Essential Submodules with respect to an Arbitrary Submodule

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Abstract. The concept of essential submodules is a well known concept. In this paper we try to replace an arbitrary submodule of $M$, say $T$, instead of $0$ in the definition of essential submodules. By this, essential submodules are precisely $\{0\}$-essential submodules. For a submodule $K$ of right $R$-module $M$, we have $K \subseteq_{\text{ess}} M$ if and only if $(K : m)$ is $\text{ann}_M(m)$-essential right ideal of $R$, for each $m \in M \setminus \{0\}$. Among other things, this generalization of essential submodules gives a necessary and sufficient condition for $\frac{M}{T}$ being finitely co-generated.

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1. Introduction

Throughout this article, all rings are associative with identity and all modules are unitary right modules. We know that the submodule $K$ of right $R$-module $M$ is called essential, denoted by $K \subseteq_{\text{ess}} M$, provided that for each submodule $L$ of $M$, $K \cap L = 0$ implies that $L = 0$. The right $R$-module $M$ is called uniform provided that every non-zero submodule of $M$ is an essential submodule. If $K$ is a submodule of right $R$-module $M$, then by Zorn’s Lemma, $S = \{L \mid L \subseteq M \text{ and } K \cap L = 0\}$

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has a maximal element which is called the complement of \( K \) in \( M \) and is denoted by \( K^c \). For each \( m \in M \), \( (K : m) = \{ r \in R \mid mr \in K \} \). In Section 2, first, the essentiality with respect to a submodule is defined and is shown, this concept is different from the concept of essentiality (Example 2.9). After that, for a submodule \( T \) of right \( R \)-module \( M \), the relationship between essential submodules of \( M \) with respect to \( T \) and essential right ideals of \( R \) with respect to \( (T : m) \), for each \( m \in M \setminus \{0\} \), will be investigated (Theorem 2.7). Moreover, it will be answered, for a submodule \( K \) of \( M \), when is \( K^c \) the largest submodule of \( M \) which has zero intersection with \( K \)?

In Section 3, for a submodule \( T \) of right \( R \)-module \( M \), the intersection of all submodules of \( M \) which containing \( T \) and also are essential with respect to \( T \) will be investigated. All unexplained terminologies and basic results on modules that are used in the sequel can be found in [3], [4] and [5].

2. \( \{\} \)-essential submodules

The reader is reminded that a submodule \( K \) of right \( R \)-module \( M \) is essential provided that \( K \) has non-zero intersection to every non-zero submodule.

**Definition 2.1.** Let \( R \) be a ring and \( T \) be a proper submodule of right \( R \)-module \( M \). The submodule \( K \) of \( M \) is called \( T \)-essential provided that \( K \not\subseteq T \) and for each submodule \( L \) of \( M \), \( K \cap L \subseteq T \) implies that \( L \subseteq T \). In this case \( K \) is denoted by \( K \lessgeq \! T M \).

**Proposition 2.2.** For each \( m, n \in \mathbb{Z} \), \( m\mathbb{Z} \leq_{n\mathbb{Z}} m\mathbb{Z} + n\mathbb{Z} \).

**Proof.** Put \( (n, m) = d \), \( [n, m] = l \). Assume that \( k\mathbb{Z} \subseteq m\mathbb{Z} + n\mathbb{Z} = d\mathbb{Z} \) such that \( m\mathbb{Z} \cap k\mathbb{Z} \subseteq n\mathbb{Z} \). Put \( (k, m) = g \), \( [k, m] = e \). It is clear that \( nm = dl \) and \( km = ge \). Since both \( d|k \) and \( d|m \), then \( d|(k, m) = g \).

On the other hand, since \( n|e \) and \( m|e \), then \( l = [n, m]|e \). Therefore \( d|g \) and \( l|e \) imply that \( dl|ge \). Thus \( nm|km \) and hence \( n|k \) which implies that \( k\mathbb{Z} \subseteq n\mathbb{Z} \). \( \square \)
At first glance, it seems that for submodules $K$ and $T(\neq M)$ of $M$, $K \leq_T M$ if and only if $\frac{K + T}{T} \subseteq \text{ess } M$. But it is not true, generally. For this, we need some assertions.

