Journal of Mathematical Extension
Vol. 16, No. 11, (2022) (9)1-11
URL: https://doi.org/10.30495/JME.2022.2181
ISSN: 1735-8299
Original Research Paper

Some Results On 3-Dimensional Lorentzian Manifold

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Abstract. In this paper, first Szabo operator related to the Ricci operator of Lorentzian 3-manifold in the algebraic setting is determined. Then, a necessary and sufficient condition for a Lorentzian 3-manifolds admitting a parallel line field with vanishing Szabo operator is obtained.

AMS Subject Classification: 53C44; 53C30; 53C80
Keywords and Phrases: Killing vector, Lorentzian manifold, Szabo operator, Walker manifold

1 Introduction

The existence of a parallel line on a Riemannian manifold gives rise to a local decomposition of the manifold as a direct product is familiar. This property extends to semi-Riemannian manifolds whenever the line is nondegenerate, i.e., it is spanned by a non-null locally defined vector field. However, there are several research on geometrical consequences of the existence of a parallel degenerate line [2, 3, 4, 5].

Received: September 2021; Accepted: June 2022
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Lorentz manifolds admitting a parallel degenerate line field are of special interest, so that, Walker [8, 9] obtained canonical coordinates for semi-Riemannian metrics admitting parallel degenerate plane fields which allowed further investigations [1, 2, 3, 4, 5]. In order to study some geometric properties, it is some times useful to have some insight from the 3-dimensional case. Let (M, g) be a 3-dimensional Lorentzian manifold admitting a parallel degenerate line field with local coordinates (t, x, y) where the Lorentzian metric tensor is expressed as:

$$g_f = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \varepsilon & 0 \\ 1 & 0 & f(t,x,y) \end{pmatrix} \tag{1}$$

for some function $f(t, x, y)$, where, $\varepsilon = \pm 1$ and the parallel degenerate line field becomes, $D = \langle \frac{\partial}{\partial t} \rangle$.

Let (M, g) be a semi-Riemannian manifold. A $(0, 4)-$tensor $R \in \otimes^4 T_pM$, for each point p in M, is said to be an algebraic curvature tensor, if R has the symmetries of the curvature of the Levi-Civita connection:

$$\begin{aligned} R(x, y, z, w) = R(z, w, x, y) = -R(y, x, z, w), \\ R(x, y, z, w) + R(y, z, x, w) + R(z, x, y, w) = 0. \end{aligned}$$

A $(0, 5)-$tensor, which we denote symbolically by $\nabla R \in \otimes^5 T_pM$, is said to be a covariant derivative algebraic curvature tensor if ∇R has the symmetries of the covariant derivative of the curvature of the Levi-Civita connection:

$$\begin{aligned} &\nabla R(a, b, c, d; e) = -\nabla R(b, a, c, d; e) = \nabla R(c, d, a, b; e), \\ &\nabla R(a, b, c, d; e) + \nabla R(a, c, d, b; e) + \nabla R(a, d, b, c; e) = 0, \\ &\nabla R(a, b, c, d; e) + \nabla R(a, b, d, e; c) + \nabla R(a, b, e, c; d) = 0. \end{aligned}$$

The Jacobi operator $J_R(x)$ and the Szabo operator $S_R(x)$ are the symmetric linear operators on T_pM defined by:

$$\langle J_R(x)y, w\rangle = R(y, x, , x, w) \text{ and } \langle S_{\nabla R}(x)y, w\rangle = \nabla R(y, x, x, w; x).$$

It is obvious the $J_R(x)$ and $S_R(x)$ are self-adjoint.
Since any 3-dimensional algebraic curvature tensor is completely determined its Ricci tensor, we consider separately the following possibilities for the Ricci operator Ric (for more information see [3, 7]):

$$\begin{aligned} &\textit{Type Ia}: \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \quad \textit{Type Ib}: \begin{pmatrix} \alpha & -\beta & 0 \\ \beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \\ &\textit{Type II}: \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 1 & \beta \end{pmatrix}, \quad \textit{Type III}: \begin{pmatrix} \alpha & 0 & 0 \\ 1 & \alpha & 0 \\ 0 & 1 & \alpha \end{pmatrix}. \end{aligned} \tag{2}$$

