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Some Inequalities Involving Laplace Transformation

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Abstract. In this paper, we prove some integral inequalities, where the right hand side of some inequalities involve the Laplace transformation.

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1. Introduction

Through this section we introduce some known statements. In fact, G.H. Hardy proved the following known Theorems:

Theorem 1.1. ([1]). *Let f be nonnegative integrable function. Define*

$$F(x) = \int_0^x f(t)\,dt,$$

then

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p dx < \Big(\frac{p}{p-1}\Big)^p \int_0^\infty f^p(x)\,dx, \qquad (p>1).$$

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Theorem 1.2. ([1]). *If* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f \geqslant 0$, $g \geqslant 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, $0 < \int_0^\infty g^q(x)dx < \infty$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{sin(\frac{\pi}{p})}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}(\int_0^\infty g^q(x)dx)^{\frac{1}{q}},$$

where the constant factor is the best possible. Its equivalent form is

$$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}dx\Big)^p dy < \Big[\frac{\pi}{sin(\frac{\pi}{p})}\Big]^p \int_0^\infty f^p(x)dx.$$

Inequality in above mentioned theorem is called Hardy-Hilbert's integral inequality, which is important in analysis and its applications(cf. Mitrinovic et al.[5]). Recently, various extensions on above mentioned inequality have appeared ([2,3,4,6,7]).

Theorem 1.3. ([1]). *If* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $k(x) > 0$ *and*

$$\int k(x)x^{s-1}dx = \phi(s),$$

then

$$\int\int k(xy)f(x)g(y)dxdy < \phi\Big(\frac{1}{p}\Big)\Big(\int x^{p-2}f^p(x)dx\Big)^{\frac{1}{p}}\Big(\int g^q(y)dy\Big)^{\frac{1}{q}},$$

$$\int\Big(\int k(xy)f(y)dy\Big)^p dx < \phi^p\Big(\frac{1}{p}\Big)\Big(\int x^{p-2}f^p(x)dx\Big),$$

$$\int x^{p-2}\Big(\int k(xy)f(y)dy\Big)^p dx < \phi^p\Big(\frac{1}{q}\Big)\Big(\int f^p(x)dx\Big).$$

In this study, by the following theorem known as generalized Minkowski's inequality we obtain some integral inequalities.

Theorem 1.4. ([1]). *If* $k > 1$, *then*

$$\Big[\int\Big\{\int f(x,y)\,dy\Big\}^k dx\Big]^{\frac{1}{k}} < \int\Big\{\int f^k(x,y)\,dx\Big\}^{\frac{1}{k}} dy,$$

unless

$$f(x,y) \equiv \phi(x)\psi(y).$$

In this work, by applying the generalized Minkowski's inequality we prove some inequalities involving Laplace transformation.

2. Main Results

At first, note that $\chi[x,\infty)(t) = \chi[0,t](x)$ when ever t and x are nonnegative variables and x is the characteristic function.

Theorem 2.1. *Let f be a nonnegative integrable function which its Laplace transformation exists and $p > 1$. Define*

$$F(x) = \int_0^x f(t)\,dt.$$

Then

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p dx < \frac{1}{p-1}\Big(\int_0^\infty L_f(s)\,ds\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big),$$

where L_f is the Laplace transformation of f.

Proof.

$$\begin{aligned}
\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p dx &= \int_0^\infty \Big(\int_0^\infty \frac{1}{x}\chi_{[0,x]}(t) f(t)\,dt\Big)^p dx \\
&< \Big(\int_0^\infty \Big[\int_0^\infty \frac{1}{x^p}\chi^p_{[0,x]}(t) f^p(t)\,dx\Big]^{\frac{1}{p}} dt\Big)^p \\
&= \Big(\int_0^\infty \Big[\int_0^\infty \chi_{[t,+\infty)}(x) x^{-p}\,dx\Big]^{\frac{1}{p}} f(t)\,dt\Big)^p \\
&= \Big(\int_0^\infty \Big[\int_t^\infty x^{-p}\,dx\Big]^{\frac{1}{p}} f(t)\,dt\Big)^p \\
&= \frac{1}{p-1}\Big(\int_0^\infty t^{-\frac{1}{q}} f(t)\,dt\Big)^p \\
&= \frac{1}{p-1}\Big(\int_0^\infty \big(\frac{f(t)}{t}\big)^{\frac{1}{q}} (f(t))^{\frac{1}{p}}\,dt\Big)^p.
\end{aligned}$$

