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The Length of Centralizers and Conjugate Type Vectors in a Finite Group with Applications

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Abstract. In this paper, we first introduce a class of finite group G_f which is covered all finite metacyclic 2-groups of negative type in Beuelre's classification. Next, we obtain the size of centralizers and also conjugate type vector of the groups. Finally, the n-th commutativity degree of G_f is obtained as a direct application of the results.

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1 Introduction

The importance of conjugacy classes in the structure of finite groups was identified in the study of groups. Some results about the relation between conjugate type vector of a finite group G and nilpotency of the group, and also the conjugacy class sizes and solvability of G were investigated [4, 5, 9, 10, 16]. Moreover, there are some works which have considered the influence of conjugacy class sizes on the prime power groups. We do not mention these results here, but the reader may refer to [1, 2, 11, 16, 18]. For instance, Ashrafi [17] computed the number of distinct centralizers of some finite groups and examined the structure of the groups with precisely six distinct centralizers.

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Recently, Erfanian *et al*. [7] using the centralizers of finite groups, computed the commutativity degree or commuting probability of n-th power of a random element in the group which is denoted by $P_n(G)$ and defined as the ratio $P_n(G) = \frac{1}{|G|^2}|\{(a,b) \in G \times G : [a^n, b] = 1\}|$. This notation can be presented as $P_{p^m}(G) = \frac{1}{|G|^2}\sum_{g\in G}|C_G(g^{p^m})|$, where $n = p^m k$ such that $\gcd(k, p) = 1$ and m is a non-negative integer. The importance of the computation of $P_n(G)$ follows from this point of view that it is a generalization of commutativity degree $P(G)$ which was introduced by Erdös and Turan [6] in 1968, and studied by some authors in deferent contexts [8, 14, 15]. In the recent results of [15], an explicit formula for $P_n(G)$, where G is a 2-generator p-group of nilpotency class two is given.

In this paper, we first introduce a family of finite groups denoted by G_f which is covered all finite metacyclic 2-group of negative type in Beuerle's classification [3]. Then the size of centralizers and the set of lengths of conjugacy classes or conjugate type vector of G_f are obtained. The n-th commutativity degree of such groups is also given as application of the results.

2 Definitions and Counting the Size of Centralizers

In this section, some basic results on the centralizers of elements in the group G_f are stated and proved. These results will be used for calculations in the subsequent sections. Moreover, a general formula for the size of centralizers of an arbitrary element in the group G_f is provided. For an element x of G, x^G denotes the conjugacy class containing x and $|x^G|$ (which is called the index of x in G) the length of x^G. The number $e_p(n)$ denotes the largest exponent of p in n, that is $p^{e_p(n)}|n$ but $p^{e_p(n)+1} \nmid n$, and $\mathrm{ctv}(G)$ considers the set of lengths of conjugacy classes.

If G is a finite metacyclic p-group , then

$$G = \langle a, b : a^{p^\alpha} = 1, b^{p^\beta} = a^{p^{\alpha-\varepsilon}}, a^b = a^\lambda \rangle$$

for some positive integers $\alpha, \beta, \gamma, \varepsilon$, where $\lambda = p^{\alpha-\gamma} \pm 1$. The group is called positive or negative type if $\lambda = p^{\alpha-\gamma} + 1$ or $\lambda = p^{\alpha-\gamma} - 1$, respectively [3].

Finite metacyclic 2-groups of negative type can be classified as follows:

$$G_f = G(\alpha, \beta, \gamma, \varepsilon) = \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, a^b = a^{2^{\alpha-\gamma}-1} \rangle, \qquad (1)$$

where $\alpha-\gamma\geq 2$ and $\gamma>0$. In this case, we have the following computations which are needed for the future reference.
If a^ib^j and $a^sb^t\in G_f$, then $a^ib^ja^sb^t=a^{i+s\lambda^j}b^{j+t}$, and

$$(a^ib^j)^k=a^{i(1+\lambda^j+\cdots+\lambda^{(k-1)j})}b^{kj}=a^{i\frac{1-\lambda^{kj}}{1-\lambda^j}}b^{kj}, \tag{2}$$

for each $1\leq i,s\leq 2^\alpha$ and $1\leq j,t\leq 2^\beta$. Moreover, $(a^ib^j)^{a^sb^t}=a^{\lambda^t(i+s-s\lambda^j)}b^j$.
Hence, the centralizer of a^ib^j in G_f is

$$C_{G_f}(a^ib^j)=\{a^sb^t:1\leq s\leq 2^\alpha,1\leq t\leq 2^\beta,s(1-\lambda^j)\lambda^t\overset{2^\alpha}{\equiv}i(1-\lambda^t)\}. \tag{3}$$

From the considerations above we have the following results.

