Journal of Mathematical Extension
Vol. 16, No. 11, (2022) (5)1-16
URL: https://doi.org/10.30495/JME.2022.2192
ISSN: 1735-8299
Original Research Paper

Homological Properties of Banach Modules on Homogeneous Spaces

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Abstract. Let G be a locally compact group and H be a compact subgroup of G. The aim of this paper is to characterize some homological properties of $L^1(G/H)$, $C_0(G/H)$ and $M(G/H)$ as left Banach $L^1(G)$-modules such as flatness, injectivity and projectivity. Moreover, we study the projectivity of $C_0(G/H)$ and $M(G/H)$ as Banach left $L^1(G/H)$-modules and $M(G)$-modules.

AMS Subject Classification: 46H25; 43A85
Keywords and Phrases: Banach module, homogeneous spaces, flatness, injectivity, locally compact group, projectivity.

1 Introduction

Homological properties of certain left Banach $L^1(G)$-modules have been studied by Dales and Polyakov in 2004 [1], and in 2008 some of those results were investigated by Ramsden for semigroup algebras [6]. However, homological properties of left Banach $L^1(G)$-modules constructed on homogeneous spaces have not been investigated so far.

Received: October 2021; Accepted: March 2022

Throughout this paper, we assume that G is a locally compact group and H is a closed subgroup of G with left Haar measures λ_G and λ_H, respectively. Also, Δ_G and Δ_H are the modular functions of G and H, respectively. Let $q : G \longrightarrow G/H$ be the natural quotient map. Consider the G-space G/H as a homogeneous space that G acts on by $x(yH) = (xy)H$. Let μ be a Radon measure on G/H. For $x \in G$, μ_x is defined by $\mu_x(E) = \mu(xE)$, where $E \subset G/H$ is a Borel set. The measure μ is said to be G-invariant, if $\mu_x = \mu$ for all $x \in G$. The Radon measure μ is said to be strongly quasi-invariant if there is a continuous function $\theta : G \times G/H \longrightarrow (0, \infty)$ such that $d\mu_x(yH) = \theta(x, yH)d\mu(yH)$ for any $x, y \in G$. A continuous function $\rho : G \longrightarrow (0, \infty)$ is called rho-function for the pair (G, H), when $\rho(x\xi) = (\frac{\Delta_H(\xi)}{\Delta_G(\xi)})\rho(x)$ for any $x \in G, \xi \in H$. For every rho-function ρ there exits a strongly quasi-invariant measure μ on G/H such that the Weil's formula holds:

$$\int_G f(x)\rho(x)d\lambda_G(x) = \int_{G/H}\int_H f(x\xi)d\lambda_H(\xi)d\mu(xH) \quad (f \in C_c(G)).$$

Furthermore, the measure μ satisfies $\frac{d\mu_x}{d\mu}(yH) = \frac{\rho(xy)}{\rho(x)}$ for any $x, y \in G$. The map $T_\rho : L^1(G) \longrightarrow L^1(G/H)$ was defined by Tavallei and et al. in [5] by

$$T_\rho f(xH) = \int_H \frac{f(x\xi)}{\rho(x\xi)} d\lambda_H(\xi) \quad (xH \in G/H, \xi \in H),$$

where μ is a strongly quasi-invariant measure on G/H which arises from a rho-function ρ and $L^1(G/H) = L^1(G/H, \mu)$. The map T_ρ is a linear, bounded and surjective map with $\|T_\rho\| \leq 1$ and $\int_{G/H} T_\rho f(xH)d\mu(xH) = \int_G f(x)d\lambda_G(x)$ for all $f \in L^1(G)$. Moreover, for every $\varphi \in L^1(G/H)$,

$$\|\varphi\|_1 = \inf\{\|f\|_1 : f \in L^1(G), \varphi = T_\rho f\}. \tag{1}$$

It turns out that the Banach space $L^1(G/H)$ isometrically isomorphic to the quotient space $\frac{L^1(G)}{Ker(T_\rho)}$ equipped with the usual quotiont norm. For a compact subgroup H of G, define

$$L^1(G : H) = \{f \in L^1(G) : R_\xi f = f, \xi \in H\}.$$

Then $L^1(G:H)$ is a left ideal of $L^1(G)$, and $L^1(G/H)$ is isometrically isomorphic to $L^1(G:H)$. Thus $L^1(G/H)$ is a Banach algebra. It was shown in [5] that

$$L^1(G:H) = \{\psi \circ q : \psi \in L^1(G/H)\}. \tag{2}$$

For more details see [5].

