

So, $(I : \mathfrak{m}^n) + (\mathfrak{m}^n : I) = R$.

Moreover, let I_1, I_2 and I_3 be non-adjacent vertices and $\mathfrak{m}$ be a maximal ideal. It is clear that $\sqrt{(I_j : \mathfrak{m})} \nsubseteq \mathfrak{m}$ for $j = 1, 2, 3$ that $I_j \nsubseteq \mathfrak{m}$. Then the subgraph induced by $\{\mathfrak{m}, I_1, I_2, I_3\}$ is isomorphic to the star graph $K_{1,3}$. This is a forbidden subgraph for a line graph. □

Corollary 2.7. *Every maximal ideal $\mathfrak{m}$ of R is a universal vertex. So, $\Delta(C^*(R)) = |\mathbb{I}(R)| - 1$. Moreover, $\mathfrak{m}$ is a central point.*

Corollary 2.8. *Let $(R, \mathfrak{m})$ be a local ring with $\mathfrak{m}^k \neq \mathfrak{m}^{k+1}$ for $1 \leq k \leq 5$. Then $C^*(R)$ is non-planar. In particular, if R is a Noetherian local ring* *which is non-Artinian containing at least* 5 *ideals, then* $C^*(R)$ *is non-planar.*

Proof. It is clear that the subgraph induced by $\{\mathfrak{m}, \mathfrak{m}^2, \cdots, \mathfrak{m}^5\}$ is isomorphc to K_5. So, we are desired. $\square$

Example 2.9. $C^*(\mathbb{Z}[X])$ is not a line graph. Suppose that $I_1 = \langle X^2\rangle$, $I_2 = \langle X^2+2\rangle$ and $I_3 = \langle X^2+5\rangle$. Then $(I_1 : I_2) = (I_1 : I_3) = I_1$, $(I_2 : I_1) = (I_2 : I_3) = I_2$, and $(I_3 : I_1) = (I_3 : I_2) = I_3$. So, $C(I_1, I_2), C(I_2, I_3)$ and $C(I_1, I_3)$ are proper ideals of $\mathbb{Z}[X]$. Now Theorem 2.6 shows the result.

Theorem 2.10. *Suppose that* $\{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$ *is a subset of* $\mathrm{Max}(R)$ *and that* I *is a non-trivial proper ideal of* R*. Then* I *is adjacent to* $\mathfrak{m}_1 \cdots \mathfrak{m}_n$*.*

Proof. Note that by the simplicity of our graph, I has to be disjoint from $\mathfrak{m}_1 \cdots \mathfrak{m}_n$. We may assume that $I \nsubseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n = \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_n$. Then $(I : \mathfrak{m}_1 \cdots \mathfrak{m}_n) \nsubseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n$. Set $J := (I : \mathfrak{m}_1 \cdots \mathfrak{m}_n)$. Hence, $\sqrt{J} \nsubseteq \mathfrak{m}_1 \cdots \mathfrak{m}_n$. One may assume that there is some $1 \leq s \leq n$ such that $\sqrt{J} \nsubseteq \mathfrak{m}_1 \cup \cdots \cup \mathfrak{m}_s$ and $\sqrt{J} \subseteq \mathfrak{m}_{s+1} \cap \cdots \cap \mathfrak{m}_n$. For $i = 1, \cdots, s$ one has $\sqrt{J} + \mathfrak{m}_i = R$. This yields $\sqrt{J} + \mathfrak{m}_1 \cdots \mathfrak{m}_s = R$. It is easy to see that $\mathfrak{m}_1 \cdots \mathfrak{m}_s \subseteq (\mathfrak{m}_1 \cdots \mathfrak{m}_n : I)$. Therefore, $\sqrt{J} + \sqrt{\mathfrak{m}_1 \cdots \mathfrak{m}_n : I} \supseteq \sqrt{J} + \mathfrak{m}_1 \cdots \mathfrak{m}_s = R$. Finally, $C(I, \mathfrak{m}_1 \cdots \mathfrak{m}_n) = R$. $\square$

Corollary 2.11. *Let* $|\mathrm{Max}(R)| = n$*. For all* $I \in \mathbb{I}^*(R)$*,* $\deg(I) \geq 2^n - 2$*.*

Proof. For all vertex I, if $I \notin \{\mathfrak{m}_{i_1} \cdots \mathfrak{m}_{i_t} | \mathfrak{m}_{i_j} \in \mathrm{Max}(R), 1 \leq t \leq n\}$, one has from the theorem that $\deg(I) \geq \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n - 1$. Otherwise, $\deg(I) \geq 2^n - 2$. $\square$

Theorem 2.12. *Let* $|\mathrm{V}(C^*(R))| \geq 3$*, then* $\mathrm{gr}(C^*(R)) = 3$*.*

Proof. It is clear that if $\mathfrak{m}_1, \mathfrak{m}_2$ and $\mathfrak{m}_3$ are disjoint maximal ideals, then $\mathfrak{m}_1 — \mathfrak{m}_2 — \mathfrak{m}_3 — \mathfrak{m}_1$ is a triangle.