**Lemma 2.3.** Let $T \subseteq K \subseteq M$ be submodules of right $R$-module $M$. Then $K \leq TM$ if and only if $\frac{K + T}{T} \subseteq \text{ess } M$.

**Proof.** The verification is immediate. □

**Proposition 2.4.** Let $K$ and $T$ be submodules of right $R$-module $M$. Then $K \leq TM$ implies that $\frac{K + T}{T} \subseteq \text{ess } M$.

**Proof.** Let $A = \frac{S}{T}$ be a non-zero submodule of $\frac{M}{T}$ such that $A \cap \frac{K + T}{T} = 0$. Therefore $K \cap A \subseteq T$ and hence the $T$-essentiality of $K$ in $M$ implies that $A \subseteq T$, as desired. □

**Definition 2.5.** Let $K$ be a submodule and $T$ be a proper submodule of right $R$-module $M$. A submodule $K'$ of $M$ is called $T$-complement to $K$ if $K'$ is maximal with respect to the property that $K \cap K' \subseteq T$.

**Proposition 2.6.** Let $C$ and $S$ be submodules of right $R$-module $M$ and $T = C \cap S$. Then $C$ is $T$-complement to $S$ if and only if $\frac{S + C}{C} \subseteq \text{ess } M$.

**Proof.** Let $\frac{S + C}{C} \subseteq \text{ess } M$ and $D$ be a submodule of $M$ such that $C \subseteq D$ and $D \cap S \subseteq T$. It is clear that $\frac{D}{C} \cap \frac{S + C}{C} = 0$ because $d + C = s + C$, for $d \in D$ and $s \in S$, implies that $s \in D \cap S \subseteq T = C \cap S \subseteq C$. The essentiality $\frac{S + C}{C}$ in $\frac{M}{C}$ implies that $C = D$. Conversely, assume that $D$ is a submodule of $M$ containing $C$ such that $\frac{D}{C} \cap \frac{S + C}{C} = 0$. If $x \in D \cap S$, then $x + C \in \frac{D}{C} \cap \frac{S + C}{C}$ and hence $x + C = C$. Therefore $D \cap S \subseteq C \cap S = T$. By assumption, $D = C$. □

By the above definition, it is easy to see that $K$ is an essential submodule of right $R$-module $M$ if and only if $K \leq_{\{0\}} M$. It is well known that if $K \subseteq \text{ess } M$, then $(K : m) \subseteq \text{ess } R$, for each $m \in M$. But the converse is not true. For an example $K = \{\bar{0}, \bar{2}, \bar{4}\}$ is not essential in $\mathbb{Z}_6$ as a $\mathbb{Z}$-module but for each $\bar{x} \in \mathbb{Z}_6$, $(K : \bar{x}) \subseteq \text{ess } \mathbb{Z}$ because $\mathbb{Z}$ is uniform. Now consider the following theorem.
Theorem 2.7. Let $M$ be an $R$-module and $K, T$ be submodules of $M$. The following assertions are equivalent

1. $K \Delta_T M$;

2. For each $m \in M \setminus T$, there exists $r \in R$ such that $mr \in K \setminus T$.

3. $(K : m) \leq (T : m) R$, for each $m \in M \setminus T$.

Proof. $1 \Rightarrow 2$ Let $m \in M \setminus T$. Since $K \Delta_T M$, then $K \cap mR \not\subseteq T$. Hence there exists $r \in R$ such that $mr \in K \setminus T$.

$2 \Rightarrow 1$ By hypotheses, $K \not\subseteq T$. Assume that $L$ is a submodule of $M$ such that $K \cap L \subseteq T$. If $L \not\subseteq T$, there exists $a \in L \setminus T$. By assumption, there is an $r \in R$ such that $ar \in K \setminus T$. On the other hand $ar \in K \cap L \subseteq T$, a contradiction.