Lemma 1.1. *Let H be a closed subgroup of G. Then $\ker T_\rho$ is a left ideal of $L^1(G)$.*

Proof. Let $f \in L^1(G)$ and $g \in \ker T_\rho$. Then one has

$$f \cdot g(xH) = T_\rho(f \star g)(xH) \quad = \int_G f(y) \int_H \frac{g(y^{-1}x\xi)}{\rho(x)\frac{\Delta_H(\xi)}{\Delta_G(\xi)}} d\lambda_H(\xi) d\lambda_G(x) = 0.$$

Since $T_\rho(g)(y^{-1}xH) = 0$, we have $\int_H \frac{g(y^{-1}x\xi)}{\frac{\Delta_H(\xi)}{\Delta_G(\xi)}} d\lambda_H(\xi) = 0$. Thus $f \star g \in \ker T_\rho$. □

Lemma 1.2. *Let G be a locally compact group and H be a closed subgroup of G. Then $L^1(G/H)$ is a left $L^1(G)$-module.*

Proof. For $\varphi \in L^1(G/H)$ there is $g_\varphi \in L^1(G)$ such that $T_\rho(g_\varphi) = \varphi$. Define a left module action of $L^1(G)$ on $L^1(G/H)$ by

$$f \cdot \varphi = T_\rho(f \star g_\varphi), \quad (f \in L^1(G)).$$

Then

$$\begin{aligned} \|f \cdot \varphi\|_{L^1(G/H)} &\leq \int_{G/H} \int_H \left|\frac{f \star g_\varphi(x\xi)}{\rho(x\xi)}\right| d\lambda_H(\xi) d\mu(xH) \\ &= \int_G |f \star g_\varphi| d\lambda_G \leq \|f\|_1 \|g_\varphi\|_1 < \infty. \end{aligned}$$

From equality (2) we have $\|f \cdot \varphi\|_{L^1(G/H)} \leq \|f\|_1 \|\varphi\|_{L^1(G/H)}$. It can be easily seen that this operation, converts $L^1(G/H)$ to a left Banach $L^1(G)$-module and $L^1(G/H)$ is essential as a left $L^1(G)$-module. □

We conclude this section with some examples of homogeneous spaces.

Example 1.3. Let $M_n(\mathbb{C})$ be the space of $n \times n$ matrices,

$$G = GL_n(\mathbb{C}) := \{T \in M_n(\mathbb{C}) : \det T \neq 0\},$$

and

$$H = U(n) := \{T \in M_n(\mathbb{C}) : T^*T = I\}.$$

Then H is a compact subgroup of G and

$$SU(n) = \{T \in U(n) : \det T = 1\},$$

is a compact subgroup of $U(n)$.

Example 1.4. : Let $G = \{f : \mathbb{N} \longrightarrow \mathbb{N} \mid f \text{ is a bijection}\}$ with discrete topology and $H_k = \{f \in G \mid f(n) = n, n > k\}$. Obviously G is a group under composition of functions and H_k is a subgroup of G. Then H_k (the symmetric group) is a finite subgroup of G.

2 The Module $L^1(G/H)$

Let E and F be two Banach spaces and $B(E,F)$ denote the space of all bounded linear operators from E to F. The dual space $B(E,\mathbb{C})$ of E is denoted by E'. We write $B(E)$ in place of $B(E,E)$. An operator $T \in B(E,F)$ is called admissible if there exists an operator $S \in B(F,E)$ such that $T \circ S \circ T = T$. Let A be a Banach algebra and E, F be left Banach A-modules. The linear space of all left A-module morphisms is denoted by ${}_AB(E,F)$. An operator $T \in {}_AB(E,F)$ is called a retraction if there exists an operator $S \in {}_AB(F,E)$ such that $T \circ S = I_F$. A left Banach A-module P is called projective if for every admissible epimorphism $T \in {}_AB(E,F)$ and any $S \in {}_AB(P,F)$, there exists $R \in {}_AB(P,E)$ such that $T \circ R = S$. A left Banach A-module J is called injective if for every admissible monomorphism $T \in {}_AB(E,F)$ and any $S \in {}_AB(E,J)$ there exists $R \in {}_AB(F,J)$ such that $R \circ T = S$. If a left (right) Banach A-module E is projective, then the right (left) Banach A-module E' is injective. Finally, let us recall that if E is a Banach left A-module, then E' is a right Banach A-module under the dual module action defined by