Applying the Holder's inequality, one may obtain

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p dx < \frac{1}{p-1}\Big(\int_0^\infty \frac{f(t)}{t}\,dt\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big).$$

Now, since

$$\int_0^\infty \frac{f(t)}{t}\,dt = \int_0^\infty L_f(s)\,ds,$$

the assertion is proved. □

Theorem 2.2. *Let f be a nonnegative integrable function and $p>1$. Define*

$$F(x) = \int_x^\infty f(t)\,dt.$$

Then

$$\int_0^\infty F^p(x)\,dx < \Big(\int_0^\infty tf(t)\,dt\Big)\Big(\int_0^\infty f(t)\,dt\Big)^{p-1}.$$

Proof.

$$\begin{aligned}\int_0^\infty F^p(x)\,dx &= \int_0^\infty \Big(\int_0^\infty \chi_{[x,+\infty)}(t)f(t)\,dt\Big)^p dx\\ &< \Big(\int_0^\infty \Big[\int_0^\infty \chi_{[x,+\infty)}(t)f^p(t)\,dx\Big]^{\frac{1}{p}} dt\Big)^p\\ &= \Big(\int_0^\infty \Big[\int_0^\infty \chi_{[0,t]}(x)f^p(t)\,dx\Big]^{\frac{1}{p}} dt\Big)^p\\ &= \Big(\int_0^\infty t^{\frac{1}{p}} f(t)\,dt\Big)^p\\ &= \Big(\int_0^\infty (tf(t))^{\frac{1}{p}}(f(t))^{\frac{1}{q}}\,dt\Big)^p\\ &\leqslant \Big(\int_0^\infty tf(t)\,dt\Big)\Big(\int_0^\infty f(t)\,dt\Big)^{p-1}. \quad \square\end{aligned}$$

Lemma 2.3. *Suppose that $p>1$, $r>\frac{p-1}{p}$ and $t>0$. Then*

$$\int_t^\infty \frac{(x-t)^{rp-p}}{x^{rp}}\,dx = t^{1-p}\beta(p-1, rp-p+1).$$

Proof. By putting $x - t = u$, one may obtain

$$\int_t^\infty \frac{(x-t)^{rp-p}}{x^{rp}}\,dx = \int_0^\infty \frac{u^{-p}}{(1+\frac{t}{u})^{rp}}\,du.$$

Finally, suppose that $\frac{t}{u} = v$.Then

$$\int_0^\infty \frac{u^{-p}}{(1+\frac{t}{u})^{rp}}\,du = t^{1-p}\int_0^\infty v^{p-2}(1+v)^{-rp}\,dv$$

$$= t^{1-p}\beta(p-1, rp-p+1). \quad \square$$

Theorem 2.4. *Suppose* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $r > \frac{p-1}{p}$ *and*

$$f_r(x) = \frac{1}{\Gamma(r)}\int_0^x (x-t)^{r-1} f(t)\,dt.$$

Then

$$\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^p\,dx < \frac{1}{(\Gamma(r))^p}\beta(rp-p+1, p-1)\Big(\int_0^\infty L_f(s)\,ds\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big).$$

Proof. First of all we have

$$\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^p\,dx = \frac{1}{(\Gamma(r))^p}\int_0^\infty \Big(\int_0^\infty \chi_{[0,x]}(t)\frac{(x-t)^{r-1}}{x^r}f(t)\,dt\Big)^p\,dx.$$