$1 \Rightarrow 3$ Assume that $K \Delta_T M$ and $m \in M \setminus T$. By 2, there exists $r \in R$ such that $mr \in K \setminus T$ or equivalently $(K : m) \not\subseteq (T : m)$. Suppose that $I$ is a right ideal of $R$ such that $(K : m) \cap I \subseteq (T : m)$. It is clear that $K \cap mI \subseteq T$ and hence $mI \subseteq T$ because $K \Delta_T M$. Now, $mI \subseteq T$ implies that $I \subseteq (T : m)$, as desired.

$3 \Rightarrow 1$ Suppose that $L$ is a submodule of $M$ such that $K \cap L \subseteq T$. If $L \not\subseteq T$, there exists $x \in L \setminus T$. By hypotheses, there exists $r \in R$ such that $xr \in K \setminus T$. It is a contradiction because $xr \in K \cap L \subseteq T$. □

Proposition 2.8. Let $\{N_i\}_{i \in I}$, $\{M_i\}_{i \in I}$ and $T$ be submodules of right $R$-module $M$ such that $N_i \triangleleft_T M_i$ for every $i \in I$. Then $\oplus_{i \in I} N_i \triangleleft \oplus_{i \in I} T \oplus M_i$.

Proof. By Theorem 2.7, assume that $\{m_i\}_{i \in I} \in \oplus M_i \setminus \oplus T$. Since $N_i \triangleleft_T M_i$ for every $i \in I$, there exists an $r \in R$ such that $\{m_i r\} \in \oplus N_i \setminus \oplus T$, as desired. □

The following example shows that the converse of Proposition 2.4, is not true, generally.

Example 2.9. It is easy to check that $\frac{6\mathbb{Z} + 12\mathbb{Z}}{12\mathbb{Z}} = \frac{6\mathbb{Z}}{12\mathbb{Z}}$ is an essential $\mathbb{Z}$-submodule of $\frac{\mathbb{Z}}{12\mathbb{Z}}$, but $6\mathbb{Z}$ is not $12\mathbb{Z}$-essential $\mathbb{Z}$-submodule of $\mathbb{Z}$. To the
contrary, assume that $6\mathbb{Z} \subseteq 12\mathbb{Z}$ \mathbb{Z}$. By Theorem 2.7, for $8 \in \mathbb{Z} \setminus 12\mathbb{Z}$ there exists an $n \in \mathbb{Z}$ such that $8n \in 6\mathbb{Z}$. Therefore $3|n$ and hence $8n \in 12\mathbb{Z}$, a contradiction.

**Corollary 2.10.** Let $K$ be a submodule of right $R$-module $M$. Then $N \subseteq_{\text{ess}} M$ if and only if $(K : m) \subseteq \text{ann}_r(m) R$, for each $m \in M \setminus \{0\}$.

**Proof.** It is clear that for each $m \in M$, $\text{ann}_r(m) = (\{0\} : m)$. By Theorem 2.7, we have $N \subseteq_{\text{ess}} M$ if and only if $N \subseteq_{\{0\}} M$ if and only if $(N : m) \subseteq_{\{0\}} R$, for each $m \in M$. □

Let $R$ be a ring. An element $x \in R$ is said to be regular provided that $\text{ann}_r(x) = \text{ann}_l(x) = 0$ and the set of all regular elements of $R$ is denoted by $\mathcal{C}_R$. For a right $R$-module $M$, put $T(M) = \{m \in M | \text{ann}_r(m) \cap \mathcal{C}_R \neq \emptyset\}$. If $T(M) = 0$, $M$ is called torsion free and if $T(M) = M$, $M$ is called torsion $R$-module. See [4, §10, Exercise 19].

**Corollary 2.11.** Let $R$ be a domain, $M$ be a right $R$-module and $K$ be a non-zero submodule of $M$. Then $K$ is an essential submodule of $M$ if and only if $M_K$ is a torsion $R$-module.