Here, we can classify the Szabo operator, according to the classification of Ricci operator is suitable orthogonal frame $\{e_1, e_2, e_3\}$ [2].
The specialties geometry of Szabo operator is investigated by Gilkey and Stavrov in [6]. They proved that:

Lemma 1.1. *Let ∇R be a covariant derivative algebraic curvature tensor on a Lorentzian vector space. If $trac\{S_{\nabla R}(0)^2\}$ is constant on $S^-(V)$, then $S_{\nabla R} = 0$.*
Where, $S^{\pm}(V)$ be the pseudo-spheres of unit space-like $(+)$ and time-like $(-)$ vectors in V:

$$S^{\pm}(V) = \{z \in V; g(z, z) = \pm 1\}.$$

Also in [6], they are proved that:

Lemma 1.2. *Let ∇R be a covariant derivative algebraic curvature tensor on a vector space of arbitrary signature. If $S_{\nabla R} = 0$, then $\nabla R = 0$.*

Here, we study some geometric and algebraic properties of Lorentzian 3-metric. The paper is organized in the following way. In Sect 2, we calculate the non-zero components of the Levi-Civita connection and curvature tensor of 3-dimensional Lorentz manifolds admitting a parallel degenerate line field. In Sect 3, Szabo operator related to the Ricci operator of 3-dimensional Lorentzian manifold is determined. Finally, in Sect 4, a necessary and sufficient condition for a function $f(t, x, y)$

of the Walker metric of 3-dimensional manifold, with vanishing Szabo operator is constructed.

2 Preliminaries

Here, we will give some necessary objects corresponding to Lorentzian geometry. It follows after a straightforward calculation that the Levi-Civita connection of any metric (1), is given by:

$$\begin{aligned}
\nabla_{\partial t}\partial y &= \frac{1}{2}f_t\partial_t,\\
\nabla_{\partial x}\partial y &= \frac{1}{2}f_x\partial_t,\\
\nabla_{\partial y}\partial y &= \frac{1}{2}(ff_t+f_y)\partial_t - \frac{1}{2\varepsilon}f_x\partial_x - \frac{1}{2}f_t\partial_y,
\end{aligned}$$

where $\partial_t, \partial_x, \partial_y$ are the coordinate vector fields $\frac{\partial}{\partial_t}, \frac{\partial}{\partial_x}, \frac{\partial}{\partial_y}$ respectively.

Hence, if (M, g) admits a parallel null vector field, then the associate Levi-Civita connection satisfies

$$\nabla_{\partial x}\partial y = \frac{1}{2}f_x\partial_t, \qquad \nabla_{\partial y}\partial y = \frac{1}{2}f_y\partial_t - \frac{1}{24}f_x\partial_x$$

Let R denote the curvature tensor taken with the sign convention:

$$R(X, Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y].$$

Therefore, for any vector field $Z \in \mathcal{X}(M)$:

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z + \nabla_{[X,Y]}Z.$$

Then, the non-zero components of the curvature tensor of any metric (1) are given by:

$$\begin{aligned}
R(\partial_t, \partial_y)\partial_t &= -\frac{1}{2}f_{tt}\partial_t,\\
R(\partial_t, \partial_y)\partial_x &= -\frac{1}{2}f_{tx}\partial_t,
\end{aligned}$$

$$
\begin{aligned}
R(\partial_t, \partial_y)\partial_y &= -\frac{1}{2} f f_{tt}\partial_t + \frac{1}{2\varepsilon} f_{tx}\partial_x + \frac{1}{2} f_{tt}\partial_y, \qquad (3)\\
R(\partial_x, \partial_y)\partial_t &= -\frac{1}{2} f_{tx}\partial_t,\\
R(\partial_x, \partial_y)\partial_x &= -\frac{1}{2} f_{xx}\partial_t,\\
R(\partial_x, \partial_y)\partial_y &= -\frac{1}{2} f f_{tx}\partial_t + \frac{1}{2\varepsilon} f_{xx}\partial_x + \frac{1}{2} f_{tx}\partial_y.
\end{aligned}
$$

