Journal of Mathematical Extension
Vol. 16, No. 10, (2022) (4)1-12
URL: https://doi.org/10.30495/JME.2022.2099
ISSN: 1735-8299
Original Research Paper

On $\mathcal{I}_{\phi}^{\mathcal{K}}-$Convergence

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Abstract. In this article, we introduce the notion of $\mathcal{I}_{\phi}^{\mathcal{K}}-$convergence of real sequences as an extension of $\mathcal{I}^{\mathcal{K}}-$convergence. We investigate various properties and implication relations of this convergence method.

AMS Subject Classification: 40A35; 40A05
Keywords and Phrases: $\phi-$convergence, $\mathcal{I}-$convergence, $\mathcal{I}^{\mathcal{K}}-$convergence, $\mathcal{I}_{\phi}^{\mathcal{K}}-$convergence

1 Introduction

In 1951, H. Fast and H. Steinhaus extended the method of usual convergence to statistical convergence independently (see [9, 24]) by involving the concept of natural density. The natural density [19] of a set $A \subseteq \mathbb{N}$ is a real number $d(A)$ lying in the interval $[0, 1]$ defined as $d(A) = \lim\limits_{k} \frac{|\{a \in A : a \leq k\}|}{k}$, (if the limit exists) where $k \in \mathbb{N}$ and the vertical bar denotes the number of elements in the set $\{a \in A : a \leq k\}$. A sequence $x = (x_k)$ is said to be statistically convergent to a number l if for every $\varepsilon > 0$, the natural density of the set of all k's for which the corresponding sequential term x_k lies outside the interval

Received: July 2021; Accepted: November 2021
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$(l-\varepsilon, l+\varepsilon)$ is zero [10]. In other words if the condition $d(A(\varepsilon)) = 0$ where $A(\varepsilon) = \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ holds for each $\varepsilon > 0$.

50 years later, in 2001 the idea of statistical convergence was further extended to two types of convergence namely, $\mathcal{I}$ and $\mathcal{I}^*-$convergence by Kostyrko et al. [14]. $\mathcal{I}-$convergence was not only the generalization of statistical convergence but also so many known convergence methods become the particular cases of $\mathcal{I}-$convergence. Several works in this direction can be found in [7, 8, 11, 16, 17, 18, 21, 23].

On the other hand, in 2011 the $\mathcal{I}^*-$convergence method was further extended to $\mathcal{I}^{\mathcal{K}}-$convergence by M. Macaj and M. Sleziak in [15], where the convergence along a set from the associated filter $\mathcal{F}(\mathcal{I})$ was considered with respect to another ideal $\mathcal{K}$ instead of ordinary convergence. In other words, a sequence $x = (x_k)$ is said to be $\mathcal{I}^{\mathcal{K}}-$convergent to a real number l, if for every $\varepsilon > 0$, there exists $M = \{m_1 < m_2 < ... < m_k < ..\} \in \mathcal{F}(\mathcal{I})$ such that $\{k \in M : |x_k - l| \geq \varepsilon\} \in \mathcal{K}$. In particular when $\mathcal{K} = \mathcal{I}_f$, where $\mathcal{I}_f$ is the ideal consisting of all finite subsets of $\mathbb{N}$, then we get $\mathcal{I}^*-$convergence. So this generalization makes sense and is found to be interesting to many mathematicians. Further investigations, findings and extensions related to $\mathcal{I}^{\mathcal{K}}-$convergence can be found in [1, 2, 3, 4, 5].

An Orlicz function [20] is a function $\phi : \mathbb{R} \to \mathbb{R}$ such that it is even, non-decreasing on $\mathbb{R}^+$, continuous on $\mathbb{R}$, and satisfying

$$\phi(x) = 0 \Longleftrightarrow x = 0 \text{ and } \phi(x) \to \infty \text{ as } x \to \infty,$$

where $\mathbb{R}$, $\mathbb{R}^+$, and ϕ stands for the set of all real numbers, set of all positive real numbers, and Orlicz function respectively.

An Orlicz function $\phi : \mathbb{R} \to \mathbb{R}$ is said to satisfy the $\triangle_2$ condition, if there exists a $K > 0$ such that $\phi(2x) \leq K \cdot \phi(x)$, for every $x \in \mathbb{R}^+$.

Example 1.1. [22] (i) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) =| x |$ is an Orlicz function.

(ii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^7$ is not an Orlicz function.

(iii) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = x^2$ is an Orlicz function satisfying the $\triangle_2$ condition.