**Proof.** For each $0 \neq m \in M$, we have $\text{ann}_r(m) = 0$ because $\mathcal{C}_R = R \setminus \{0\}$ and

$$T(M) = \{x \in M | \text{ann}_r(x) \cap (R \setminus \{0\}) \neq \emptyset\} = \{x \in M | \text{ann}_r(x) \neq 0\} = \{0\}.$$ 

By Theorem 2.7, $K \subseteq_{\text{ess}} M$ if and only if $K \subseteq_{\{0\}} M$ if and only if $(K : m) \subseteq (0 : m), \forall m \in M \setminus \{0\}$ if and only if $(K : m) \subseteq \text{ann}_r(m) = 0, \forall m \in M \setminus \{0\}$ if and only if $M_K$ is a torsion $R$-module. □

**Proposition 2.12.** Let $K, L$ and $T$ be submodules of right $R$-module $M$. Then
1. If $K$ and $L$ are $T$-essential submodules of $M$, then $K \cap L$ is $T$-essential too.
2. Let $K \subseteq L \subseteq M$. Then $K \preceq_T M$ if and only if $K \preceq_T L$ and $L \preceq_T M$.

**Proof.** The verification is immediate. □

**Theorem 2.13.** Let $T_1 \leq K_1 \leq M_1 \leq M$ and $T_2 \leq K_2 \leq M_2 \leq M$.
such that \( M_1 \cap M_2 = T_1 \cap T_2 \). Then, \( K_1 + K_2 \leq_{(T_1 + T_2)} M_1 + M_2 \) if and only if \( K_1 \leq_{T_1} M_1 \) and \( K_2 \leq_{T_2} M_2 \).

**Proof.** Assume that \( K_1 + K_2 \leq_{(T_1 + T_2)} M_1 + M_2 \) and \( L_1 \) is a submodule of \( M_1 \) such that \( K_1 \cap L_1 \subseteq T_1 \). It is clear that \((K_1 + K_2) \cap L_1 \subseteq T_1 + T_2\). If \( x \in K_1 \), \( y \in K_2 \) and \( z \in L_1 \) such that \( x + y = z \), then \( x - z = -y \in M_1 \cap M_2 = T_1 \cap T_2 \). Hence \( y \in T_1 \subseteq K_1 \). Therefore \( z = x + y \in K_1 \cap L_1 \subseteq T_1 \). In the other hand \( x - z \in T_1 \) implies that \( x \in T_1 \). Thus \( x + y \in T_1 + T_2 \). By hypothesis, \( L_1 \subseteq T_1 + T_2 \). It implies that \( L_1 \subseteq T_1 \). Similarly, we can show that \( K_2 \leq_{T_2} M_2 \). Conversely, suppose that \( x + y \in M_1 + M_2 \setminus T_1 + T_2 \), where \( x \in M_1 \) and \( y \in M_2 \). Either \( x \notin T_1 \) or \( y \notin T_2 \). Assume that \( x \in M_1 \setminus T_1 \). There exists \( r \in R \) such that \( xr \in K_1 \setminus T_1 \). If \( yr \in K_2 \), then the proof is completed (\( (x+y)r \in K_1 + K_2 \setminus T_1 + T_2 \)). If \( yr \in M_2 \setminus K_2 \subseteq M_2 \setminus T_2 \), then there exists \( s \in R \) such that \( yrs \in K_2 \setminus T_2 \). Hence \( (x+y)rs \in K_1 + K_2 \setminus T_1 + T_2 \). \( \square \)

**Theorem 2.14.** Let \( M \) and \( N \) be \( R \)-modules, \( T \subseteq N \) and \( f \in \text{Hom}_R(M, N) \) such that \( \text{Im}f \not\subseteq T \). Then \( \text{Im}f \leq_T N \) if and only if, for each homomorphism \( h \), if \( \ker h \cap \text{Im}f \subseteq T \), then \( \ker h \subseteq T \).

**Proof.** The “only if” part is clear. Conversely, let \( K \) be a submodule of \( N \) such that \( \text{Im}f \cap K \subseteq T \). Define the map \( h : (\text{Im}f + K) \longrightarrow \frac{M}{f^{-1}(T)} \), with \( h(f(m) + k) = m + f^{-1}(T) \), for each \( m \in M \) and \( k \in K \). It is clear that \( h \) is an \( R \)-homomorphism such that \( \ker h \cap \text{Im}f \subseteq T \). By hypotheses, \( K \subseteq \ker h \subseteq T \). \( \square \)

**Lemma 2.15.** Let \( M \) and \( N \) be right \( R \)-modules, \( T \) and \( K \) be submodules of \( N \) and \( f \in \text{Hom}_R(M, N) \). If \( \leq_T N \), then \( f^{-1}(K) \leq_{f^{-1}(T)} M \).