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are maximal ideals and I is a non-maximal ideal, then $\mathfrak{m}_1 — \mathfrak{m}_2 — I — \mathfrak{m}_1$ is a triangle.

Let $\mathfrak{m}$ be the only maximal ideal and I and J be non-maximals. Then $I — \mathfrak{m} — J$ is a path. If I and J are comparable, then $\mathrm{gr}(C^*(R)) = 3$. Otherwise, we may assume that I is finitely generated as an R-module. So, by Nakayama Lemma $\mathfrak{m}I \neq I$. This shows that $\mathfrak{m} — I — \mathfrak{m}I — \mathfrak{m}$ is a triangle and we are done. $\square$

Remark 2.13. Let Q be a primary ideal and let I be an ideal that $I \nsubseteq Q$. Then $Q \subseteq (Q : I) \subseteq \sqrt{Q}$. In particular, if Q is a prime ideal does not containing I, then $(Q : I) = Q$.

Proposition 2.14. *Let Q_1 and Q_2 be non-comparable primary ideals of R. Then $Q_1 — Q_2$ is an edge in $C(R)$ if and only if it is an edge in $C^*(R)$.*

Proof. It is enough for us to prove the "necessary" part. To this end, we note that $Q_i \subseteq (Q_i : Q_j) \subseteq \sqrt{Q}_i$ for $1 \leq i, j \leq 2$ with $i \neq j$, by Remark 2.13. Assume that $Q_1 - Q_2$ is an edge in $C^*(R)$. Then $(Q_1 : Q_2) + (Q_2 : Q_1) = R$. Thus, $\sqrt{Q}_1 + \sqrt{Q}_2 = R$ which implies that $Q_1 + Q_2 = R$. $\square$

Theorem 2.15. *Let Q_i be a $\mathfrak{m}_i$-primary ideal for $i = 1, 2$, where $\mathfrak{m}_i \in \mathrm{Max}(R)$.*

(i) If $\mathfrak{m}_1 \neq \mathfrak{m}_2$, then Q_1 is adjacent to Q_2. In particular, $\mathfrak{m}_1^n - \mathfrak{m}_2^k$ is an edge for all positive integers n, k.

(ii) If $\mathfrak{m}_1 = \mathfrak{m}_2$, then Q_1 is adjacent to Q_2 if and only if they are comparable.

Proof. (i) We may assume that Q_1 and Q_2 are non-comparable. So, $Q_i \subseteq (Q_i : Q_j) \subseteq \sqrt{Q}_i = \mathfrak{m}_i$ for $1 \leq i, j \leq 2$ with $i \neq j$, by Remark 2.13. By the fact that $\mathfrak{m}_1 + \mathfrak{m}_2 = R$, one has $Q_1 + Q_2 = R$. Hence, Q_1 and Q_2 are adjacent in $C^*(R)$.

(ii) Let $\mathfrak{m}_1 = \mathfrak{m}_2$. If they are comparable there is no something to prove. Assume that Q_1 and Q_2 are not comparable. Then $(Q_1 : Q_2) + (Q_2 : Q_1) \subseteq \mathfrak{m}_1 = \mathfrak{m}_2$ is a proper ideal. So, they are non-adjacent. $\square$

Example 2.16. Suppose that $R = F[X, Y]$ is the polynomial ring in variables X and Y and coefficients in the field F. Let $I = \langle X^2, Y\rangle$ and $J = \langle X, Y^2\rangle$. These ideals are both $\langle X, Y\rangle$-primary. It is easy to see that I is not adjacent to J in $C^*(R)$. Therefore, the ideals are not adjacent in $C(R)$ too.