(iv) The function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(x) = e^{|x|} - | x | - 1$ is an Orlicz function not satisfying the $\triangle_2$ condition.

In [20], Rao and Ren described the important roles and applications of Orlicz function in various fields like economics, stochastic problems, etc.

In 2019, Khusnussaadah and Supama [12] introduced the concept of $\phi-$convergence using the Orlicz function ϕ. Later on, in this direction, Savas and Debnath introduced lacunary statistically $\phi-$convergence [22] and Debnath and Choudhury introduced $\mathcal{I}-$statistically $\phi-$convergence [6].

In this paper, by using $\mathcal{I}^\mathcal{K}-$convergence and $\phi-$convergence we introduce a new idea called $\mathcal{I}_\phi^\mathcal{K}-$convergence mainly as a generalization of $\mathcal{I}^\mathcal{K}-$convergence.

2 Definitions and Preliminaries

Definition 2.1. *[13] A family $\mathcal{I} \subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if and only if (i) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (Additive) and (ii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$ (Hereditary).*

If $\forall x \in X$, $\{x\} \in \mathcal{I}$, then $\mathcal{I}$ is said to be admissible. Also, $\mathcal{I}$ is said to be non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Some standard examples of ideal are given below:
(i) The set $\mathcal{I}_f = \{A \subseteq \mathbb{N} : |A| < \infty\}$ is an admissible ideal in $\mathbb{N}$ where $|A|$ represents the cardinal number of set A.
(ii) The set $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$ is an admissible ideal in $\mathbb{N}$ where $d(A)$ is the natural density of A.
(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.
(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ be a decomposition of $\mathbb{N}$ such that $D_i \cap D_j = \emptyset$ satisfies for $i \neq j$. Then, the set $\mathcal{I} = \{A \subseteq \mathbb{N} : |\{p : A \cap D_p \neq \emptyset\}| < \infty\}$ forms an ideal in $\mathbb{N}$.

More important examples can be found in [11] and [13].

Definition 2.2. *[13] A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is said to be a filter in X if and only if (i) $\emptyset \notin \mathcal{F}$ (ii) $M, N \in \mathcal{F}$ implies $M \cap N \in \mathcal{F}$ and (iii) $M \in \mathcal{F}, N \supset M$ implies $N \in \mathcal{F}$.*

If $\mathcal{I}$ is a proper non-trivial ideal in X, then $\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists\, A \in$

$\mathcal{I}$ *s.t* $M = X \setminus A\}$ is a filter in X. It is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.3. *[14] A sequence $x = (x_k)$ is said to be $\mathcal{I}-$convergent to l if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ belongs to $\mathcal{I}$. In this case, the real number l is called the $\mathcal{I}-$limit of the sequence $x = (x_k)$. Symbolically, $\mathcal{I} - \lim_{k\to\infty} x_k = l$.*

Definition 2.4. *[15] Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}^{\mathcal{K}}-$convergent to l if there exists $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $y = (y_k)$ defined by $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is $\mathcal{K}-$convergent to l.*

Definition 2.5. *[12] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A sequence $x = (x_k)$ is said to be $\phi-$convergent to l if $\lim_k \phi(x_k - l) = 0$. In this case, l is called the $\phi-$limit of (x_k) and it is denoted by $\phi - \lim x = l$.*

Definition 2.6. *[6] Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. A real sequence $x = (x_k)$ is said to be $\mathcal{I}_\phi-$convergent to a real number l if for every $\varepsilon > 0$, the set $A(\varepsilon) = \{k \in \mathbb{N} : \phi(x_k - l) \geq \varepsilon\}$ belongs to $\mathcal{I}$. Symbolically we write $\mathcal{I}_\phi - \lim_{k\to\infty} x_k = l$.*

Remark 2.7. *[1] If $\mathcal{I}$ and $\mathcal{K}$ are two ideals in $\mathbb{N}$ then the set $\mathcal{I} \vee \mathcal{K} = \{A \cup B : A \in \mathcal{I}, B \in \mathcal{K}\}$ forms an ideal in $\mathbb{N}$. Further, if $\mathcal{I} \vee \mathcal{K}$ is non-trivial then the dual filter of $\mathcal{I} \vee \mathcal{K}$ is denoted and defined by $\mathcal{F}(\mathcal{I} \vee \mathcal{K}) = \{M \cap N : M \in \mathcal{F}(\mathcal{I}), N \in \mathcal{F}(\mathcal{K})\}$.*

Throughout the paper, unless stated, the symbols $\mathcal{I}, \mathcal{K}, \mathcal{I} \vee \mathcal{K}, \mathcal{I}_1, \mathcal{I}_2$, $\mathcal{K}_1$, and $\mathcal{K}_2$ stands for non-trivial admissible ideal in $\mathbb{N}$, and the sequences that we have considered are real sequences.