Lemma 2.1. *Let $G_f=\langle a,b:a^{2^\alpha}=1,b^{2^\beta}=a^{2^{\alpha-\varepsilon}},a^b=a^{2^{\alpha-\gamma}-1}\rangle$, where $\alpha-\gamma\geq 2$. Then $Z(G_f)=\langle a^{2^{\alpha-1}},b^{2^\gamma}\rangle$. If $\lambda=2^{\alpha-\gamma}-1$, then for each natural number k, we have*

$$e_2(1-\lambda^k)=\begin{cases}\alpha-\gamma+e_2(k), & k\text{ even},\\ 1, & k\text{ odd}.\end{cases}$$

Proof. Let $k=2^{e_2(k)}k'$ for each natural number k, where k' is odd. Then

$$\begin{aligned}1-\lambda^k&=1-(-1+2^{\alpha-\gamma})^k\\&=1-(-1)^k+2^{\alpha-\gamma+e_2(k)}\left(k'-\frac{k'(k-1)}{2}2^{\alpha-\gamma}+\cdots\pm\frac{k'(k-1)!}{k!}2^{(\alpha-\gamma)j}\right)\\&=1-(-1)^k+2^{\alpha-\gamma+e_2(k)}\ell,\end{aligned}$$

for some integer ℓ. Now if k is even, then $1-\lambda^k=2^{\alpha-\gamma+e_2(k)}\ell$ and $e_2(1-\lambda^k)=\alpha-\gamma+e_2(k)$. If k is odd, then $1-\lambda^k=2+2^{\alpha-\gamma+e_2(k)}\ell_1=2\ell_2$ for some integers ℓ_1 and ℓ_2, hence $e_2(1-\lambda^k)=1$. □

Lemma 2.2. *Let $G_f=\langle a,b:a^{2^\alpha}=1,b^{2^\beta}=a^{2^{\alpha-\varepsilon}},a^b=a^{2^{\alpha-\gamma}-1}\rangle$, where $\alpha-\gamma\geq 2$. If $\lambda=2^{\alpha-\gamma}-1$, then $a^sb^t\in C_{G_f}(a^ib^j)$ if and only if*

(i) $2^{e_2(j)+e_2(s)+\alpha-\gamma}s'l_j\lambda^t\overset{2^\alpha}{\equiv}2^{e_2(i)+e_2(t)+\alpha-\gamma}i'l_t$ *if j and t are even,*

(ii) $2^{e_2(j)+e_2(s)+\alpha-\gamma}s'l_j\lambda^t\overset{2^\alpha}{\equiv}2^{e_2(i)+1}i'l_t$ *if j is even and t is odd,*

(iii) $2^{e_2(s)+1}s'l_j\lambda^t \overset{2^\alpha}{\equiv} 2^{e_2(i)+e_2(t)+\alpha-\gamma}i'l_t$ *if j is odd and t is even,*

(iv) $2^{e_2(s)+1}s'l_j\lambda^t \overset{2^\alpha}{\equiv} 2^{e_2(i)+1}i'l_t$ *if j and t are odd,*

where $i^{'}, s^{'}, l_j, l_t, \lambda^t$ are odd positive integers and $1 \leq i, s \leq 2^\alpha$ and $1 \leq j, t \leq 2^\beta$.

Proof. To prove (i), suppose $k = 2^{e_2(k)}k'$ for each natural number k. Then $1-\lambda^k = 2^{e_2(1-\lambda^k)}l_k$ where l_k is an odd number. In Equation 3 for each $a^sb^t \in C_{G_f}(a^ib^j)$, $s(1-\lambda^j)\lambda^t \overset{2^\alpha}{\equiv} i(1-\lambda^t)$. Thus $2^{e_2(s)}s'2^{e_2(1-\lambda^j)}l_j\lambda^t \overset{2^\alpha}{\equiv} 2^{e_2(i)}i'2^{e_2(1-\lambda^t)}l_t$, where $s^{'}, l_j, \lambda^t, i^{'}, l_t$ are odd positive integers. Now if j and t are even, then using Lemma 2.1, $2^{e_2(j)+e_2(s)+\alpha-\gamma}l_js'\lambda^t \overset{2^\alpha}{\equiv} 2^{e_2(i)+e_2(t)+\alpha-\gamma}i'l_t$. A similar method can be used to prove another parts and we omit it here. $\square$

In the following two theorems we compute the size of centralizers of an arbitrary element in the groups G_f. This process also provides a count for each conjugacy class sizes.

We treat each case of Lemma 2.2 to obtain the size of centralizers of elements of the groups G_f.

Theorem 2.3. *Let G_f be a group of the form $G_f = \langle a, b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, a^b = a^{2^{\alpha-\gamma}-1}\rangle$ where $\alpha-\gamma \geq 2$. Then the size of the centralizer of a non-central element $g = a^ib^j$ of G_f for which $1 \leq i \leq 2^\alpha$ and $1 \leq j \leq 2^\beta$ is*

(1) $2^{\alpha+\beta-\gamma+e_2(i)}$, *if j is even, $e_2(j) \geq \gamma$ and $e_2(i) < \gamma$,*

(2) $2^{\alpha+\beta-\gamma+e_2(j)}$, *if j is even and either $e_2(i) \geq \alpha-1$ or ($e_2(i)-e_2(j) \geq \alpha-\gamma-1$ and $e_2(i) \geq \gamma$),*