**Proof.** Assume that \( L \) be a submodule of \( M \) such that \( f^{-1}(K) \cap L \subseteq f^{-1}(T) \). It is clear that \( K \cap f(L) \subseteq T \) and hence \( f(L) \subseteq T \). Thus \( L \subseteq f^{-1}(T) \), as desired. \( \square \)

**Corollary 2.16.** Let \( M \) and \( N \) be right \( R \)-modules, \( K \) be a submodule of \( N \) and \( f \in \text{Hom}_R(M, N) \). If \( K \subseteq_{\text{ess}} N \), then \( f^{-1}(K) \leq_{\text{ker} f} M \). Moreover, if \( f \) is an epimorphism, then \( K \subseteq_{\text{ess}} N \) if and only if \( f^{-1}(K) \leq_{\text{ker} f} M \).
Proof. The first part is immediate consequence of Lemma 2.15, because $f^{-1}(0) = \ker f$. Now suppose that $L$ be a submodule of $N$ such that $K \cap L = 0$. It is obvious that $f^{-1}(K) \cap f^{-1}(L) \subseteq \ker f$. Thus $f^{-1}(L) \subseteq \ker f$ since $f^{-1}(K) \subseteq \ker f$. If $y \in L$, there exists $x \in M$ such that $y = f(x)$. Therefore $x \in f^{-1}(L) \subseteq \ker f$ and hence $y = f(x) = 0$. □

Lemma 2.17. Let $K$ and $T$ be submodules of right $R$-module $M$. If $K \trianglelefteq_T M$, then $K^c \subseteq T$. Moreover, if $K \trianglelefteq_T M$ and $K \cap T = 0$, then $K^c = T$.

Proof. The verification is immediate. □

The following proposition shows that when the complement of the submodule $K$ of a right $R$-module $M$, is the largest submodule which has zero intersection with $K$.

Proposition 2.18. Let $K$ be a submodule of right $R$-module $M$. The following assertions are equivalent.

1. $K$ is $K^c$-essential in $M$;

2. For each submodule $N$ of $M$, $K \cap N = 0$ implies that $N \subseteq K^c$;

3. For each $x \in M \setminus K^c$ there exists $r \in R$ such that $0 \neq xr \in K$.

Proof. 1$\Rightarrow$2 It is clear by definition.

1$\Rightarrow$3 By Theorem 2.7, For each $x \in M \setminus K^c$ there exists $r \in R$ such that $xr \in K \setminus K^c = K \setminus \{0\}$.

2$\Rightarrow$1 Let $N$ be a submodule of $M$ such that $K \cap N \subseteq K^c$. Then $K \cap N \subseteq K \cap K^c = \{0\}$ and by hypotheses $N \subseteq K^c$.

3$\Rightarrow$1 it is clear by Theorem 2.7. □

As an application of the Proposition 2.18, we have the following theorem.