3 Main Results

Definition 3.1. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}_\phi^{\mathcal{K}}-$convergent to l if there exists $M \in \mathcal{F}(\mathcal{I})$ such that the* *sequence* $y=(y_k)$ *defined by* $y_k=\begin{cases} x_k, & k\in M\\ l, & k\notin M\end{cases}$ *is* $\mathcal{K}_\phi-$*convergent to* l. *Symbolically we write* $\mathcal{I}_\phi^\mathcal{K}-\lim\limits_{k\to\infty} x_k=l$.

If we consider $\phi(x)=|x|$, then we get $\mathcal{I}^\mathcal{K}-$convergence. So, $\mathcal{I}_\phi^\mathcal{K}-$convergence is a generalization of $\mathcal{I}^\mathcal{K}-$convergence.

Example 3.2. Let $\phi:\mathbb{R}\to\mathbb{R}$ be an Orlicz function defined as $\phi(x)=|x|$. Consider the decomposition of $\mathbb{N}$ given by $\mathbb{N}=\bigcup\limits_{p=1}^{\infty} D_p$, where $D_p=\{2^{p-1}(2s-1):s=1,2,3,..\}$. Let $\mathcal{I}$ be the ideal consisting of all subsets of $\mathbb{N}$ which intersects a finite number of D_p's. Consider the sequence $x=(x_k)$ defined by $x_k=\frac{1}{p}$ if $k\in D_p$. Then the sequence is $\mathcal{I}_\phi^\mathcal{I}-$convergent to 0.
Justification: Let $M=\mathbb{N}\setminus D_1$. Then $M\in\mathcal{F}(\mathcal{I})$ and it is easy to verify that the sequence $y=(y_k)$ defined by $y_k=\begin{cases} x_k, & k\in M\\ 0, & k\notin M\end{cases}$ is $\mathcal{I}_\phi-$convergent to 0. Thus $\mathcal{I}_\phi^\mathcal{I}-\lim\limits_{k\to\infty} x_k=0$.

Theorem 3.3. *Let* $\phi:\mathbb{R}\to\mathbb{R}$ *be a convex Orlicz function with* $\triangle_2$ *condition. Suppose* $x=(x_k)$ *be a sequence such that* $\mathcal{I}_\phi^\mathcal{K}-\lim\limits_{k\to\infty} x_k=l$. *Then* l *is unique.*

Proof. Since ϕ satisfies $\triangle_2$ condition, so there exists $K>0$ such that $\phi(2x)\le K\cdot\phi(x)$. If possible suppose there exists $l_1,l_2\in\mathbb{R}$, $l_1\ne l_2$ such that

$$\mathcal{I}_\phi^\mathcal{K}-\lim_{k\to\infty} x_k=l_1 \text{ and } \mathcal{I}_\phi^\mathcal{K}-\lim_{k\to\infty} x_k=l_2.$$

So, there exists $M,N\in\mathcal{F}(\mathcal{I})$ such that the sequences $y=(y_k)$ and $z=(z_k)$ defined as follows

$$y_k=\begin{cases} x_k, & k\in M\\ l_1, & k\notin M\end{cases} \text{ and } z_k=\begin{cases} x_k, & k\in N\\ l_2, & k\notin N\end{cases}$$

are $\mathcal{K}_\phi-$convergent to l_1 and l_2 respectively. Thus for every $\varepsilon>0$, the sets $A(\varepsilon),B(\varepsilon)\in\mathcal{K}$, where $A(\varepsilon)=\{k\in\mathbb{N}:\phi(y_k-l_1)\ge\frac{\varepsilon}{K}\}$ and $B(\varepsilon)=\{k\in\mathbb{N}:\phi(z_k-l_2)\ge\frac{\varepsilon}{K}\}$. Now, we claim that the following inclusion is true

$$(\mathbb{N}\setminus A(\varepsilon))\cap(\mathbb{N}\setminus B(\varepsilon))\subseteq\{k\in\mathbb{N}:\phi((y_k-z_k)-(l_1-l_2))<\varepsilon\}. \quad (1)$$