Now, applying Theorem 1.4, one obtains

$$\int_0^\infty \Big(\frac{f_r(x)}{x^r}\Big)^p\,dx < \frac{1}{(\Gamma(r))^p}\Big(\int_0^\infty \Big[\int_0^\infty \chi_{[t,\infty)}(x)\frac{(x-t)^{rp-p}}{x^{rp}}\,dx\Big]^{\frac{1}{p}} f(t)\,dt\Big)^p$$

$$= \frac{1}{(\Gamma(r))^p}\Big(\int_0^\infty \Big[\int_t^\infty \frac{(x-t)^{rp-p}}{x^{rp}}\,dx\Big]^{\frac{1}{p}} f(t)\,dt\Big)^p$$

$$= \frac{1}{(\Gamma(r))^p}\beta(rp-p+1, p-1)\Big(\int_0^\infty t^{\frac{1}{p}-1} f(t)\,dt\Big)^p$$

$$= \frac{1}{(\Gamma(r))^p}\beta(rp-p+1, p-1)\Big(\int_0^\infty (\frac{f(t)}{t})^{\frac{1}{q}}(f(t))^{\frac{1}{p}}\,dt\Big)^p$$

$$\leqslant \frac{1}{(\Gamma(r))^p}\beta(rp-p+1,p-1)\Big(\int_0^\infty L_f(s)\,ds\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big). \quad \square$$

Theorem 2.5. *Suppose $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $r>\frac{p-1}{p}$ and*

$$f^r(x)=\frac{1}{\Gamma(r)}\int_x^\infty (t-x)^{r-1}f(t)\,dt.$$

Then

$$\int_0^\infty (f^r(x))^p\,dx<\frac{1}{\Gamma(r)(rp-p+1)}\Big(\int_0^\infty (tf(t))^{rp-p+1}\,dt\Big)\Big(\int_0^\infty (f(t))^{q-rq+1}\,dt\Big)^{p-1}.$$

Proof. By using assumption and Holder's inequality we have

$$\int_o^\infty (f^r(x))^p\,dx=\frac{1}{\Gamma(r)}\int_0^\infty\Big(\int_0^\infty \chi_{[x,+\infty)}(t)(t-x)^{r-1}f(t)\,dt\Big)^p\,dx$$

$$<\frac{1}{\Gamma(r)}\Big(\int_0^\infty\Big[\int_0^\infty \chi_{[0,t]}(x)(t-x)^{rp-p}\,dx\Big]^{\frac{1}{p}}f(t)\,dt\Big)^p$$

$$=\frac{1}{\Gamma(r)(rp-p+1)}\Big(\int_0^\infty t^{r-1+\frac{1}{p}}f(t)\,dt\Big)^p$$

$$=\frac{1}{\Gamma(r)(rp-p+1)}\Big(\int_0^\infty (tf(t))^{r-1+\frac{1}{p}}(f(t))^{1-r+\frac{1}{q}}\,dt\Big)^p$$

$$=\frac{1}{\Gamma(r)(rp-p+1)}\Big(\int_0^\infty (tf(t))^{rp-p+1}\,dt\Big)\Big(\int_0^\infty (f(t))^{q-rq+1}\,dt\Big)^{p-1}. \quad \square$$

Remark 2.6. *By putting $r=1$ in Theorems 2.4 and 2.5 respectively, we obtain Theorems 2.1 and 2.2.*

Theorem 2.7. *Suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and f is nonnegative integrable function which its Laplace transformation exists. Define*

$$F(x)=\int_0^x f(t)\,dt.$$

Then

$$\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p\,dx < \frac{1}{p^p}\Big(\int_0^\infty L_F(s)\,ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty L_f(s)\,ds\Big)^{\frac{1}{q}},$$

where L_f *is the Laplace transformation of* f.

Proof. Note that

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt,$$

so

$$\begin{aligned}
\int_0^\infty \Big(\frac{F(x)}{x}\Big)^p\,dx &= \int_0^\infty \Big[\Big(\int_0^\infty e^{-xt}\,dt\Big)\Big(\int_0^\infty \chi_{[0,x]} f(s)\,ds\Big)\Big]^p\,dx \\
&= \int_0^\infty \Big(\int\int e^{-xt}\chi_{[0,x]}(s) f(s)\,dv\Big)^p\,dx \\
&< \int\int\Big[\int_0^\infty e^{-pxt}\chi_{[s,+\infty)}(x) f^p(s)\,dx\Big]^{\frac{1}{p}}\,dv \\
&= \frac{1}{p^p}\Big(\int_0^\infty\int_0^\infty \frac{e^{-st}}{t^{\frac{1}{p}}} f(s)\,ds\,dt\Big) \\
&\leqslant \frac{1}{p^p}\Big(\int_0^\infty\int_0^\infty \frac{e^{-st}}{t} f(s)\,ds\,dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty\int_0^\infty e^{-st} f(s)\,ds\,dt\Big)^{\frac{1}{q}} \\
&= \frac{1}{p^p}\Big(\int_0^\infty \frac{L_f(t)}{t}\,dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty L_f(t)\,dt\Big)^{\frac{1}{q}} \\
&= \frac{1}{p^p}\Big(\int_0^\infty L_F(s)\,ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty L_f(s)\,ds\Big)^{\frac{1}{q}}. \quad \square
\end{aligned}$$