Theorem 2.19. Let $R$ be a commutative ring and $M = \oplus_{i \in F} M_i$ be an $R$-module, where $M_i$’s are non-isomorphic simple submodules of $M$ and $F = \{1, 2, \cdots, n\}$. Then, for each $I \subseteq F$, $\oplus_{i \in I} M_i \trianglelefteq_T M$, where $T = \oplus_{j \in F \setminus I} M_j$.
Proof. Let $K$ be a submodule of $M$ such that $(\oplus_{i \in I} M_i) \cap K = 0$. We must show that $K \subseteq T$. By [1, Lemma 9.2], there exists a subset $J \subseteq F$ such that $M = (\oplus_{i \in I} M_i) \oplus K \oplus (\oplus_{j \in J} M_j)$. Hence
\[
\text{ann}(K) = \text{ann}(\oplus_{t \in F \setminus (I \cup J)} M_t) \supseteq \text{ann}(\oplus_{i \in I} M_i) = \bigcap_{t \in F \setminus I} \text{ann}(M_t).
\]
In the other hand for each disjoint $i, j \in F \setminus I$, ann$(M_i)$ and ann$(M_j)$ are coprime and hence
\[
\bigcap_{t \in F \setminus I} \text{ann}(M_t) = \prod_{t \in F \setminus I} \text{ann}(M_t),
\]
by [2, Proposition 1.10]. Therefore for each $x \in K$, $x = m_1 + m_2 + \cdots + m_r$, where $0 \neq m_i \in M_{j_i}$. Hence
\[
\prod_{t \in F \setminus I} \text{ann}(M_t) \subseteq \text{ann}(x) \subseteq \text{ann}(m_i) \ (\forall i),
\]
therefore there exists $t_i \in F \setminus I$ such that ann$(M_{t_i}) \subseteq \text{ann}(m_i) = \text{ann}(M_{j_i})$. By maximality of ann$(M_{t_i})$’s we have ann$(M_{t_i}) = \text{ann}(M_{j_i})$. Thus $M_{t_i} \cong M_{j_i}$ and hence $M_{t_i} = M_{j_i}$. Therefore $x \in \oplus_{i \in F \setminus I} M_i$, as desired. □

3. The $\{\}$-Socle

In this section, for a proper submodule $T$ of right $R$-module $M$, the intersection of all submodules of $M$ which containing $T$ and simultaneously are $T$-essential is investigated.

Lemma 3.1. Let $K$ and $T(\neq M)$ be submodules of right $R$-module $M$ such that $T \subseteq K$. Then there exists a submodule $K'$ of $M$ such that $K + K' \subseteq_T M$ and $\frac{K + K'}{T} = \frac{K}{T} \oplus \frac{K' + T}{T}$.

Proof. Define $S = \{N \mid N$ is a submodule of $M$ and $N \cap K \subseteq T\}$. By Zorn’s Lemma, $S$ has a maximal element, say $K'$. Assume that $L$ is a submodule of $M$ such that $(K + K') \cap L \subseteq T$. We clime that $K \cap (K' + L) \subseteq T$. For, suppose that $x \in K$, $y \in K'$, and $z \in L$ such that
x = y + z. Thus \( x - y = z \in (K + K') \cap L \subseteq T \subseteq K \). Hence \( y = x - z \in K \cap K' \subseteq T \) and hence \( x \in T \), as desired. The maximality of \( K' \) in \( S \) implies that \( L \subseteq K' \) and hence \( L \subseteq T \). For the second part it is enough to show that \( \frac{K}{T} \cap \frac{K' + T}{T} = 0 \). Assume that \( x \in K \) and \( y \in K' \) such that \( x + T = y + T \). Thus \( x - y \in T \subseteq K \) and hence \( y \in K \cap K' \subseteq T \), as desired. □

**Definition 3.2.** Let \( K \) and \( T \) be submodules of right \( R \)-module \( M \). \( K \) is called \( T \)-simple submodule of \( M \) provided that \( \frac{K + T}{T} \) is a simple \( R \)-module. Moreover,

\[
\text{Soc}_T(M) = \sum \{ K : K \text{ is a } T \text{-simple submodule of } M \}.
\]

**Lemma 3.3.** Let \( T \) be a submodule of right \( R \)-module \( M \) and

\[
\text{S}_T(M) = \bigcap \{ L : T \subseteq L \text{ and } L \preceq_T M \}.
\]

Then \( \frac{\text{S}_T(M)}{T} \) is a semisimple right \( R \)-module.

**Proof.** Let \( H \) be a submodule of \( \frac{\text{S}_T(M)}{T} \). By Lemma 3.1, there exists a submodule \( H' \) of \( M \) such that \( H + H' \preceq_T M \). Then \( H \preceq H' + H' + T \). Then

\[
\text{S}_T(M) = \frac{\text{S}_T(M)}{T} \cap \left( \frac{H}{T} \oplus \frac{H' + T}{T} \right) = \frac{H}{T} \oplus \left( \frac{\text{S}_T(M)}{T} \cap \frac{H' + T}{T} \right). \quad \Box
\]