Theorem 2.17. *Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \cdots, \mathfrak{m}_n\}$. Then $C^*(R)$ has a complete n-partite subgraph.*

Proof. For $i = 1, \cdots, n$, let V_i be the set of all $\mathfrak{m}_i$-primary ideals where are not pairwise comparable. Hence, non of two vertices in V_i are adjacent, by Theorem 2.15. Assume that $I \in V_i$ and $J \in V_j$ for $i \neq j$. Then $\sqrt{I} = \mathfrak{m}_i$ and $\sqrt{J} = \mathfrak{m}_j$. Thus, $\sqrt{(I : J)} + \sqrt{(J : I)} \supseteq \sqrt{I} + \sqrt{J} = R$. This yields that $I — J$ is an edge. $\square$

Corollary 2.18. *In the light of Theorem 2.17, if $n \geq 2$ and $|V_i| \geq 3$, then $C^*(R)$ is non-planar.*

Corollary 2.19. *If $|\mathrm{Max}(R)| = n$, then $\chi(C^*(R)) \geq n$.*

Theorem 2.20. *Let $I, J \in \mathbb{I}^*(R)$ where $\mathrm{Ann}(I) + \mathrm{Ann}(J) = R$, then I is adjacent to J in $C^*(R)$. Moreover, if R is not a field and I is a minimal non-nilpotent ideal of R, then $\deg(I) \geq 1$.*

Proof. The first assertion is clear, because $\mathrm{Ann}(I) + \mathrm{Ann}(J) \subseteq C(I, J)$. The second part conclude that from this fact $I = Re$ where $e^2 = e$ and by first part $C(Re, R(1 - e)) = R$. $\square$

3 The Comaximal Colon Ideal Graph of a Module

In this section, we will discuss some fundamental properties of the graph $C^*(M)$. The results will show that $C^*(M)$ has not properties similar to that of the comaximal graph $C(R)$, which is defined and studied by Meng Ye et al. in [15]. We begin with some notation and definitions.

Definition 3.1. *Let M be an R-module. We define the* comaximal colon ideal graph *$C^*(M)$ whose its vertices are all proper non-trivial submodules of M and two distinct vertices N and K are adjacent if and only if*

$$C(N, K) = (N : K) + (K : N) = R,$$

i.e., $(N : K)$ and $(K : N)$ are comaximal ideals of R. We note that in this case, $(N : K)(K : N) = (N : K) \cap (K : N)$.

Note that this graph contains inclusion graph (two distinct submodules N and K are adjacent if and only if N and K are comparable).

Clearly, if there exists a submodule $N \in \mathbb{S}^*(M) \setminus \mathrm{Min}(M)$, then $C^*(M)$ is a non-empty graph.

Our starting point is the following lemma and we list some basic properties concerning to $C^*(M)$.

Lemma 3.2. *Let M be an R-module. Then the following assertions hold.*

(i) Suppose that N, K are distinct comparable non-trivial submodules of M, then $N — K$ is an edge in $C^*(M)$.

(ii) If $(R, \mathfrak{m})$ is a local ring and $N, K \in \mathbb{S}^*(M)$ are non-comparable, then N is not adjacent to K in $C^*(M)$. In this case, M is a uniserial module if and only if $C^*(M)$ is a complete graph.

(iii) If N is a non-trivial submodule of M with a minimal set of at least 3 generators, then $\mathrm{gr}(C^*(M)) = 3$.

(iv) Suppose that N and K are submodules of M where $(N : K)$ and $(K : N)$ are distinct maximal ideals of R, then $N — K$ is an edge of $C^*(M)$.

(v) Let $N_1 \subset N_2 \subset \ldots \subset N_s$ be a finite ascending chain of non-trivial submodules of M, then $C^*(M)$ has a subgraph isomorphic to K_s.

(vi) If $\omega(C^*(M)) < \infty$, then M is both Noetherian and Artinian module.

Proof. (i) It is clear.

(ii) Let $(R, \mathfrak{m})$ be a local ring, then for every two non-comparable submodules N, K of M, we have $C(N, K) \subseteq \mathfrak{m} \subsetneq R$. This implies that N is not adjacent to K in $C^*(M)$. The second part is clear by (i).

(iii) Let $N \supseteq Rx + Ry + Rz$, where $\{x, y, z\}$ is a minimal set of generators, then by (i), $Rx — Rx + Ry — N — Rx$ is a 3-cyclic in $C^*(M)$.

(iv) Suppose that $\mathfrak{m}_1 = (N : K)$ and $\mathfrak{m}_2 = (K : N)$ are distinct maximal ideals of R, then $C(N, K) = R$ and hence $N — K$ is an adge of $C^*(M)$.

(v) It is clear.