For any $p \in (\mathbb{N} \setminus A(\varepsilon)) \cap (\mathbb{N} \setminus B(\varepsilon))$, we have $\phi(y_p - l_1) < \frac{\varepsilon}{K}$ and $\phi(z_p - l_2) < \frac{\varepsilon}{K}$. Therefore, the following inequality holds because of ϕ is even, convex and ϕ has $\triangle_2-$condition

$$\begin{aligned}\phi((y_p - z_p) - (l_1 - l_2)) &= \phi(\frac{1}{2}(2y_p - 2l_1) + \frac{1}{2}(-2z_p + 2l_2)) \\ &\leq \frac{1}{2}\phi(2(y_p - l_1)) + \frac{1}{2}\phi(2(z_p - l_2)) \\ &\leq \frac{K}{2}\phi(y_p - l_1) + \frac{K}{2}\phi(z_p - l_2) \\ &< \frac{K}{2} \cdot \frac{\varepsilon}{K} + \frac{K}{2} \cdot \frac{\varepsilon}{K} = \varepsilon.\end{aligned}$$

Consequently, the inclusion (1) holds, and eventually we can say that the sequence $y - z = (y_k - z_k)$ defined as

$$y_k - z_k = \begin{cases} 0, & k \in M \cap N \\ x_k - l_2, & k \in M \setminus N \\ l_1 - x_k, & k \in N \setminus M \\ l_1 - l_2, & k \in M^c \cap N^c \end{cases}$$

is $\mathcal{K}_\phi-$convergent to $l_1 - l_2$. In other words,

$$\forall \varepsilon > 0,\ \{k \in \mathbb{N} : \phi((y_k - z_k) - (l_1 - l_2)) \geq \varepsilon\} \in \mathcal{K}. \tag{2}$$

Choose $\varepsilon := \phi(\frac{l_1 - l_2}{2})$. Then, from Equation (2) we get

$$\{k \in \mathbb{N} : \phi((y_k - z_k) - (l_1 - l_2)) \geq \phi(\frac{l_1 - l_2}{2})\} \in \mathcal{K}.$$

Now as the inclusion

$$M \cap N \subseteq \{k \in \mathbb{N} : \phi((y_k - z_k) - (l_1 - l_2)) \geq \phi(\tfrac{l_1 - l_2}{2})\}$$

holds, so by hereditary of $\mathcal{K}$, $M \cap N \in \mathcal{K}$ which implies $\mathbb{N} \setminus (M \cap N) \in \mathcal{F}(\mathcal{K})$. Again as $M, N \in \mathcal{F}(\mathcal{I})$, so $M \cap N \in \mathcal{F}(\mathcal{I})$. Now $\mathbb{N} \setminus (M \cap N) \in \mathcal{F}(\mathcal{K})$ and $M \cap N \in \mathcal{F}(\mathcal{I})$ implies $(\mathbb{N} \setminus (M \cap N)) \cap (M \cap N) \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$ i.e $\emptyset \in \mathcal{F}(\mathcal{I} \vee \mathcal{K})$, a contradiction. $\square$

Theorem 3.4. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function with $\triangle_2$ condition. Let $\mathcal{I}$, $\mathcal{K}$, and $\mathcal{I} \vee \mathcal{K}$ be non-trivial ideal in $\mathbb{N}$ such that $\mathcal{I}_\phi^\mathcal{K} - \lim\limits_{k\to\infty} x_k = l_1$ and $\mathcal{I}_\phi^\mathcal{K} - \lim\limits_{k\to\infty} y_k = l_2$. Then,*
(i) $\mathcal{I}_\phi^\mathcal{K} - \lim\limits_{k\to\infty} (x_k + y_k) = l_1 + l_2$ and (ii) $\mathcal{I}_\phi^\mathcal{K} - \lim\limits_{k\to\infty} (x_k y_k) = l_1 l_2$.

Proof. (i) Suppose $\mathcal{I}_\phi^\mathcal{K} - \lim\limits_{k\to\infty} x_k = l_1$ and $\mathcal{I}_\phi^\mathcal{K} - \lim y_k = l_2$. Then by definition there exists $M, N \in \mathcal{F}(\mathcal{I})$ such that the sequences $u = (u_k)$ defined by

$$u_k = \begin{cases} x_k, & k \in M \\ l_1, & k \notin M \end{cases}$$

and $v = (v_k)$ defined by

$$v_k = \begin{cases} y_k, & k \in N \\ l_2, & k \notin N \end{cases}$$

are respectively $\mathcal{K}_\phi-$convergent to l_1 and l_2. Then, it is quite easy to prove that the sequence $u + v = (u_k + v_k)$ defined by

$$u_k + v_k = \begin{cases} x_k + y_k, & k \in M \cap N \\ x_k + l_2, & k \in M \setminus N \\ y_k + l_1, & k \in N \setminus M \\ l_1 + l_2, & k \in M^c \cap N^c \end{cases}$$

is $\mathcal{K}_\phi-$convergent to $l_1 + l_2$. In other words

$$\forall\, \varepsilon > 0,\ \{k \in \mathbb{N} : \phi((u_k + v_k) - (l_1 + l_2)) \geq \varepsilon\} \in \mathcal{K}. \tag{3}$$