(3) $2^{\alpha+\beta-\gamma+e_2(j)-1}$, *if j is even, $e_2(i) \geq \gamma$, $e_2(i) < \alpha-1$ and $e_2(i)-e_2(j) < \alpha-\gamma-1$,*

(4) $2^{\alpha+\beta-\gamma+e_2(j)}$, *if j is even, $e_2(j) \leq e_2(i) < \gamma$ and $e_2(i)-e_2(j) \geq \alpha-\gamma-1$,*

(5) $2^{\alpha+\beta-\gamma+e_2(j)-1}$, *if j is even, $e_2(j) \leq e_2(i) < \gamma$ and $e_2(i)-e_2(j) < \alpha-\gamma-1$,*

(6) $2^{\alpha+\beta-\gamma+e_2(i)}$, *if j is even and $e_2(i) < e_2(j) < \gamma$,*

(7) $2^{\beta+1}$, *if j is odd.*

Proof. To obtain the size of centralizers of elements of G_f, we proceed in some cases. We let $a^s b^t \in C_{G_f}(a^i b^j)$ such that $1 \leq s \leq 2^{\alpha}$ and $1 \leq t \leq 2^{\beta}$.
(**a**) First we suppose that $e_2(i) \geq \gamma$. Thus we have the following cases:

(a.1) If j, t are even, then using Lemma 2.2, $\gamma - e_2(j) \leq e_2(s) \leq \alpha$ and the number of such cases is

$$2^{\beta-1}\Big(1 + \sum_{e_2(s)=\gamma-e_2(j)}^{\alpha-1} \varphi(2^{\alpha-e_2(s)})\Big) = 2^{\beta-1}\Big(1 + 2^{\alpha-\gamma+e_2(j)-1} + \cdots + 1\Big)$$
$$= 2^{\alpha+\beta-\gamma+e_2(j)-1}.$$

(a.2) If j is even and t is odd, then we have two cases. If $e_2(i) \geq \alpha - 1$, then $\gamma - e_2(j) \leq e_2(s) \leq \alpha$ and the number of such cases is

$$2^{\beta-1}\Big(1 + \sum_{e_2(s)=\gamma-e_2(j)}^{\alpha-1} \varphi(2^{\alpha-e_2(s)})\Big) = 2^{\alpha+\beta-\gamma+e_2(j)-1},$$

and if $e_2(i) < \alpha - 1$, then $e_2(s) = e_2(i) - e_2(j) + 1 + \gamma - \alpha \geq 0$, which implies that $e_2(i) - e_2(j) \geq \alpha - \gamma - 1$. The number of such cases is

$$2^{\beta-1} \cdot 2^{\alpha-\gamma+e_2(j)} = 2^{\alpha+\beta-\gamma+e_2(j)-1}.$$

(a.3) If j is odd and t is even, then $e_2(s) \geq \alpha - 1$ and the number of these cases is

$$2^{\beta-1}(1+1) = 2^{\beta}.$$

(a.4) If j, t are odd, then we have two cases. If $e_2(i) \geq \alpha - 1$, then $e_2(s) \geq \alpha - 1$ and the number of such cases is $2^{\beta-1}(1+1) = 2^{\beta}$, and if $e_2(i) < \alpha - 1$ and $s = 2^{e_2(s)}s'$, then $e_2(s) = e_2(i)$. It follows that $s' \overset{2^{\alpha-e_2(j)-1}}{\equiv} i l_t l_j^{-1} \lambda^{-t}$. Thus $s' = 2^{\alpha-e_2(j)-1}x + i l_t l_j^{-1} \lambda^{-t}$, and hence $1 \leq x \leq 2$. Therefore the number of these cases is $2^{\beta-1} \cdot 2 = 2^{\beta}$. Applying the above results, we obtain $|C_G(a^i b^j)| =$

$$\begin{cases} 2^{\alpha+\beta-\gamma+e_2(j)}, & j \text{ is even and } (e_2(i) \geq \alpha - 1 \text{ or } e_2(i) - e_2(j) \geq \alpha - \gamma - 1), \\ 2^{\alpha+\beta-\gamma+e_2(j)-1}, & j \text{ is even and } e_2(i) < \alpha - 1 \text{ and } e_2(i) - e_2(j) < \alpha - \gamma - 1), \\ 2^{\beta+1}, & j \text{ is odd.} \end{cases}$$

(**b**) Next, suppose that $e_2(j) \geq \gamma$ and $e_2(i) < \gamma$. In this case, j is even. If t is even, then $\gamma - e_2(i) \leq e_2(t) \leq \beta$ and the number of such cases is

$$2^{\alpha}\Big(1+\sum_{e_2(t)=\gamma-e_2(i)}^{\beta-1}\varphi(2^{\beta-e_2(t)})\Big)=2^{\alpha+\beta-\gamma+e_2(i)}.$$

If t is odd, then $e_2(i) \geq \alpha - 1 > \gamma$, which is impossible. Therefore,

$$|C_G(a^ib^j)| = 2^{\alpha+\beta-\gamma+e_2(i)}.$$

(**c**) Now, we assume that $e_2(i), e_2(j) < \gamma$ and j is even. If $e_2(i) \geq e_2(j)$, then we have the following two cases:

(c.1) If t is even, then either $1 \leq e_2(t) < \gamma - e_2(i)$ or $\gamma - e_2(i) \leq e_2(t) \leq \beta$. In the former,

$$e_2(j)+e_2(s)+\alpha-\gamma = e_2(i)+e_2(t)+\alpha-\gamma,$$

which implies that $e_2(s) = e_2(i) - e_2(j) + e_2(t)$. Let $s = 2^{e_2(s)}s'$. Then we have $s' \overset{2^{\gamma-e_2(j)-e_2(s)}}{\equiv} i'l_tl_j^{-1}\lambda^{-t}$, and hence $s' = 2^{\gamma-e_2(j)-e_2(s)}x + i'l_tl_j^{-1}\lambda^{-t}$. Thus $1 \leq x \leq 2^{\alpha-\gamma+e_2(j)}$ and the number of such cases is

$$\sum_{e_2(t)=1}^{\gamma-e_2(i)-1}\varphi(2^{\beta-e_2(t)})\cdot 2^{\alpha-\gamma+e_2(j)} = 2^{\alpha+\beta-2\gamma+e_2(i)+e_2(j)}(2^{\gamma-e_2(i)-1}-1).$$

In the latter, $\gamma - e_2(j) \leq e_2(s) \leq \alpha$ and the number of such cases is

$$\Big(1+\sum_{e_2(t)=\gamma-e_2(i)}^{\beta-1}\varphi(2^{\beta-e_2(t)})\Big)\Big(1+\sum_{e_2(s)=\gamma-e_2(j)}^{\alpha-1}\varphi(2^{\alpha-e_2(s)})\Big)=2^{\alpha+\beta-2\gamma+e_2(i)+e_2(j)}.$$

(c.2) If t is odd, then $e_2(s) = e_2(i) - e_2(j) + 1 + \gamma - \alpha \geq 0$, which implies that $e_2(i) - e_2(j) \geq \alpha - \gamma - 1$. The number of such cases is $2^{\beta-1}\cdot 2^{\alpha-\gamma+e_2(j)} = 2^{\alpha+\beta-\gamma+e_2(j)-1}$. Utilizing the above results we have

$$|C_G(a^ib^j)| = \begin{cases} 2^{\alpha+\beta-\gamma+e_2(j)-1}, & e_2(i)-e_2(j) < \alpha-\gamma-1, \\ 2^{\alpha+\beta-\gamma+e_2(j)}, & e_2(i)-e_2(j) \geq \alpha-\gamma-1. \end{cases}$$

Now consider the case that $e_2(i) < e_2(j)$. We have two possibilities.

(c.3) If t is even, then either $1 \leq e_2(t) < \gamma - e_2(i)$ or $\gamma - e_2(i) \leq e_2(t) \leq \beta$. In the former, $e_2(s) = e_2(i) - e_2(j) + e_2(t) \geq 0$, which implies that $e_2(j) -$

$e_2(i) \le e_2(t) < \gamma - e_2(i)$. Let $s = 2^{e_2(s)}s'$. Then $s' \overset{2^{\gamma-e_2(j)-e_2(s)}}{\equiv} i'l_tl_j^{-1}\lambda^{-t}$. This gives $s' = 2^{\gamma-e_2(j)-e_2(s)}x + i'l_tl_j^{-1}\lambda^{-t}$. Hence $1 \le x \le 2^{\alpha-\gamma+e_2(j)}$ and the number of such cases is

$$\sum_{e_2(t)=e_2(j)-e_2(i)}^{\gamma-e_2(i)-1} \varphi(2^{\beta-e_2(t)}) \cdot 2^{\alpha-\gamma+e_2(j)} = 2^{\alpha+\beta-2\gamma+e_2(i)+e_2(j)}(2^{\gamma-e_2(j)}-1).$$

In the latter, $\gamma - e_2(i) \le e_2(s) \le \alpha$ and the number of such cases is

$$\Big(1+\sum_{e_2(t)=\gamma-e_2(i)}^{\beta-1} \varphi(2^{\beta-e_2(t)})\Big)\Big(1+\sum_{e_2(s)=\gamma-e_2(j)}^{\alpha-1} \varphi(2^{\alpha-e_2(s)})\Big) = 2^{\alpha+\beta-2\gamma+e_2(i)+e_2(j)}.$$

(c.4) If t is odd, then $e_2(s) = e_2(i) - e_2(j) + 1 + \gamma - \alpha < 0$, which is impossible. Therefore $|C_G(a^ib^j)| = 2^{\alpha+\beta-\gamma+e_2(i)}$.

(**d**) Finally, consider the case $e_2(i), e_2(j) < \gamma$ and j is odd. Thus we deal with two cases.