(vi) Suppose that M is not a Notherian module, then there exists a strict ascending chain of submodules of M which is not stationary. By virtue of (i), this implies that $C^*(M)$ has a subgraph isomorphic to the complete graph K_∞, which is a contradiction. A similar argument can be applied for the Artinian case. $\square$

Corollary 3.3. *Let M be an R-module which has a strict ascending or descending chain of submodules with length ≥ 5, then $C^*(M)$ is not a planar graph. In particular, if $\dim R \geq 5$, then $C^*(R)$ is not planar.*

Proof. Suppose that there exists a strict ascending or descending chain of submodules of M with length ≥ 5, then $C^*(M)$ has a subgraph isomorphic to the complete graph K_5. Therefore by Kuratowski's theorem $C^*(M)$ is not planar. In particular, if $\dim R \geq 5$, then there exists a strict ascending chain of prime submodules of M with length 5 and this implies that $C^*(M)$ is not planar. $\square$

In [15, Proposition 2.1 (1)], it is shown that, if $(R, \mathfrak{m})$ is a local ring, then $C(R)$ is an empty graph. The following example shows that this is not the case in $C^*(M)$.

Example 3.4. (i) Consider $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$-module, then $\mathrm{V}(C^*(M)) = \{\langle\bar{2}\rangle, \langle\bar{3}\rangle, \langle\bar{4}\rangle, \langle\bar{6}\rangle\}$. Clearly $C^*(\mathbb{Z}_{12})$ is the complete graph K_4.

(ii) Consider $M = \mathbb{Z}_8$ as a $\mathbb{Z}_8$-module, where $\mathbb{Z}_8$ is a local ring with only maximal ideal $\mathfrak{m} = \langle\bar{2}\rangle$. Then $\mathrm{V}(C^*(\mathbb{Z}_8)) = \{\langle\bar{2}\rangle, \langle\bar{4}\rangle\}$ and $C^*(\mathbb{Z}_8)$ is the complete graph K_2 with only edge $\langle\bar{2}\rangle — \langle\bar{4}\rangle$.

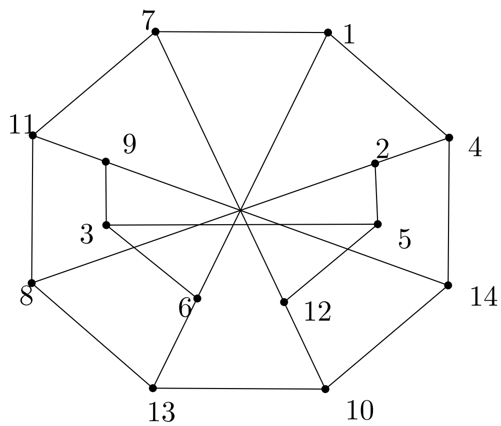
(iii) We consider $M = \mathbb{Z}_2 \times \mathbb{Z}_2$ as a $\mathbb{Z}$-module, then the set of vertices is $\mathrm{V}(C^*(M)) := \{\mathbb{Z}_2 \times 0, 0 \times \mathbb{Z}_2, \langle(\bar{1}, \bar{1})\rangle\}$. One can check that $C^*(M)$ is an empty graph.

(iv) Consider $M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as a $\mathbb{Z}$-module, then $\mathrm{V}(C^*(M))$ has

14 submodules as follows:

$$
\begin{aligned}
N_1 :&= \mathbb{Z}_2 \times \mathbb{Z}_2 \times 0 = \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{1},\overline{0}),(\overline{1},\overline{0},\overline{0}),(\overline{1},\overline{1},\overline{0})\}\\
&= N_4 + N_6 = N_4 + N_7 = N_6 + N_7,\\
N_2 :&= \mathbb{Z}_2 \times 0 \times \mathbb{Z}_2 = \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{0},\overline{1}),(\overline{1},\overline{0},\overline{0}),(\overline{1},\overline{0},\overline{1})\}\\
&= N_4 + N_5 = N_4 + N_8 = N_5 + N_8,\\
N_3 :&= 0 \times \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{0},\overline{1}),(\overline{0},\overline{1},\overline{0}),(\overline{0},\overline{1},\overline{1})\}\\
&= N_6 + N_9 = N_5 + N_6 = N_5 + N_9,\\
N_4 :&= \langle(\overline{1},\overline{0},\overline{0})\rangle = \mathbb{Z}_2 \times 0 \times 0,\\
N_5 :&= \langle(\overline{0},\overline{0},\overline{1})\rangle = 0 \times 0 \times \mathbb{Z}_2,\\
N_6 :&= \langle(\overline{0},\overline{1},\overline{0})\rangle = 0 \times \mathbb{Z}_2 \times 0,\\
N_7 :&= \langle(\overline{1},\overline{1},\overline{0})\rangle = \{(\overline{0},\overline{0},\overline{0}),(\overline{1},\overline{1},\overline{0})\},\\
N_8 :&= \langle(\overline{1},\overline{0},\overline{1})\rangle = \{(\overline{0},\overline{0},\overline{0}),(\overline{1},\overline{0},\overline{1})\},\\
N_9 :&= \langle(\overline{0},\overline{1},\overline{1})\rangle = \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{1},\overline{1})\},\\
N_{10} :&= \langle(\overline{1},\overline{1},\overline{1})\rangle = \{(\overline{0},\overline{0},\overline{0}),(\overline{1},\overline{1},\overline{1})\},\\
N_{11} :&= \{(\overline{0},\overline{0},\overline{0}),(\overline{1},\overline{1},\overline{0}),(\overline{1},\overline{0},\overline{1}),(\overline{0},\overline{1},\overline{1})\}\\
&= N_7 + N_8 = N_7 + N_9 = N_8 + N_9,\\
N_{12} :&= \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{0},\overline{1}),(\overline{1},\overline{1},\overline{0}),(\overline{1},\overline{1},\overline{1})\}\\
&= N_5 + N_7 = N_5 + N_{10} = N_7 + N_{10},\\
N_{13} :&= \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{1},\overline{0}),(\overline{1},\overline{0},\overline{1}),(\overline{1},\overline{1},\overline{1})\}\\
&= N_6 + N_8 = N_6 + N_{10} = N_8 + N_{10},\\
N_{14} :&= \{(\overline{0},\overline{0},\overline{0}),(\overline{0},\overline{1},\overline{1}),(\overline{1},\overline{0},\overline{0}),(\overline{1},\overline{1},\overline{1})\},\\
&= N_4 + N_9 = N_4 + N_{10} = N_9 + N_{10}.
\end{aligned}
$$