$$\langle x, \lambda \cdot a\rangle = \langle a \cdot x, \lambda\rangle$$

for $\lambda \in E'$, $x \in E$ and $a \in A$.

A left (right) Banach A-module E is called flat if E' is injective as a right (left) A-module. For two Banach spaces E and F, we denote by $E\hat{\otimes}F$ their projective tensor product. The projective tensor norm on $E\hat{\otimes}F$ is denoted by $\|.\|_\pi$, where

$$\|u\|_\pi = \inf\{\sum_1^\infty \|x_n\|\|y_n\| : u = \sum_1^\infty x_n \otimes y_n, (x_n \in X, y_n \in Y)\}$$

Let A be a Banach algebra, E a Banach A-bimodule and F a left Banach A-module. Then $E\hat{\otimes}F$ becomes a left A-module with the following module action:

$$a \cdot (x \otimes y) = (a \cdot x) \otimes y \quad (a \in A, x \in X, y \in Y).$$

For a Banach algebra A we denote by $A^\flat$ the Banach algebra formed by adjoining an identity to A. The morphism $\Pi \in {}_AB(A^\flat\hat{\otimes}E, E)$ is defined by

$$\Pi(a \otimes x) = a \cdot x \qquad (a \in A^\flat, x \in E).$$

We shall use the following theorem from [4, IV.1.1, IV.1.2].

Theorem 2.1. *Let A be a Banach algebra and E be a left A-module. Then E is projective if and only if the morphism $\Pi \in {}_AB(A^\flat\hat{\otimes}E, E)$ is a retraction. In the case that E is essential, E is projective if and only if the morphism $\Pi \in {}_AB(A\hat{\otimes}E, E)$ is a retraction.*

Remark 2.2. It is well known that $L^1(G)\hat{\otimes}L^1(G/H)$ is a left $L^1(G)$-module. If f, $f_1 \in L^1(G)$, $\varphi \in L^1(G/H)$, then for $F = f_1 \otimes \varphi \in L^1(G \times G/H)$ we have

$$\begin{aligned} f \cdot F(x, zH) &= (f \star f_1)(x)\varphi(zH) \\ &= \int_G f(y)(f_1 \otimes \varphi)(y^{-1}x, zH)d\lambda_G(y) \\ &= \int_G f(y)F(y^{-1}x, zH)d\lambda_G(y) \quad (x, z \in G). \end{aligned}$$

Thus this formula holds for any $F \in L^1(G \times G/H)$.

Theorem 2.3. *Let H be a compact subgroup of G, and let μ be the G-invariant measure on G/H arising from the constant rho-function $\rho = 1$. Then $L^1(G/H, \mu)$ is projective as a left $L^1(G)$-module ($M(G)$-module), and hence flat as a left $L^1(G)$-module ($M(G)$-module).*

Proof. Let

$$\Pi : L^1(G)\hat{\otimes}L^1(G/H) \longrightarrow L^1(G/H), \quad f \otimes \varphi \mapsto f \cdot \varphi,$$

where

$$f \cdot \varphi(xH) = \int_G f(y)\varphi(y^{-1}xH)d\lambda_G(y).$$

Let also E be a compact symmetric subset of G such that $\lambda_G(HE) > 0$. We can choose a Haar measure on G such that $\lambda_G(HE) = 1$. If $K = q(E) \subset G/H$, then K is compact. Define

$$\rho : L^1(G/H) \longrightarrow L^1(G)\hat{\otimes}L^1(G/H) = L^1(G \times G/H),$$