Theorem 2.8. *Suppose that* $p > 1$, $k(x, y)$ *is nonnegative and homogeneous of degree -1 and*

$$C = \Big\{\int_0^\infty k^p(1,t)dt\Big\}^{\frac{1}{p}}.$$

Also assume that all of the following integrals converge. Then

$$a)\ \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{q}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{p}}\|g\|_q,$$

$$a')\ \int_0^\infty\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy < C^p\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{p-1}\Big(\int_0^\infty f(x)dx\Big),$$

$$b)\ \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < C\Big(\int_0^\infty \frac{g(y)}{y}dy\Big)^{\frac{1}{p}}\Big(\int_0^\infty g(y)dy\Big)^{\frac{1}{q}}\|f\|_p.$$

Moreover, If f and g have Laplace transformation, then

$$i)\ \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < C\Big(\int_0^\infty L_f(s)ds\Big)^{\frac{1}{q}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{p}}\|g\|_q,$$

$$i')\ \int_0^\infty\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy < C^p\Big(\int_0^\infty L_f(s)ds\Big)^{p-1}\Big(\int_0^\infty f(x)dx\Big),$$

$$ii)\ \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < C\Big(\int_0^\infty L_g(s)ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty g(y)dy\Big)^{\frac{1}{q}}\|f\|_p.$$

(c) *Inequalities (a) and (i) are equivalent to* (a′) *and* (i′), *respectively.*

Proof. (a).

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy \leqslant \Big(\int_0^\infty\Big\{\int_0^\infty k(x,y)f(x)dx\Big\}^p dy\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(y)dy\Big)^{\frac{1}{q}}$$
$$< \Big(\int_0^\infty\Big\{\int_0^\infty k^p(x,y)f^p(x)dy\Big\}^{\frac{1}{p}}dx\Big)\|g\|_q.$$

By taking $y = tx$, one may obtain

$$\int_0^\infty\Big\{\int_0^\infty k^p(x,y)f^p(x)dy\Big\}^{\frac{1}{p}}dx = \Big(\int_0^\infty k^p(1,t)dt\Big)^{\frac{1}{p}}\Big(\int_0^\infty x^{\frac{1-p}{p}}f(x)dx\Big)$$
$$= C\Big(\int_0^\infty (f(x))^{\frac{1}{p}}(\frac{f(x)}{x})^{\frac{1}{q}}dx\Big)$$

$$\leqslant C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{q}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{p}}.$$

(a').

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy &< \Big(\int_0^\infty\Big\{\int_0^\infty f^p(x)k^p(x,y)dy\Big\}^{\frac{1}{p}}dx\Big)^p\\ &= \Big(\int_0^\infty\Big\{\int_0^\infty f^p(x)k^p(x,tx)xdt\Big\}^{\frac{1}{p}}dx\Big)^p\\ &= C^p\Big(\int_0^\infty (f(x))^{\frac{1}{p}}(\frac{f(x)}{x})^{\frac{1}{q}}dx\Big)^p\\ &\leqslant C^p\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{p-1}\Big(\int_0^\infty f(x)dx\Big).\end{aligned}$$

By the identity

$$\int_0^\infty \frac{f(x)}{x}dx = \int_0^\infty L_f(s)ds,$$

if f and g have Laplace transformation, then inequalities (i), (i') and (ii) are obtained.