Clearly for all $k \in \mathbb{Z}$ and submodules N_i, N_j, $kN_i \subseteq N_j$ if and only if $N_i \subseteq N_j$. So, two vertices are adjacent if and only if the corresponding submodules are comparable. $C^*(M)$ is a 3-regular graph and it is a bipartite graph with a bipartition of two sets as $X := \{N_1, N_2, N_3, N_{11}, N_{12}, N_{13}, N_{14}\}$ and $Y := \{N_4, N_5, N_6, N_7, N_8, N_9, N_{10}\}$. Hence, $\chi(C^*(M)) = 2$ and $\alpha(C^*(M)) = 7$. In the Figure 1, we have taken k for denoting the vertex N_k.



Theorem 3.5. *Let M be a finitely generated faithful multiplication R-module. Then $C^*(R) \cong C^*(M)$.*

Proof. Consider the map $\phi : \mathbb{I}^*(R) \to \mathbb{S}^*(M)$ where $\phi(I) = IM$. Clearly ϕ is a bijection between $\mathrm{V}(C^*(R))$ and $\mathrm{V}(C^*(M))$. Assume that $N = IM$ and $K = JM$ are distinct submodules of M, we must show that $N — K$ is an edge in $C^*(M)$ if and only if $I — J$ is an edge in $C^*(R)$. Obviously, $(N : K)M = (I : J)M$ and $(K : N)M = (J : I)M$. Suppose that $C(N, K) = R$, then $(N : K)M + (K : N)M = M$, since M is a cancellation module. This implies that $C(I, J) = R$. The converse is clear. $\square$

Remark 3.6. A module M is called *coatomic* if every proper submodule of M is contained in a maximal submodule of M. In this case, if $|\mathrm{V}(C^*(M)| \geq 2$, then $|\mathrm{E}(C^*(M)| \geq 1$. It is well-known that co-semisimple modules, finitely generated modules and multiplication modules are coatomic. By Lemma 3.2 (ii), in a coatomic R-module M with at least two non-trivial submodules, $C^*(M)$ has no isolated vertex. We recall that an R-module M is said to be *sum-irreducible* precisely when it is non-zero and cannot be expressed as the sum of two proper submodules of itself.

In the following theorem, we will give some relations between the graph-theoretic properties of $C^*(M)$ and the module-theoretic properties of M.

Theorem 3.7. *Let M be an R-module. Then the following assertions hold.*

(i) If M is coatomic, then $\deg(\mathrm{Rad}(M))$ is at least equal to the number of essential submodules of M.

(ii) If $N - K$ is an edge in $C^*(M)$ and $(N\colon K)$ is a small ideal of R, then N is a proper submodule of K. In particular, if R is a hollow ring and $C^*(M)$ is a complete graph, then M is a uniserial module.

(iii) If $N \cap K = 0$ and $\mathrm{Ann}(K) = 0$, then N is not adjacent to K in $C^*(M)$. Moreover, if N and K are distinct simple submodules of M and either $\mathrm{Ann}(N) = 0$ or $\mathrm{Ann}(K) = 0$, then N is not adjacent to K in $C^*(M)$.