Further, note that the existence of parallel null field simplifies (3) as follows:

$$R(\partial_x, \partial_y)\partial_x = -\frac{1}{2} f_{xx}\partial_t, \qquad R(\partial_x, \partial_y)\partial_y = \frac{1}{2\varepsilon} f_{xx}\partial_x.$$

As a matter of notation, let *Ric* and *Sc* denote the *Ricci* tensor and the scalar curvature of (M, g), defined by $Ric(X, Y) = trace\{Z \to R(X, Z)Y\}$ and $Sc = \text{trace Ric}$, respectively.
Moreover, let $\hat{Ric}$ be the *Ricci* operator defined by:
$\langle \hat{Ric}(X), Y\rangle = Ric(X, Y)$. Then the *Ricci* tensor of any metric (1) satisfies:

$$Ric = \begin{pmatrix} 0 & 0 & \frac{1}{2} f_{tt} \\ 0 & 0 & \frac{1}{2} f_{tx} \\ \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & \frac{1}{2\varepsilon}(\varepsilon f f_{tt} - f_{xx}) \end{pmatrix},$$

when expressed in the local coordinate basis. Moreover, the *Ricci* operator $\hat{Ric}$ of a metric (1), when expressed in the coordinate basis, takes the form:

$$\hat{Ric} = \begin{pmatrix} \frac{1}{2} f_{tt} & \frac{1}{2} f_{tx} & -\frac{1}{2\varepsilon} f_{xx} \\ 0 & 0 & \frac{1}{2\varepsilon} f_{tx} \\ 0 & 0 & \frac{1}{2} f_{tt} \end{pmatrix}. \qquad (4)$$

Hence, the *Ricci* operator has eigenvalues:

$$\lambda_1 = 0, \lambda_2 = \lambda_3 = \frac{1}{2} f_{tt},$$

and the scalar curvature satisfies

$$Sc = f_{tt}.$$

3 Szabo operators on 3-dimensional Lorentzian manifolds

Since any 3-dimensional algebraic curvature tensor is completely determined by its *Ricci* tensor. By (2), we can characterize the Szabo operator, according to the classification of *Ricci* operator in suitable orthogonal frame $\{e_1, e_2, e_3\}$ [3]. So, by a direct and straightforward computation according to definition of Szabo operator we will have the following

Proposition 3.1. *The non-zero component of Szabo operator of each type of classification are explained as:*
(1) When $\{e_1, e_2, e_3\}$ *is unite orthogonal, one of the following cases occurs.*
Type Ia:

$$\begin{aligned}\nabla R_{12113} = \nabla R_{13112} &= \frac{1}{2}(\beta-\gamma)(-\alpha+\beta+\gamma)^2,\\ \nabla R_{21223} = \nabla R_{23221} &= -\frac{1}{2}(\alpha-\gamma)(\alpha-\beta+\gamma)^2,\\ \nabla R_{31332} = \nabla R_{32331} &= -\frac{1}{2}(\alpha-\beta)(\alpha+\beta-\gamma)^2.\end{aligned}$$

Type Ib:

$$\begin{aligned}\nabla R_{12112} = \nabla R_{13113} &= (\alpha-2\beta)\beta(\alpha-2\gamma),\\ \nabla R_{21223} = \nabla R_{23221} &= \frac{1}{2}(2\beta(\beta-\gamma-\alpha^3)+\alpha(\beta-2\beta^2-\gamma))+\alpha^2(1+\gamma)),\\ \nabla R_{31332} = \nabla R_{32331} &= \frac{1}{2}(\alpha^2(-1+\gamma)-2\beta(\beta+\gamma)-\alpha^3)+\alpha(\beta-2\beta^2-\gamma)).\end{aligned}$$