(d.1) t is even. If $e_2(s) \ge \alpha - 1$, then $\gamma - e_2(i) \le e_2(t) \le \beta$ and the number of such cases is

$$2\Big(1+\sum_{e_2(t)=\gamma-e_2(i)}^{\beta-1} \varphi(2^{\beta-e_2(t)})\Big) = 2^{\beta-\gamma+e_2(i)+1}.$$

Also, if $e_2(s) < \alpha - 1$, then $e_2(s) = e_2(i) + e_2(t) + \alpha - \gamma - 1 < \alpha - 1$, which implies that $1 \le e_2(t) < \gamma - e_2(i)$. Let $s = 2^{e_2(s)}s'$. Then $s' = 2^{\alpha-e_2(s)-1}x + i'l_tl_j^{-1}\lambda^{-t}$, from which it follows that $1 \le x \le 2$. Hence the number of such cases is

$$2\Big(\sum_{e_2(t)=1}^{\gamma-e_2(i)-1} \varphi(2^{\beta-e_2(t)})\Big) = 2^{\beta-\gamma+e_2(i)+1}(2^{\gamma-e_2(i)-1}-1).$$

(d.2) t is odd. If $e_2(s) \ge \alpha - 1$, then $e_2(i) \ge \alpha - 1$, which is a contradiction. Thus $e_2(s) < \alpha - 1$, which implies that $e_2(s) = e_2(i)$. Let $s = 2^{e_2(s)}s'$. Then $s' = 2^{\alpha-e_2(s)-1}x + i'l_tl_j^{-1}\lambda^{-t}$, from which it follows that $1 \le x \le 2$. Hence the number of such cases is $2 \cdot 2^{\beta-1} = 2^\beta$. Therefore, $|C_G(a^ib^j)| = 2^{\beta+1}$, as claimed. $\square$

The following corollary is a direct consequence of Theorem 2.3 to obtain the set of lengths of conjugacy classes of G_f, ctv(G_f).

Corollary 2.4. *If G_f is the group in the Equation* (1)*, then*

$$\mathrm{ctv}(G_f) = \{1, 2, 2^2, \ldots, 2^\gamma, 2^{\alpha-1}\}.$$

3 n-th Commutativity Degrees

In this section, the n-th commutativity degrees of the groups G_f are computed. If G_f is a 2-group and $a^ib^j \in G_f$, then

$$a^{i_m}b^{j_m} = (a^ib^j)^{2^m} = a^{i(1+\lambda^j+\cdots+\lambda^{(2^m-1)j})}b^{2^m j} = a^{i\frac{1-\lambda^{2^m j}}{1-\lambda^j}}b^{2^m j}. \tag{4}$$

Thus, $e_2(i_m) = e_2\left(i\frac{1-\lambda^{p^m j}}{1-\lambda^j}\right)$. It is easy to show that $P_n(G_f) = P_{2^m}(G_f)$, where $n = 2^m.\bar{n}$ such that $\gcd(2,\bar{n}) = 1$. Thus $P_n(G_f) = P_{2^{e_2(n)}}(G_f)$, in what follows we assume that $n = 2^m$ is a prime power. Now, we compute the n-th commutativity degree, $P_{2^m}(G_f)$ of groups in Equation (1).

Theorem 3.1. *Let G_f be a group of the form $G_f = \langle a,b : a^{2^\alpha} = 1, b^{2^\beta} = a^{2^{\alpha-\varepsilon}}, a^b = a^{2^{\alpha-\gamma}-1}\rangle$, where $\alpha-\gamma \geq 2$. Then the n-th commutativity degree of G_f is given by:*

$$P_n(G_f) = P_{2^m}(G_f) = \begin{cases} \frac{3}{2^{\alpha+1}} + \frac{3}{2^{\gamma+2}} - \frac{1}{2^{2\gamma+1}}, & m = 0, \\ \frac{5}{2^{\gamma-m+2}} + \frac{1}{2^{\alpha-m+1}} - \frac{1}{2^{2\gamma-2m+1}}, & 1 \leq m < \gamma, \\ \frac{3}{4} + \frac{1}{2^{\alpha-m+1}}, & \gamma \leq m < \alpha-1, \\ 1, & m \geq \alpha-1. \end{cases}$$

Proof. The situation is different for $m = 0$, $1 \leq m < \gamma$, $\gamma \leq m < \alpha-1$ and $m \geq \alpha-1$. We will deal with each case separately. In what follows, the notation (k) stands for those elements a^ib^j of G_f, which appear in the k-th condition in Theorem 2.3, where $1 \leq k \leq 7$. Also we let $e_2^*(k) = \min\{\gamma, e_2(k)\}$ for each natural number k. Now we consider the three cases:

Case 1. Suppose that $m = 0$. Then we have

$$P_1(G_f) = \frac{1}{2^{2\alpha+2\beta}}\left[\sum_{(1)}+\sum_{(2)}+\sum_{(3)}+\sum_{(4)}+\sum_{(5)}+\sum_{(6)}+\sum_{(7)}\right]|C_G(a^ib^j)|.$$