Proof. (i) Assume that $N \in \mathrm{V}(C^*(M))$, then $N \ll M$ if and only if $N \subseteq \mathrm{Rad}(M)$. Therefore $N - \mathrm{Rad}(M)$ is an edge in $C^*(M)$ and the proof is complete.

(ii) Since $N - K$ is an edge in $C^*(M)$, therefore $C(N, K) = R$. By hypothesis $(N\colon K) \ll R$, then $(K : N) = R$ which implies that $N \subset K$. For the second part, since R is a hollow ring hence every ideal of R is a small ideal and the proof is clear by the first part.

(iii) Assume that $N - K$ is an edge in $C^*(M)$. Since $N \cap K = 0$ we conclude that $(N : K) = \mathrm{Ann}(K)$ and $(K : N) = \mathrm{Ann}(N)$. This implies that $\mathrm{Ann}(N) + \mathrm{Ann}(K) = R$ and by hypothesis since $\mathrm{Ann}(K) = 0$ hence $\mathrm{Ann}(N) = R$ which is a contradiction. The second part is clear. $\square$

Corollary 3.8. *Let $N, K \in \mathbb{S}^*(M)$ and $N \cap K = 0$. Then $N - K$ is an edge in $C^*(M)$ if and only if $\mathrm{Ann}(N) + \mathrm{Ann}(K) = R$.*

The condition $\mathrm{Ann}(K) = 0$ in Theorem 3.7 (iii), can not be omitted, because we consider the $\mathbb{Z}$-module $M = \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $N = \mathbb{Z}_3 \oplus 0$, $K = 0 \oplus \mathbb{Z}_5$. Then we have $(N : K) = 5\mathbb{Z}$ and $(K : N) = 3\mathbb{Z}$, therefore $(N : K) + (K : N) = 5\mathbb{Z} + 3\mathbb{Z} = \mathbb{Z}$. We infer that $N - K$ is an edge of $C^*(M)$.

Theorem 3.9. *If* $|\mathrm{V}(C^*(M))| \geq 2$*, then* $C^*(M)$ *is an empty graph if and only if* $M = N \oplus K$ *where* $N \cong K$ *is a simple module.*

Proof. If $C^*(M)$ is the empty graph, then no two submodules of M are comparable. So every non-trivial submodule is both a maximal and a minimal submodule. Let N, K be two vertices of $C^*(M)$, then there exist $x \in N$ and $y \in K$ such that $N = Rx$ and $K = Ry$. It conclude that $R/\mathrm{Ann}(x) \cong Rx$ and $R/\mathrm{Ann}(y) \cong Ry$. If N and K are two non-isomorphic non-trivial submodules of M, then $\mathrm{Ann}(N) \neq \mathrm{Ann}(K)$ are two maximal ideals, since otherwise $N \cong K$ which is a contradiction. Now since $\mathrm{Ann}(N)$ and $\mathrm{Ann}(K)$ are two distinct maximal ideals of R hence $R = \mathrm{Ann}(N) + \mathrm{Ann}(K) \subseteq (K : N) + (N : K) \subseteq R$. Therefore N and K are adjacent in $C^*(M)$. This contradiction shows that all non-trivial submodules of M are isomorphic to a simple module N. It is easy to see that $M = N \oplus K$. $\square$

Corollary 3.10. *Let* M *be an* R*-module and* $|\mathrm{V}(C^*(M))| \geq 2$*. The following assertions hold.*

(i) If M *is a hollow module, then* $C^*(M)$ *is not empty.*

(ii) If $C^*(M)$ *is empty, then* M *is a two dimensional vector space.*

Proof. (i) It is clear.

(ii) Note that in the notation of Theorem 3.9, $F = R/\mathrm{Ann}(M)$ is a field. On the other hand, $N = Rx$ and $K = Ry$ are cyclic modules. Hence, M is a two dimensional vector space over F. $\square$

Theorem 3.11. *If* $C^*(M)$ *is non-empty, then* $\mathrm{diam}(C^*(M)) \leq 3$*.*

Proof. Let N and K are two different submodule of $\mathbb{S}^*(M)$. If N and K are adjacent, then $\mathrm{d}(N, K) = 1$. Assume N and K are non-adjacent. If $N + K \subsetneqq M$ then $N — N + K — K$ is a path which implies $\mathrm{d}(N, K) = 2$. Hence assume $N + K = M$. If $N \cap K \neq 0$, then $N — N \cap K — K$ is a path which implies $\mathrm{d}(N, K) = 2$. Hence assume that $N \cap K = 0$ and $M = N \oplus K$. If N is not simple, then N has a non-trivial submodule N_1. So $N — N_1 — N_1 + K — K$ is a path which implies $\mathrm{d}(N, K) \leq 3$. So, we may assume that N and K are simple. If N and K are non-isomorphic, then $\mathrm{Ann}(N) \neq \mathrm{Ann}(K)$ are two different maximal ideals. Hence N and K are adjacent in $C^*(M)$. So assume that $N \cong K$, then by Theorem 3.9, $C^*(M)$ is an empty graph which is a contradiction. $\square$