Type II:

$$\begin{aligned}\nabla R_{12112} = \nabla R_{23223} = -\nabla R_{12113} = -\nabla R_{13112} &= -\frac{1}{2}(\beta-2\gamma)^2,\\ \nabla R_{21223} = \nabla R_{23221} &= \frac{1}{4}\alpha(-4\beta+\alpha(3-2\alpha+2\beta)),\\ \nabla R_{31332} = \nabla R_{32331} &= \frac{1}{4}\alpha(-\alpha(3+2\alpha)+2\beta(2+\alpha)).\end{aligned}$$

Type III:

$$\nabla R_{12112} = \nabla R_{12113} = \nabla R_{13112} = \nabla R_{13113} = 2\alpha,$$
$$\nabla R_{21221} = \nabla R_{31331} = \frac{-2+\alpha^2}{\sqrt{2}},$$
$$\nabla R_{23223} = \nabla R_{32332} = \frac{-\alpha^2}{\sqrt{2}},$$
$$\nabla R_{31332} = \nabla R_{32331} = -\nabla R_{21223} = -\nabla R_{23221} = \alpha.$$

(2) When, $\{e_1, e_2, e_3\}$ is not unite orthogonal, one of the following cases occurs:
Type IV1: $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1\rangle = -\langle e_2, e_2\rangle = -\langle e_3, e_3\rangle$ and the structure constant satisfies $\alpha\gamma - \beta\delta = 0$.
Type IV2: $\{e_1, e_2, e_3\}$ is orthonormal with $\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = -\langle e_3, e_3\rangle$ and the structure constant satisfies $\alpha\gamma + \beta\delta = 0$.
Type IV3: $\{e_1, e_2, e_3\}$ is pseudo-orthonormal with

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$$

and the structure constant satisfies $\alpha\gamma = 0$. Then according the this classification, we have:
Type IV1:

$$\nabla R_{12113} = \nabla R_{13112} = \frac{1}{4}(\alpha+\beta)(-\beta^2+\gamma^2+2\alpha(\alpha+\delta)),$$
$$\nabla R_{21223} = \nabla R_{23221} = -\frac{1}{2}(\beta+\gamma)((\beta-\gamma)\gamma-\alpha\delta+\delta^2),$$
$$\nabla R_{31332} = \nabla R_{32331} = -\frac{1}{4}(\beta-\gamma)(-2\alpha^2+(\beta-\gamma)^2-2\delta^2).$$

Type IV2:

$$\nabla R_{12113} = \nabla R_{13112} = \frac{1}{4}(\alpha+\gamma)(-\beta^2+\gamma^2+2\alpha(\alpha+\delta)),$$
$$\nabla R_{21223} = \nabla R_{23221} = -\frac{1}{2}(\beta+\gamma)((\beta-\gamma)\gamma-\alpha\delta+\delta^2),$$

$$\nabla R_{31332} = \nabla R_{32331} = -\frac{1}{4}(\beta - \gamma)(\alpha^2 + \beta^2 - \gamma^2 + \delta^2).$$

Type IV3:

$$\begin{aligned}
\nabla R_{13113} &= \gamma(\alpha^2 + \beta\gamma - \alpha\delta), \\
\nabla R_{31331} &= 2\gamma(-\beta\gamma + \alpha - \alpha + \delta), \\
\nabla R_{31332} &= \nabla R_{32331} = \frac{1}{2}\gamma(\alpha^2 + 3\beta\gamma - \alpha\delta).
\end{aligned}$$

4 Szabo Operators of Walker Metrics on 3-Manifolds

Let (M, g) be a 3-dimensional Lorentzian manifold admitting a parallel one-dimensional degenerate plane with the local coordinates (t, x, y), where the Lorentzian metric tensor expresses by (1).