In the sequel, we compute $\sum_{(k)} |C_G(a^ib^j)|$, $k = 1,2,...,7$. We have

$$\begin{aligned}
\sum_{(1)} |C_G(a^ib^j)| &= 2^{\alpha+\beta-\gamma} \sum_{(1)} 2^{e_2(i)} = 2^{\alpha+\beta-\gamma} \cdot 2^{\beta-\gamma} \sum_{0 \le e_2(i) < \gamma} 2^{e_2(i)} \\
&= 2^{\alpha+2\beta-2\gamma} \Big(\varphi(2^{\alpha}) 2^0 + \cdots + \varphi(2^{\alpha-\gamma+1}) 2^{\gamma-1} \Big) \\
&= 2^{\alpha+2\beta-2\gamma} \Big(2^{\alpha-1} + \cdots + 2^{\alpha-1} \Big) \\
&= 2^{2\alpha+2\beta-2\gamma-1} \gamma.
\end{aligned}$$

Summing over (2) and (4) and adding the size of the centralizer of a^ib^j, we obtain

$$\begin{aligned}
\sum_{(2)+(4)} |C_{G_f}(a^ib^j)| &= 2^{\alpha+\beta-\gamma} \sum_{\substack{e_2(i) \ge \alpha-1 \text{ or} \\ 1 \le e_2(j) \le e_2(i)+\gamma+1-\alpha}} 2^{e_2^*(j)} \\
&= 2^{\alpha+\beta-\gamma} \Big[2 \sum_{e_2(j) \ge 1} 2^{e_2^*(j)} + \sum_{e_2(i)=\alpha-\gamma}^{\alpha-2} \sum_{e_2(j)=1}^{e_2(i)+\gamma+1-\alpha} 2^{e_2^*(j)} \Big] \\
&= 2^{\alpha+2\beta-\gamma} \big[\gamma+1+(\gamma+1-\alpha)(2^{\gamma-1}-1)+2^{\alpha-1}-\alpha \\
&\qquad -2^{\gamma-1}(2^{\alpha-\gamma}-\alpha+\gamma-1) \big] \\
&= 2^{\alpha+2\beta-\gamma} \cdot 2^{\gamma} \\
&= 2^{\alpha+2\beta}.
\end{aligned}$$

Now, the summation of the size of the centralizers of an arbitrary element $a^ib^j \in G_f$ over the conditions of (6) is computed as follows:

$$\begin{aligned}
\sum_{(6)} |C_{G_f}(a^ib^j)| &= 2^{\alpha+\beta-\gamma} \sum_{0 \le e_2(i) < e_2(j) < \gamma} 2^{e_2(i)} \\
&= 2^{\alpha+\beta-\gamma} \sum_{s=0}^{\gamma-2} 2^s \varphi(2^{\alpha-s})(2^{\beta-s-1} - 2^{\beta-\gamma}) \\
&= 2^{2\alpha+2\beta-2\gamma-1}(2^{\gamma} - \gamma - 1).
\end{aligned}$$

Next, for Cases (3) and (5) we have the following computations:

$$\sum_{(3)+(5)} |C_{G_f}(a^i b^j)| = 2^{\alpha+\beta-\gamma-1}\Big[\sum_{\substack{\gamma\le e_2(i)<\alpha-1\\ \gamma<e_2(j)\le\beta}} 2^{e_2^*(j)} + \sum_{e_2(j)=1}^{\gamma}\sum_{e_2(i)=e_2(j)}^{e_2(j)+\alpha-\gamma-2} 2^{e_2^*(j)}\Big]$$

$$= 2^{\alpha+\beta-\gamma-1}\Big[2^{\gamma}\cdot 2^{\beta-\gamma-1}(2^{\alpha-\gamma}-2)$$

$$+\sum_{s=1}^{\gamma} 2^{s}\varphi(2^{\beta-s})(2^{\alpha-s}-2^{\gamma+1-s})\Big]$$

$$= 2^{2\alpha+2\beta-\gamma-2}-2^{\alpha+2\beta-1}.$$

Finally, for Case (7) we have the following:

$$\sum_{(7)}|C_{G_f}(a^i b^j)| = 2^{\beta+1}\cdot 2^{\beta-1}\cdot 2^{\alpha} = 2^{\alpha+2\beta}.$$

Therefore, from the computations above the commutativity degree of G_f in the case $m=0$ is

$$P_1(G_f) = \frac{3}{2^{\alpha+1}} + \frac{3}{2^{\gamma+2}} - \frac{1}{2^{2\gamma+1}}.$$

Case 2. If $m \ge 1$, then we have

$$e_2(i_m) = \begin{cases} e_2(i)+m, & j \text{ even},\\ e_2(i)+\alpha-\gamma-1+m, & j \text{ odd}.\end{cases}$$

First we consider the case where $m \ge \gamma$. Then $e_2(i_m), e_2(j_m) \ge m \ge \gamma$.

If $m \ge \alpha-1$, then

$$P_{2^m}(G_f) = 1.$$

If $\gamma \le m < \alpha-1$, then we have two cases. Note that, as $e_2(i_m) \ge \gamma$ only conditions (2) and (3) can hold for $a^{i_m}b^{j_m}$.