Corollary 3.12. *Suppose that M is a sum-irreducible R-module, then* $\operatorname{diam}(C^*(M)) \leq 2$. *Moreover, in this case if M has an essential submodule, then* $\operatorname{gr}(C^*(M)) \leq 4$.

Theorem 3.13. *Let M be a faithful prime R-module and $N \in \operatorname{Min}(M)$. If $K — N$ is an edge in $C^*(M)$, then $N \subsetneq K$.*

Proof. Since K is adjacent to N in $C^*(M)$, hence $C(N, K) = R$. We note that $(N : K)K \subseteq N$ and $N \in \operatorname{Min}(M)$ hence either $(N : K)K = 0$ or $(N : K)K = N$. If $(N : K)K = 0$, then $(N : K) \subseteq \operatorname{Ann}(K) = \operatorname{Ann}(M) = 0$, then $(K : N) = R$ and we are desired. Otherwise, we have $(N : K)K = N \subsetneq K$. $\square$

Corollary 3.14. *Let M be a faithful prime R-module and $X = \operatorname{Min}(M)$. Then the induced subgraph of $C^*(M)$ generated by X is empty.*

Assume that $d(n)$ is the number of divisors of a positive integer number n except 1 and n. In particular, $d(p) = 0$ for every prime number $p \in \mathbb{P}$.

Theorem 3.15. *Consider $\mathbb{Z}_n$ as a $\mathbb{Z}$-module. If $n \notin \mathbb{P}$, then $C^*(\mathbb{Z}_n)$ is the complete graph $K_{d(n)}$.*

Proof. Suppose that $N = \langle \bar{t} \rangle$ and $K = \langle \bar{s} \rangle$ are submodules of $\mathbb{Z}_n$. We show that $N — K$ is an edge in $C^*(\mathbb{Z}_n)$.

case 1. If $(t, s) = 1$, then $C(\langle \bar{t} \rangle, \langle \bar{s} \rangle) = t\mathbb{Z} + s\mathbb{Z} = \mathbb{Z}$.

case 2. If $(t, s) = d > 1$, then $t = t_1 d$, $s = s_1 d$ and $(t_1, s_1) = 1$, therefore

$$(\langle \bar{t} \rangle : \langle \bar{s} \rangle) = (\frac{t}{gcd(t,s)})\mathbb{Z} = t_1 Z, (\langle \bar{s} \rangle : \langle \bar{t} \rangle) = (\frac{s}{gcd(t,s)})\mathbb{Z} = s_1\mathbb{Z}.$$

It follows that $C(N, K) = t_1\mathbb{Z} + s_1\mathbb{Z} = \mathbb{Z}$.

Therefore $C^*(\mathbb{Z}_n)$ is the complete graph $K_{d(n)}$. $\square$

Definition 3.16. Let M be an R-module.

(i) M is called a *prüfer module* if every non-zero finitely generated submodule of M is invertible.

(ii) M is called a *dedekind module*, if each non-zero submodule of M is invertible in M.

For an R-module M we denote the collection of all finitely generated (resp. cyclic) submodules of M by $\mathrm{FG}(M)$ (resp. $\mathrm{CY}(M)$). The following theorem is a consequence of [3, Theorem 2.3].

Theorem 3.17. *Let R be an integral domain and M a faithful multiplication R-module. The following assertions are equivalent.*

(i) M is a prüfer domain.

(ii) The induced subgraph of $C^*(M)$ generated by $\mathrm{FG}(M)$ is a complete graph.

(iii) The induced subgraph of $C^*(M)$ generated by $\mathrm{CY}(M)$ is a complete graph.

(iv) For every $P \in \mathrm{Spec}(R)$, $C^*(M_P)$ is a complete graph.

In the following theorem we will give some necessary and sufficient conditions for that $C^*(M)$ is a complete graph.

Theorem 3.18. *Let M be a module on a simple ring R. Then the following assertions are equivalent.*

(i) $C^*(M)$ is a complete graph.

(ii) M is a uniserial module.

(iii) For every submodule N of M and all $L \in \mathrm{Max}(N)$, $L \ll N$.