**Proposition 3.4.** Let \( T \) be a submodule of right \( R \)-module \( M \). Then

\[
\text{Soc}_T(M) = \bigcap \{ L : T \subseteq L \text{ and } L \preceq_T M \}.
\]

**Proof.** Let \( S \) be a \( T \)-simple submodule of \( M \) and \( L \) be a submodule of \( M \) containing \( T \) such that \( L \preceq_T M \). Since \( (\frac{S}{T} \cap L + T) \) is a submodule of \( \frac{S + T}{T} \), then either \( (S \cap L) + T = T \) or \( (S \cap L) + T = S + T \). But \( (S \cap L) + T = T \) and \( L \preceq_T M \) imply that \( S \subseteq T \), a contradiction. Thus \( (S \cap L) + T = S + T \). At the other hand \( L \cap (T + S) = T + (L \cap S) \) and
hence $S + T \subseteq L$. Therefore $S \subseteq L$ and hence $\text{Soc}_T(M) \subseteq \bigcap\{L : T \subseteq L \text{ and } L \unlhd_T M\} = S_T(M)$. In the other hand by Lemma 3.3,

$$\frac{S_T(M)}{T} = \sum_{i \in I} \frac{S_i}{T} = \frac{\sum_{i \in I} S_i}{T},$$

where $S_i$'s are simple $R$-modules. Then for each $i \in I$, $S_i$ is a $T$-simple submodule of $M$ and hence $S_T(M) \subseteq \text{Soc}_T(M)$. □

The following theorem gives a necessary and sufficient condition under which $\frac{M}{T}$ is finitely co-generated.

**Theorem 3.5.** Let $T$ be a submodule of right $R$-module $M$. Then $\frac{M}{T}$ is finitely co-generated if and only if $\frac{\text{Soc}_T(M)}{T}$ is finitely co-generated and $\text{Soc}_T(M) \unlhd_T M$.

**Proof.** Let $\{\frac{L_i}{T}\}_{i \in I}$ be a family of submodules of $\frac{M}{T}$ such that $\bigcap_{i \in I} \frac{L_i}{T} = 0$. Then $\bigcap_{i \in I} \frac{L_i \cap \text{Soc}_T(M)}{T} = 0$. since $\frac{\text{Soc}_T(M)}{T}$ is finitely co-generated, then $\bigcap_{i \in I_0} \frac{L_i \cap \text{Soc}_T(M)}{T} = 0$, for some finite subset $I_0$ of $I$. Therefore $(\bigcap_{i \in I_0} L_i) \cap \text{Soc}_T(M) \subseteq T$. Since $\text{Soc}_T(M) \unlhd_T M$, then $(\bigcap_{i \in I_0} L_i) \subseteq T$ or equivalently $\bigcap_{i \in I_0} \frac{L_i}{T} = 0$. Conversely, assume that $K$ be a submodule of $M$ such that $\text{Soc}_T(M) \cap K \subseteq T$. By Proposition 3.4, we have $(\bigcap\{L : T \subseteq L \text{ and } L \unlhd_T M\}) \cap K \subseteq T$. Since $\frac{M}{T}$ is finitely co-generated, then so $(\bigcap_{i=1}^n L_i) \cap K \subseteq T$ for finite number $L_i \in \{L : T \subseteq L \text{ and } L \unlhd_T M\}$. By Proposition 2.12, $\bigcap_{i=1}^n L_i \unlhd_T M$ and hence $K \subseteq T$. □

**Corollary 3.6.** Let $T$ be a submodule of right $R$-module $M$. Then $\frac{M}{T}$ is finitely co-generated if and only if $\frac{\text{Soc}_T(M)}{T}$ is finitely generated and $\text{Soc}_T(M) \unlhd_T M$.

**Proof.** By [1, Corllary 10.16], finitely co-generated semisimple $R$-modules are precisely finitely generated semisimple $R$-modules. Now by Lemma 3.3 and Proposition 3.4, $\frac{\text{Soc}_T(M)}{T}$ is semisimple, hence $\frac{\text{Soc}_T(M)}{T}$ is finitely co-generated if and only if it is finitely generated. □

**Definition 3.7.** Let $T$ be a proper submodule of right $R$-module $M$. $M$ is called $T$-uniform provided that for each submodule $K$ of $M$, if $K \not\subseteq T$, then $K \unlhd_T M$. 