First suppose that (2) holds. If j is even, then $e_2(i_m) = e_2(i)+m$. If $e_2(i_m) \ge \alpha-1$, then $e_2(i) \ge \alpha-m-1$ and the sum of the size of centralizers of such elements is

$$\sum_{\substack{j \text{ even}\\ e_2(i)\ge\alpha-m-1}} 2^{\alpha+\beta-\gamma+e_2^*(j_m)} = 2^{\alpha+\beta}\cdot 2^{\beta-1}\cdot 2^{m+1} = 2^{\alpha+2\beta+m}.$$

Also if $e_2(i_m)-e_2(j_m)\geq \alpha-\gamma-1$, then $e_2(i)\geq \alpha-\gamma-1+e_2(j)\geq \alpha-m-1$, which is already considered in the previous case.

If j is odd, then $e_2(j)=0$. If $e_2(i_m)-e_2(j_m)\geq \alpha-\gamma-1$, then $e_2(i_m)\geq \alpha-\gamma-1+e_2(j_m)\geq \alpha-1$. Thus $e_2(i)+\alpha-\gamma-1+m\geq \alpha-1$ and it follows that $e_2(i)\geq \gamma-m$. Thus the sum over the sizes of centralizers of such elements is

$$\sum_{\substack{j \text{ odd} \\ 1\leq i<2^\alpha}} 2^{\alpha+\beta-\gamma+e_2^*(j_m)} = 2^{\alpha+\beta}\cdot 2^{\beta-1}\cdot 2^{\alpha} = 2^{2\alpha+2\beta-1}.$$

If $e_2(i_m)\geq \alpha-1$, then $e_2(i)\geq \gamma-m$ and this is exactly the previous case.

Now suppose that (3) holds. If j is even, then as $e_2(i_m)<\alpha-1$, we obtain $e_2(i)<\alpha-m-1$. Also from $e_2(i_m)-e_2(j_m)<\alpha-\gamma-1$, it follows that $e_2(i_m)<\alpha-\gamma-1+e_2(j_m)$. This gives $e_2(i)+m<\alpha-\gamma-1+e_2(j)+m$ which implies $e_2(i)<\alpha-\gamma-1+e_2(j)$, and hence the sum over such cases is

$$\begin{aligned}\sum_{\substack{j \text{ even} \\ 0\leq e_2(i)<\alpha-m-1}} 2^{\alpha+\beta-\gamma+e_2^*(j_m)-1} &= 2^{\alpha+\beta-1}\cdot 2^{\beta-1}\cdot\Big(\varphi(2^\alpha)+\cdots+\varphi(2^{m+2})\Big)\\ &= 2^{\alpha+2\beta+m-1}(2^{\alpha-m-1}-1).\end{aligned}$$

On the other hand, if j is odd, then from $e_2(i_m)<\alpha-1$ it follows that $e_2(i)+\alpha-\gamma-1+m<\alpha-1$. Thus $e_2(i)<\gamma-m\leq 0$, which is impossible. Now, by using the above results, we obtain

$$\begin{aligned}P_{2^m}(G_f) &= \frac{1}{2^{2\alpha+2\beta}}\Big[2^{\alpha+2\beta+m}+2^{2\alpha+2\beta-1}+2^{\alpha+2\beta+m-1}(2^{\alpha-m-1}-1)\Big]\\ &= \frac{1}{2^{2\alpha+2\beta}}\Big[2^{\alpha+2\beta+m}+2^{2\alpha+2\beta-1}+2^{2\alpha+2\beta-2}-2^{\alpha+2\beta+m-1}\Big].\end{aligned}$$

Therefore,

$$P_{2^m}(G_f)=\frac{3}{4}+\frac{1}{2^{\alpha-m+1}}.$$

Case 3. Now we consider the last case. Suppose that $1\leq m<\gamma$. In what follows, (k') and (k'') stand for the set of all elements a^ib^j in condition (k) in such a way that j is even and odd, respectively. First suppose that j is even.

Using similar computation as before we have,

$$\begin{aligned}\sum_{(1')}|C_{G_f}(a^{i_m}b^{j_m})| &= 2^{\alpha+\beta-\gamma}\sum_{(1)}2^{e_2(i_m)} = 2^{\alpha+\beta-\gamma}\cdot 2^{\beta-\gamma+m}\sum_{0\leq e_2(i)<\gamma-m}2^{e_2(i)+m}\\ &= 2^{\alpha+2\beta-2\gamma+2m}\left(\varphi(2^{\alpha})2^0+\cdots+\varphi(2^{\alpha-\gamma+m+1})2^{\gamma-m-1}\right)\\ &= 2^{2\alpha+2\beta-2\gamma+2m-1}(\gamma-m).\end{aligned}$$