Now by definition of $u + v$ we have,

$$\begin{aligned} \{k \in \mathbb{N} : \phi((u_k + v_k) - (l_1 + l_2)) \geq \varepsilon\} \\ = \{k \in M \cap N : \phi((x_k + y_k) - (l_1 + l_2)) \geq \varepsilon\} \\ \cup \{k \in M \setminus N : \phi(x_k - l_1) \geq \varepsilon\} \\ \cup \{k \in N \setminus M : \phi(y_k - l_2) \geq \varepsilon\}. \end{aligned} \tag{4}$$

Clearly $M \cap N \in \mathcal{F}(\mathcal{I})$. Now consider the sequence $w = (w_k)$ defined as $w_k = \begin{cases} x_k + y_k, & k \in M \cap N \\ l_1 + l_2, & k \notin M \cap N \end{cases}$. Then from Equation (3), (4) and by definition of w,

$$\begin{aligned}
&\{k \in \mathbb{N} : \phi(w_k - (l_1 + l_2)) \geq \varepsilon\} \\
&\qquad = \{k \in M \cap N : \phi(w_k - (l_1 + l_2)) \geq \varepsilon\} \\
&\qquad\quad \cup \{k \in \mathbb{N} \setminus (M \cap N) : \phi(w_k - (l_1 + l_2)) \geq \varepsilon\} \\
&\qquad = \{k \in M \cap N : \phi((x_k + y_k) - (l_1 + l_2)) \geq \varepsilon\} \\
&\qquad \subseteq \{k \in \mathbb{N} : \phi((u_k + v_k) - (l_1 + l_2)) \geq \varepsilon\} \in \mathcal{K}. \quad (5)
\end{aligned}$$

From Equation (5), it is clear that w is $\mathcal{K}_\phi-$convergent to $l_1 + l_2$. Hence $(x_k + y_k)$ is $\mathcal{I}_\phi^\mathcal{K}-$convergent to $l_1 + l_2$.
(ii) We omitted the proof as it can be obtained by applying the similar technique. □

Theorem 3.5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function. Then, $\mathcal{K}_\phi - \lim_{k\to\infty} x_k = l$ implies $\mathcal{I}_\phi^\mathcal{K} - \lim_{k\to\infty} x_k = l$.*

Proof. Since $\mathcal{K}_\phi - \lim_{k\to\infty} x_k = l$, so for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : \phi(x_k - l) \geq \varepsilon\} \in \mathcal{K}. \tag{6}$$

Choose $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. Consider the sequence $y = (y_k)$ defined by $y_k = x_k$ for $k \in M$. Then, using (6), we get for every $\varepsilon > 0$, $\{k \in \mathbb{N} : \phi(y_k - l) \geq \varepsilon\} \in \mathcal{K}$ i.e $y = (y_k)$ is $\mathcal{K}_\phi-$convergent to l. Hence $\mathcal{I}_\phi^\mathcal{K} - \lim_{k\to\infty} x_k = l$. □

Remark 3.6. *The converse of Theorem 3.5 is not necessarily true.*

Example 3.7. Let $\phi : \mathbb{R} \to \mathbb{R}$ be defined as $\phi(x) = |x|$. Consider the ideals $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ and $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Let $x = (x_k)$ be the sequence defined as

$$x_k = \begin{cases} 1, & k \text{ is prime} \\ 0, & k \text{ is not prime} \end{cases}.$$

Then, there exists $M =$ set of all non-prime numbers $\in \mathcal{F}(\mathcal{I}_{\rm d})$ such that the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ 0, & k \notin M \end{cases}$ is the null sequence and therefore $\mathcal{I}_{c\,\phi}-$convergent to 0. Hence $x = (x_k)$ is $\mathcal{I}_{d\,\phi}^{\mathcal{I}_c}-$convergent to 0.
But we claim that $x = (x_k)$ is not $\mathcal{I}_{c\,\phi}-$convergent to 0. For if $\mathcal{I}_{c\,\phi} - \lim\limits_{k\to\infty} x_k = 0$, then for $\varepsilon = \frac{1}{2}$, the set $\{k \in \mathbb{N} : \phi(x_k - 0) \geq \frac{1}{2}\} =$ set of all prime numbers $\in \mathcal{I}_c$, a contradiction.