Lemma 3.8. Let $T$ be a proper submodule of right $R$-module $M$. Then $M$ is $T$-uniform if and only if for each two submodules $K$ and $N$ of $M$, $K \cap N \subseteq T$ implies that either $K \subseteq T$ or $N \subseteq T$.

Proof. Let $K$ and $N$ be two submodules of $M$ such that $K \cap N \subseteq T$ and $K \not\subseteq T$. By hypotheses, $K \subseteq_T M$ and hence $L \subseteq T$. Conversely, assume that $K$ and $N$ are submodules of $M$ such that $K \not\subseteq T$ and $K \cap L \subseteq T$. Then $L \subseteq T$, as desired. □

The right $R$-module $M$ is said to be uniserial provided that the lattice of all submodules of $M$ is totally ordered with inclusion.

Proposition 3.9. The right $R$-module $M$ is uniserial if and only if for each proper submodule $T$, $M$ is $T$-uniform.

Proof. Let $T$ be proper submodule of $M$. Assume that $N$ and $K$ are submodules of $M$ such that $K \cap N \subseteq T$. Since $M$ is uniserial, either $N \subseteq K$ or $K \subseteq N$. Hence either $K \cap N = K$ or $K \cap N = N$. Conversely, assume that $N$ and $K$ are submodules of $M$ such that $K \not\subseteq T$. Hence $K \not\subseteq (K \cap N)$ and by assumption $K \not\subseteq_{(K \cap N)} M$. On the other hand $K \cap N \subseteq K \cap N$. Thus $N \subseteq K \cap N$ and hence $N \subseteq K$. □

Note that if $R$-module $M$ is $T$-uniform, then $\frac{M}{T}$ is a uniform $R$-module but the converse is not true. For instance, assume that $R = \mathbb{Z}_2$ and $M = R \oplus R$ as an $R$-module. We know that $T = \{ (x, x) | x \in R \}$ is a maximal submodule of $M$, hence $\frac{M}{T}$ is uniform. But $R \oplus 0 \not\subseteq T$ and $R \oplus 0$ is not $T$-essential submodule of $M$ because $(0, 1) \in M \setminus T$ and for each $r \in R$, $(0, 1)r \not\in (R \oplus 0) \setminus T$.

Example 3.10. 1. Uniform $R$-modules are precisely 0-uniform $R$-module.
2. If $P$ is a prime ideal of a commutative ring $R$, then $R$ is a $P$-uniform $R$-module. Moreover, $P$ is a prime ideal of $R$ if and only if $R$ is a $P$-uniform $R$-module. Moreover, $P$ is a semi-prime ideal of $R$ if and only if $\frac{R}{P}$ is uniform and $P$ is a semi-prime ideal of $R$.

Proposition 3.11. For each positive integer number $n$, $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a uniform $\mathbb{Z}$-module if and only if $\mathbb{Z}$ is an $n\mathbb{Z}$-uniform $\mathbb{Z}$-module.
Proof. The “if” part is always true. For the “only if” part, assume that $\frac{\mathbb{Z}}{n\mathbb{Z}}$ is a uniform $\mathbb{Z}$-module. It is clear that there exist a positive integer number $k$ and a prime number $p$ such that $n = p^k$. Suppose that $m \in \mathbb{Z}$ such that $m\mathbb{Z} \not\subseteq n\mathbb{Z}$ (or equivalently $n \nmid m$). If $t \in \mathbb{Z} \setminus n\mathbb{Z}$, then there exist integer numbers $0 \leq r, s < k$ and prime numbers $p_1, p_2, \ldots, p_a$ such that

$$m = p^r p_1^{m_1} p_2^{m_2} \cdots p_a^{m_a} \quad \text{and} \quad t = p^s p_1^{m_1} p_2^{m_2} \cdots p_a^{m_a}.$$  

It is clear that there exists integer number $b$ such that $tb \in m\mathbb{Z} \setminus n\mathbb{Z}$ and by Lemma 2.7, proof is complete. □

References


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