(c).

$$\begin{aligned}\int_0^\infty\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy &= \int_0^\infty\Big(\int_0^\infty k(x,y)f(x)dx\Big)^{p-1}\Big(\int_0^\infty k(x,y)f(x)dx\Big)dy\\ &= \int_0^\infty g(y)\Big(\int_0^\infty k(x,y)f(x)dx\Big)dy\\ &= \int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy\\ &< C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{q}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{p}}\|g\|_q,\end{aligned}$$

where

$$g(y) = \Big(\int_0^\infty k(x,y)f(x)dx\Big)^{p-1}.$$

Note that

$$\|g\|_q = \Big(\int_0^\infty \Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy\Big)^{\frac{1}{q}}.$$

On the other hand

$$\begin{aligned}
\int_0^\infty \int_0^\infty k(x,y)f(x)g(y)dxdy &= \int_0^\infty \Big(\int_0^\infty k(x,y)f(x)\Big)g(y)dxdy \\
&\leqslant \Big(\int_0^\infty \Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy\Big)^{\frac{1}{p}} \|g\|_q \\
&< \left(C^p\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{p-1}\Big(\int_0^\infty f(x)dx\Big)\right)^{\frac{1}{p}} \|g\|_q \\
&= C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{q}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{p}} \|g\|_q. \quad \square
\end{aligned}$$

Corollary 2.9. *Suppose that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and f, g are nonnegative integrable functions which have Laplace transformation. Then*

(i) $$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{x+y}dx\Big)^p dy < \frac{1}{p-1}\Big(\int_0^\infty L_f(s)\,ds\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big),$$

(ii) $$\int_0^\infty \Big(\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\Big)^p dy < q^p\Big(\int_0^\infty L_f(s)\,ds\Big)^{p-1}\Big(\int_0^\infty f(t)\,dt\Big).$$

One may generalize Theorem 2.8 as follows:

Theorem 2.10. *Suppose that $p > 1$, $\lambda > 0$, $k(x,y)$ is nonnegative and homogeneous of degree $-\lambda$ and*

$$C(p,\lambda) = \Big\{\int_0^\infty t^{(p-1)(\lambda-1)}k^p(1,t)dt\Big\}^{\frac{1}{p}}.$$

Also assume that all the following integrals converge. Then

$$\int_0^\infty y^{(p-1)(\lambda-1)}\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy < C^p(p,\lambda)\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{p-1}\Big(\int_0^\infty x^{1-\lambda}f(x)dx\Big).$$

Proof.

$$\begin{aligned}\int_0^\infty y^{(p-1)(\lambda-1)}\Big(\int_0^\infty k(x,y)f(x)dx\Big)^p dy &< \Big(\int_0^\infty \Big\{\int_0^\infty y^{(p-1)(\lambda-1)}k^p(x,y)f^p(x)dy\Big\}^{\frac{1}{p}}dx\Big)^p\\ &= \Big(\int_0^\infty f(x)\Big\{\int_0^\infty (tx)^{(p-1)(\lambda-1)}k^p(x,tx)xdt\Big\}^{\frac{1}{p}}dx\Big)^p\\ &= C^p(p,\lambda)\Big(\int_0^\infty x^{\frac{2-\lambda-p}{p}}f(x)dx\Big)^p\\ &= C^p(p,\lambda)\Big(\int_0^\infty \Big(\frac{f(x)}{x}\Big)^{\frac{1}{q}}\times\Big(\frac{f(x)}{x^{\lambda-1}}\Big)^{\frac{1}{p}}dx\Big)^p\\ &\leqslant C^p(p,\lambda)\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{p-1}\Big(\int_0^\infty x^{1-\lambda}f(x)dx\Big). \ \square\end{aligned}$$

Remark 2.11. *By taking* $\lambda = 1$ *and*

$$k(x,y) = \begin{cases} y^{-1} & 0 \leqslant x \leqslant y\\ 0 & x > y\end{cases}$$

in the above mentioned theorem one may obtain Theorem 2.1.