Theorem 3.8. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be an Orlicz function and suppose $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$ satisfying $\mathcal{I} \subseteq \mathcal{K}$. Let $x = (x_k)$ be a real sequence such that $\mathcal{I}_{\phi}^{\mathcal{K}} - \lim\limits_{k\to\infty} x_k = l$. Then $\mathcal{K}_{\phi} - \lim\limits_{k\to\infty} x_k = l$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{K}$ holds and the sequence $x = (x_k)$ is $\mathcal{I}_{\phi}^{\mathcal{K}}-$convergent to l. So by definition, there exists $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is $\mathcal{K}_{\phi}-$convergent to l, which immediately implies

$$\forall\, \varepsilon > 0,\ \{k \in M : \phi(x_k - l) \geq \varepsilon\} \in \mathcal{K}. \tag{7}$$

Thus $\{k \in \mathbb{N} : \phi(x_k - l) \geq \varepsilon\} \subseteq \{k \in M : \phi(x_k - l) \geq \varepsilon\} \cup (\mathbb{N} \setminus M) \in \mathcal{K}$, by (7) and since as per our assumption $\mathcal{I} \subseteq \mathcal{K}$.
Hence, $\mathcal{K}_{\phi} - \lim\limits_{k\to\infty} x_k = l$. □

Remark 3.9. *If a sequence is $\mathcal{I}_{\phi}^{\mathcal{K}}-$convergent then it may not be $\mathcal{I}_{\phi}-$convergent.*

Example 3.10. Let us consider $\phi(x) = |x|$. Let $\mathcal{I}$ denote the ideal which considered in Example 3.2 and suppose $\mathcal{I}_c$ is the ideal given by $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$. Let $M = \{k \in \mathbb{N} : k = 2^p$ for some non-negative integer p$\}$. Consider the sequence $x = (x_k)$ defined as

$$x_k = \begin{cases} 1, & k \in M \\ 0, & k \notin M \end{cases}.$$

Then, it is easy to verify that x is $\mathcal{I}_{\phi}^{\mathcal{I}_c}-$convergent to 0 but x is not $\mathcal{I}_{\phi}-$convergent to 0.

Remark 3.11. *If a sequence is $\mathcal{I}_{\phi}-$convergent then it may not be $\mathcal{I}_{\phi}^{\mathcal{K}}-$convergent. Let us consider $\phi(x) = |x|$. Consider the ideal $\mathcal{I}$ and the sequence $x = (x_k)$ defined in Example 3.2. Then by virtue of Example 2.1 of [13] one can show that $\mathcal{I}_{\phi}^{\mathcal{I}_f} - \lim\limits_{k\to\infty} x_k \neq 0$ although $\mathcal{I}_{\phi} - \lim\limits_{k\to\infty} x_k = 0$.*

Theorem 3.12. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex Orlicz function and suppose $\mathcal{I}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}, \mathcal{K}_1,$ and $\mathcal{K}_2$ be ideals on $\mathbb{N}$ satisfying $\mathcal{I}_1 \subseteq \mathcal{I}_2$ and $\mathcal{K}_1 \subseteq \mathcal{K}_2$. Let $x = (x_k)$ be a real sequence. Then,*
(i) $\mathcal{I}_{\phi}^{\mathcal{K}_1} - \lim\limits_{k\to\infty} x_k = l$ implies $\mathcal{I}_{\phi}^{\mathcal{K}_2} - \lim\limits_{k\to\infty} x_k = l$;
(ii) ${\mathcal{I}_1}^{\mathcal{K}}_{\phi} - \lim\limits_{k\to\infty} x_k = l$ implies ${\mathcal{I}_2}^{\mathcal{K}}_{\phi} - \lim\limits_{k\to\infty} x_k = l$.

Proof. The proof follows from Definition 3.1 and so is omitted. □

Acknowledgements
The authors thank the anonymous referee for their constructive comments and suggestions to improve the quality of the paper. The second author is grateful to the **University Grants Commission, India** for their fellowships funding under the **UGC-JRF** scheme **(F. No. 16-6(DEC. 2018)/2019(NET/CSIR))** during the preparation of this paper.

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