Proof. (i $\Rightarrow$ ii) Let $C^*(M)$ is complete, then for every distinct submodules N, K of M, $C(N, K) = R$. Since R is a simple ring, hence $(N : K) = R$ or $(K : N) = R$. Therefore $N \subset K$ or $K \subset N$, we infer that M is a uniserial module.
(ii $\Rightarrow$ i) The proof is obvious.
(ii $\Rightarrow$ iii) Let N is a submodule of M and $L \in \mathrm{Max}(N)$. Let $L+T = N$, where L and T are submodules of N. Now $L \subseteq T$ or $T \subseteq L$. If $L \subseteq T$, then $N = L + T = T$, this implies that $L \ll N$. If $T \subseteq L$, then $N = L + T = L$, which is impossible, since L is a maximal submodule

of N.
(iii $\Rightarrow$ ii) Assume that L and K are submodules of M and let $L \nsubseteq K$. Then there exists an element $x \in L \setminus K$. Let y is an arbitrary element of K. We show that $y \in L$. Let $N = Rx + Ry$. If $N = Ry$, then $Rx \subseteq Rx + Ry = N = Ry \subseteq K$. So $x \in K$, which is impossible. Hence Ry is a proper submodule of $N = Rx + Ry$, and since N is a f.g. R-module, so there exists a maximal submodule $L \in \mathrm{Max}(N)$ such that $Ry \subseteq L$. We have $N = Rx+Ry \subseteq Rx+L \subseteq N$. Therefore $N = Rx+L$ and by our assumption $L \ll N$, this implies that so $N = Rx$. Therefore, $Ry \subseteq Rx + Ry = N = Rx \subseteq L$, and hence $y \in L$. This implies that $K \subseteq L$ hence M is a uniserial module. $\square$

Acknowledgements
The authors are grateful to the referee for helpful suggestions which have resulted in an improvement to the article.

References

[1] A. Abbasi, and Sh. Habibi, The total graph of a commuttive ring with respect to proper ideals, *J. Korean Math. Soc.*, 49(2012), 85-98.

[2] M. Afkhami, S. Hoseini, and K. Khashyarmanesh, The annihilator ideal graph of a commutative ring , *Note di Matematica*, 36(2016), 1-10

[3] M. M. Ali, Invertibility of multiplication modules, *New Zealand J. Math.*, 35 (2006), 17–29.

[4] D. F. Anderson, and A. Badawi, The total graph of a commutative ring, *J. Algebra*, 320, no. 7 (2008), 2706-2719.

[5] F. W. Anderson, and K. R. Fuller, *Rings and Categories of Modules*, Second Edition, 1992.

[6] D. D. Anderson, and Al. Sh. Yousef, Multiplication modules and the ideal $\theta(M)$, *Comm. in Algebra*, 30 (7) (2002), 3383-3390.

[7] D. F. Anderson, and P. F. Livingston, The zero-divisor graph of a commutative ring, *J. Algebra*, 217, no. 2 (1999), 437-447.

[8] A. Barnard, Multiplication modules, *J. Algebra*, 71 (1981), 174-178.

[9] H. R. Dorbidi, and R. Manaviyat, Some results on the comaximal ideal graph of a commutative ring, *Trans. Comb.*, 5 No. 4 (2016), 9-20.

[10] P. Fluery, Hollow modules and local endomorphism rings, *Pacific J. Math.*, 53 (1974), 379-385.

[11] R. P. Grimaldi, Graphs from rings, *Proceedings of the 20th South-eastern Conference on Combinatorics, Graph Theory, and Computing* (Boca Raton, FL, 1989). 71(1990), pp. 95–103.

[12] P. K. Sharma, and S. M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra*, 176 (1995), 124-127.

[13] M. Sivagami, and Ch. T. Tamizh Chelvam, On the trace graph of matrices, *Acta Math. Hungar.*, 158(2019), 235–250.

[14] F. Smith, Some remarks on multiplication modules, *Arch. der Math*, 50 (1988), 223-235.

[15] M. Ye, M., and T. S. Wu, Comaximal ideal graphs of commutative rings, *J. Algebra Appl.*, 11(6) (2012), : 1250114 (14 pages).

Saeed Rajaee
Assistant Professor of Mathematics
Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Payame Noor (PNU), P.O.Box 19395-3697, Tehran, Iran.
E-mail: saeed_rajaee@pnu.ac.ir

Ahmad Abbasi
Associate Professor of Mathematics
1. Department of Pure Mathematics
Faculty of Mathematical Sciences
University of Guilan

2. Center of Excellence for Mathematical Modeling, Optimization and Combinatorial Computing (MMOCC)
University of Guilan
Rasht, Iran
E-mail: aabbasi@guilan.ac.ir