Theorem 2.12. *Suppose that* $p > 1$, $\lambda > 0$, $k(x,y)$ *is nonnegative and homogeneous of degree* $-\lambda$ *and*

$$C = \Big\{\int_0^\infty k^p(1,t)dt\Big\}^{\frac{1}{p}}.$$

Then

$$\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy < C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{q}}\Big(\int_0^\infty x^{p(1-\lambda)}f(x)dx\Big)^{\frac{1}{p}}\|g\|_q.$$

Proof.

$$\begin{aligned}\int_0^\infty\int_0^\infty k(x,y)f(x)g(y)dxdy &\leqslant \Big(\int_0^\infty\Big\{\int_0^\infty k(x,y)f(x)dx\Big\}^p dy\Big)^{\frac{1}{p}}\Big(\int_0^\infty g^q(y)dy\Big)^{\frac{1}{q}}\\ &< \Big(\int_0^\infty\Big\{\int_0^\infty k^p(x,y)f^p(x)dy\Big\}^{\frac{1}{p}}dx\Big)\|g\|_q.\end{aligned}$$

By taking $y = tx$, one may obtain

$$\int_0^\infty \Big\{ \int_0^\infty k^p(x,y) f^p(x) dy \Big\}^{\frac{1}{p}} dx = \Big(\int_0^\infty k^p(1,t) dt \Big)^{\frac{1}{p}} \Big(\int_0^\infty x^{\frac{1-p\lambda}{p}} f(x) dx \Big)$$

$$= C \Big(\int_0^\infty \Big(x^{1-\lambda} f^{\frac{1}{p}}(x) \Big) \Big(\frac{f(x)}{x} \Big)^{\frac{1}{q}} dx \Big)$$

$$\leqslant C \Big(\int_0^\infty \frac{f(x)}{x} dx \Big)^{\frac{1}{q}} \Big(\int_0^\infty x^{p(1-\lambda)} f(x) dx \Big)^{\frac{1}{p}}. \square$$

Theorem 2.13. *Suppose* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $k(x) > 0$, *and*

$$C = \Big(\int_0^\infty k^p(x) dx \Big)^{\frac{1}{p}}.$$

Also assume that all of the following integrals converge. Then

a) $\displaystyle \int_0^\infty \int_0^\infty k(xy) f(x) g(y) dx dy < C \Big(\int_0^\infty \frac{f(x)}{x} dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty f(x) dx \Big)^{\frac{1}{q}} \|g\|_q.$

a′) $\displaystyle \int_0^\infty \Big(\int_0^\infty k(xy) f(x) dx \Big)^p dy < C^p \Big(\int_0^\infty \frac{f(x)}{x} dx \Big) \Big(\int_0^\infty f(x) dx \Big)^{p-1}.$

Moreover, If f and g have Laplace transformation, then

i) $\displaystyle \int_0^\infty \int_0^\infty k(xy) f(x) g(y) dx dy < C \Big(\int_0^\infty L_f(s) ds \Big)^{\frac{1}{p}} \Big(\int_0^\infty f(x) dx \Big)^{\frac{1}{q}} \|g\|_q.$

i′) $\displaystyle \int_0^\infty \Big(\int_0^\infty k(xy) f(x) dx \Big)^p dy < C^p \Big(\int_0^\infty L_f(s) ds \Big) \Big(\int_0^\infty f(x) dx \Big)^{p-1}.$

(b)*Inequalities (a) and (i) are equivalent to* (a′) *and* (i′), *respectively.*

Proof. (a).

$$\int_0^\infty \int_0^\infty k(xy) f(x) g(y) dx dy \leqslant \Big(\int_0^\infty \Big\{ \int_0^\infty k(xy) f(x) dx \Big\}^p dy \Big)^{\frac{1}{p}} \Big(\int_0^\infty g^q(y) dy \Big)^{\frac{1}{q}}$$

$$< \Big(\int_0^\infty \Big\{ \int_0^\infty k^p(xy) f^p(x) dy \Big\}^{\frac{1}{p}} dx \Big) \|g\|_q.$$

By taking $y = \frac{t}{x}$, one may obtain

$$\begin{aligned}\int_0^\infty \Big\{ \int_0^\infty k^p(xy) f^p(x) dy \Big\}^{\frac{1}{p}} dx &= \Big(\int_0^\infty k^p(t) dt \Big)^{\frac{1}{p}} \Big(\int_0^\infty x^{-\frac{1}{p}} f(x) dx \Big) \\ &= C \Big(\int_0^\infty (f(x))^{\frac{1}{q}} (\frac{f(x)}{x})^{\frac{1}{p}} dx \Big) \\ &\leqslant C \Big(\int_0^\infty \frac{f(x)}{x} dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty f(x) dx \Big)^{\frac{1}{q}}.\end{aligned}$$