Then, we can classify the Szabo operator in a suitable unit orthogonal frame $\{e_1, e_2, e_3\}$, where $e_1 = \partial_t, e_2 = \partial_x, e_3 = \partial_y$. So, the non-zero component of the Szabo operator are:

$$\begin{aligned}
\nabla R_{13313} &= \frac{1}{2} f_{tty}, \\
\nabla R_{13323} &= \nabla R_{23313} = \frac{1}{2} f_{txy} - \frac{1}{4} f_x f_{tt} + \frac{1}{4} f_t f_{tx}, \\
\nabla R_{13333} &= \nabla R_{33313} = \frac{1}{4} f f_t f_{tt} + \frac{1}{4} f_y f_{tt} + \frac{1}{4\varepsilon} f_x f_{tx}, \\
\nabla R_{23323} &= \frac{1}{2} f_{xxy} - \frac{1}{4} f_x f_{tx} + \frac{1}{4} f_t f_{tx}, \\
\nabla R_{23333} &= \nabla R_{33323} = \frac{1}{4\varepsilon} f_x f_{xx} - \frac{1}{4} f f_t f_{tx} - \frac{1}{4} f_y f_{tx}.
\end{aligned}$$

Theorem 4.1. *Let (M, g) be a three-dimensional Lorentzian manifold admitting a parallel one-dimensional degenerate plane field. Suppose S be Szabo operator of Walker metric. Then, $S \equiv 0$ if and only if the function f is of the form*

$$f(t, x, y) = G(x, t) + H(x, y)t + K(x, y) + c, \tag{5}$$

for any functions $G(x, t)$, $H(x, y)$, $K(x, y)$ and c (constant) satisfy the following relations:

$$2(H_{xy} - H_{xxy}t - K_{xxy}) + (G_x + H_x t + K_x)(G_{tx} + H_x - G_{tt}) = 0,$$

$$(G_x + H_x t + K_x)\Big(G_{tx}^2 + H_x^2 + 2G_{tx}H_x + G_{xx}G_{tt} + H_{xx}G_{tt}t + K_{xx}G_{tt}\Big) = 0.$$

Proof. Using the condition in Szabo operator, then we have five equations:

$$\tfrac{1}{2}f_{tty} = 0,$$

$$\tfrac{1}{2}f_{txy} - \tfrac{1}{4}f_x f_{tt} + \tfrac{1}{4}f_t f_{tx} = 0,$$

$$+\tfrac{1}{4}f f_t f_{tt} + \tfrac{1}{4}f_y f_{tt} + \tfrac{1}{4\varepsilon}f_x f_{tx} = 0,$$

$$\tfrac{1}{2}f_{xxy} - \tfrac{1}{4}f_x f_{tx} + \tfrac{1}{4}f_t f_{tx} = 0,$$

$$\tfrac{1}{4\varepsilon}f_x f_{xx} - \tfrac{1}{4}f f_t f_{tx} - \tfrac{1}{4}f_y f_{tx} = 0.$$

From the first equation, $\frac{1}{2}f_{tty} = 0$, we conclude:

$$f(t,x,y) = G(x,t) + H(x,y)t + K(x,y) + c.$$

By the second equation, the fourth equations, and (5) we concluded:
$2(H_{xy} - H_{xxy}t - K_{xxy}) + (G_x + H_x t + K_x)(G_{tx} + H_x - G_{tt}) = 0.$
Also by the third equation, fifth equations and (5), the proof is completed. □

Now, if (M,g) admits a parallel null vector field, then the non-zero component of Szabo operator are:

$$\nabla R_{23323} = \frac{1}{2}f_{xxy},$$
$$\nabla R_{23333} = \frac{1}{4\varepsilon}f_x f_{xx}. \qquad (6)$$

Theorem 4.2. *Let (M,g) be a three-dimensional Lorentzian manifold admitting a parallel null one-dimensional degenerate plane field. Then, $S \equiv 0$ if and only if the function f is of the from*

$$f(x,y) = P(y) \qquad or \qquad f(x,y) = xQ(y) + c$$

for any functions $P(y), Q(y)$ and c (constant).

Proof. Using the condition in Szabo operator, then we have two equations.

$$f_{xxy} = 0 \qquad \text{and} \qquad f_x f_{xx} = 0.$$

From the above equations (6), the proof is completed. □

Acknowledgments
Authors would like to thank referees for those sharp and nice comments.

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