Next, summing the size of the centralizers of $a^{i_m}b^{j_m}$ over conditions $(2^{'})$ and $(4^{'})$ yield:

$$\begin{aligned}\sum_{(2')+(4')}|C_{G_f}(a^{i_m}b^{j_m})| =& 2^{\alpha+\beta-\gamma}\sum_{(2')+(4')}2^{e_2^*(j_m)}\\ =& 2^{\alpha+\beta-\gamma}\left[2^{m+1}\sum_{e_2(j)=1}^{\beta}2^{e_2^*(j_m)}+\sum_{e_2(i)=\alpha-\gamma}^{\alpha-2-m}\sum_{e_2(j)=1}^{e_2(i)+\gamma+1-\alpha}2^{e_2^*(j_m)}\right]\\ =& 2^{\alpha+2\beta-\gamma+m}[2^m(\gamma-m+1)+(\gamma+1-\alpha)2^m(2^{\gamma-m-1}-1)\\ &+2^m(2^{\alpha-m-1}-\alpha+m)-2^{\gamma-1}(2^{\alpha-\gamma}-\alpha+\gamma-1)]\\ =& 2^{\alpha+2\beta+m}.\end{aligned}$$

Summing over $(6^{'})$ we have the following identities:

$$\begin{aligned}\sum_{(6')}|C_{G_f}(a^{i_m}b^{j_m})| &= 2^{\alpha+\beta-\gamma}\sum_{0\leq e_2(i_m)<e_2(j_m)<\gamma}2^{e_2(i_m)}\\ &= 2^{\alpha+\beta-\gamma+m}\cdot 2^{\alpha-1}\left(2^{\beta-1}\sum_{s=0}^{\gamma-m-2}\frac{1}{2^s}-2^{\beta-\gamma+m}(\gamma-m-1)\right)\\ &= 2^{2\alpha+2\beta-2\gamma+2m-1}(2^{\gamma-m}-(\gamma-m+1)).\end{aligned}$$

Next, we sum over condition $(3^{'})$ and $(5^{'})$ for the size of the centralizers of each element of the form $a^{i_m}b^{j_m}$ in G_f:

$$\begin{aligned}\sum_{(3')+(5')}|C_{G_f}(a^{i_m}b^{j_m})| &= 2^{\alpha+\beta-\gamma-1}\Big[\sum_{\substack{\gamma\le e_2(i_m)<\alpha-1\\ \gamma<e_2(j_m)}} 2^{e_2^*(j_m)}\\
&+\sum_{e_2(j_m)=1}^{\gamma}\ \sum_{e_2(i_m)=e_2(j_m)}^{e_2(j_m)+\alpha-\gamma-2} 2^{e_2^*(j_m)}\Big]\\
&= 2^{\alpha+\beta-\gamma-1}\Big[2^{\beta+2m}(2^{\alpha-\gamma-1}-1)\\
&+\sum_{s=1}^{\gamma-m}2^{\beta+m-1}(2^{\alpha-s}-2^{\gamma+1-s})\Big]\\
&= 2^{\alpha+2\beta+m-1}(2^{\alpha-\gamma-1}-1).\end{aligned}$$

In the last case we obtain $\sum_{(7')}|C_{G_f}(a^{i_m}b^{j_m})| = 0$. Finally, if j is odd then $e_2(j)=0$ that is $e_2(j_m)=m<\gamma$. We have the following sums over $(1'')-(7'')$: $\sum_{(1'')}|C_{G_f}(a^{i_m}b^{j_m})| = 0$, $\sum_{(6'')}|C_{G_f}(a^{i_m}b^{j_m})| = 0$, $\sum_{(3'')+(5'')}|C_G(a^{i_m}b^{j_m})| = 0$ and $\sum_{(7'')}|C_{G_f}(a^{i_m}b^{j_m})| = 0$. Similarly, summing over $(2'')$ and $(4'')$ yields:

$$\begin{aligned}\sum_{(2'')+(4'')}|C_{G_f}(a^{i_m}b^{j_m})| &= 2^{\alpha+\beta-\gamma+m}\sum_{\substack{e_2(j_m)\ge\alpha-1 \text{ or}\\ 1\le e_2(j_m)\le e_2(i_m)+\gamma+1-\alpha}} 1\\
&= 2^{\alpha+\beta-\gamma+m}\Big[\sum_{e_2(j)=0}\ \sum_{e_2(i_m)\ge\alpha-1} 1\\
&+\sum_{e_2(i_m)=\alpha-\gamma}^{\alpha-2}\ \sum_{e_2(j_m)=1}^{e_2(i_m)+\gamma+1-\alpha} 1\Big]\\
&= 2^{\alpha+\beta-\gamma+m}\Big[2^{\alpha+\beta-\gamma+m-1}+2^{\beta-1}\Big(\varphi(2^{\alpha})+\cdots\\
&+\varphi(2^{\alpha-\gamma+m+1})\Big)\Big]\\
&= 2^{2\alpha+2\beta-\gamma+m-1}.\end{aligned}$$

Thus, the 2^m-th commutativity degree of G_f in the case $1\le m<\gamma$ is

$$P_{2^m}(G_f) = \frac{5}{2^{\gamma-m+2}}+\frac{1}{2^{\gamma-m+1}}-\frac{1}{2^{2\gamma-2m+1}}.$$

Hence, concerning the above computations the result holds. □

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