(a').

$$\begin{aligned}\int_0^\infty \Big(\int_0^\infty k(xy) f(x) dx \Big)^p dy &< \Big(\int_0^\infty \Big\{ \int_0^\infty f^p(x) k^p(xy) dy \Big\}^{\frac{1}{p}} dx \Big)^p \\ &= \Big(\int_0^\infty \Big\{ \int_0^\infty f^p(x) k^p(t) \frac{1}{x} dt \Big\}^{\frac{1}{p}} dx \Big)^p \\ &= C^p \Big(\int_0^\infty (f(x))^{\frac{1}{q}} (\frac{f(x)}{x})^{\frac{1}{p}} dx \Big)^p \\ &\leqslant C^p \Big(\int_0^\infty \frac{f(x)}{x} dx \Big) \Big(\int_0^\infty f(x) dx \Big)^{p-1}.\end{aligned}$$

By the identity

$$\int_0^\infty \frac{f(x)}{x} dx = \int_0^\infty L_f(s) ds,$$

if f and g have Laplace transformation, then inequalities (i) and (i') are obtained.

(b).

$$\begin{aligned}\int_0^\infty \Big(\int_0^\infty k(xy) f(x) dx \Big)^p dy &= \int_0^\infty \Big(\int_0^\infty k(xy) f(x) dx \Big)^{p-1} \Big(\int_0^\infty k(xy) f(x) dx \Big) dy \\ &= \int_0^\infty g(y) \Big(\int_0^\infty k(xy) f(x) dx \Big) dy \\ &= \int_0^\infty \int_0^\infty k(xy) f(x) g(y) dx dy \\ &< C \Big(\int_0^\infty \frac{f(x)}{x} dx \Big)^{\frac{1}{p}} \Big(\int_0^\infty f(x) dx \Big)^{\frac{1}{q}} \|g\|_q,\end{aligned}$$

where

$$g(y)=\Big(\int_0^\infty k(xy)f(x)dx\Big)^{p-1}.$$

Note that

$$\|g\|_q=\Big(\int_0^\infty\Big(\int_0^\infty k(xy)f(x)dx\Big)^p dy\Big)^{\frac{1}{q}}.$$

On the other hand

$$\begin{aligned}\int_0^\infty\int_0^\infty k(xy)f(x)g(y)dxdy &= \int_0^\infty\Big(\int_0^\infty k(xy)f(x)\Big)g(y)dxdy\\ &\leqslant \Big(\int_0^\infty\Big(\int_0^\infty k(xy)f(x)dx\Big)^p dy\Big)^{\frac{1}{p}}\|g\|_q\\ &< \left(C^p\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)\Big(\int_0^\infty f(x)dx\Big)^{p-1}\right)^{\frac{1}{p}}\|g\|_q\\ &= C\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{q}}\|g\|_q. \quad\square\end{aligned}$$

Remark. 2.14. *In special case, by taking $k(x)=e^{-x}$, in the above mentioned theorem, one may obtain the following inequalities:*

a) $\displaystyle\int_0^\infty L_f(y)g(y)dy<\frac{1}{\sqrt[p]{p}}\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)^{\frac{1}{p}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{q}}\|g\|_q.$

a′) $\displaystyle\int_0^\infty L_f^p(y)dy<\frac{1}{p}\Big(\int_0^\infty \frac{f(x)}{x}dx\Big)\Big(\int_0^\infty f(x)dx\Big)^{p-1}.$

Moreover, if f and g have Laplace transformation, then

i) $\displaystyle\int_0^\infty L_f(y)g(y)dy<\frac{1}{\sqrt[p]{p}}\Big(\int_0^\infty L_f(s)ds\Big)^{\frac{1}{p}}\Big(\int_0^\infty f(x)dx\Big)^{\frac{1}{q}}\|g\|_q.$

i′) $\displaystyle\int_0^\infty L_f^p(y)dy<\frac{1}{p}\Big(\int_0^\infty L_f(s)ds\Big)\Big(\int_0^\infty f(x)dx\Big)^{p-1}.$

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