by

$$\rho(\varphi)(s, tH) = \chi_K(tH)\varphi(stH).$$

Then

$$\begin{aligned}
\|\rho(\varphi)\|_1 &= \int_G \int_{G/H} |\chi_K(tH)\varphi(stH)| d\lambda_G(s)d\mu(tH) \\
&= \int_G \int_{G/H} |\chi_K(zH)\varphi(tH)| d\lambda_G(s)d\mu(tH) \quad (z = \xi s^{-1}t, \xi \in H) \\
&= \lambda_G(HE) \int_{G/H} |\varphi(tH)| d\mu(tH) < \infty.
\end{aligned}$$

Moreover, for $g \in L^1(G/H)$ and $x \in G$ we have

$$\begin{aligned}
\Pi(\rho(\varphi))(xH) &= \int_G \rho(\varphi)(y, y^{-1}xH)d\lambda_G(y) \\
&= \varphi(xH) \int_G \chi_K(y^{-1}H)d\lambda_G(y) = \varphi(xH)\lambda_G(HE) = \varphi(xH).
\end{aligned}$$

Let $f \in L^1(G)$, $\varphi \in L^1(G/H)$ and $s, t \in G$. Then

$$\begin{aligned} \rho(f \cdot \varphi)(s, tH) &= \chi_K(tH) \int_G f(y)\varphi(y^{-1}stH)d\lambda_G(y) \\ &= f \cdot \rho(\varphi)(s, tH). \end{aligned}$$

Since $L^1(G/H)$ is unital as a left Banach $M(G)$-module, the result follows by Theorem 2.1 and [6, Theorem 3.1.1]. □

Lemma 2.4. *Let G be a locally compact group, H a closed subgroup of G and μ_1, μ_2 two strongly quasi-invariant measures on G/H that they arising from rho-functions ρ_1, ρ_2, respectively. Then $L^1(G/H, \mu_1)$ is isometrically isomorphic to $L^1(G/H, \mu_2)$ as left Banach $L^1(G)$-modules.*

Proof. Let $\eta : G/H \longrightarrow (0, \infty)$ be defined by $\eta(xH) = \frac{\rho_1(x)}{\rho_2(x)}$. Then according to [3, Theorem 2.59], $d\mu_1 = \eta d\mu_2$. Thus the mapping

$$T : L^1(G/H, \mu_1) \longrightarrow L^1(G/H, \mu_2), \quad \varphi \mapsto \eta\varphi.$$

is well defined and $\int_{G/H} |\varphi| d\mu_1 = \int_{G/H} |\varphi|\eta d\mu_2 = \int_{G/H} |T(\varphi)| d\mu_2 < \infty$. Moreover, if $\varphi \in L^1(G/H, \mu_2)$, then $T(\frac{1}{\eta}\varphi) = \varphi$. Therefore, T is a surjective linear isometry. For $\varphi \in L^1(G/H, \mu_1)$ there exists $g_\varphi \in L^1(G)$ such that $T_{\rho_1}(g_\varphi) = \varphi$ and thus $T_{\rho_1}(g_\varphi) = \frac{1}{\eta} T_{\rho_2}(g_\varphi)$. Finally for $f \in L^1(G)$, $\varphi \in L^1(G/H, \mu_1)$ we have

$$\begin{aligned} T(f \cdot \varphi) = T(T_{\rho_1}(f \star g_\varphi)) &= T(\frac{1}{\eta} T_{\rho_2}(f \star g_\varphi)) \\ &= f \cdot T_{\rho_2}(g_\varphi) \\ &= f \cdot T(\varphi). \end{aligned}$$

□

Corollary 2.5. *Let G, H, μ_1 and μ_2 be as in Lemma 2.4. If $L^1(G/H, \mu_1)$ is projective as a left $L^1(G)$-module, then $L^1(G/H, \mu_2)$ is projective as a left $L^1(G)$-module too.*

Corollary 2.6. *Let H be a compact subgroup of G. Then $L^1(G/H)$ is projective as a left Banach $L^1(G/H)$-module.*

Proof. Since $L^1(G/H)$ is an $L^1(G/H)$-bimodule and T_ρ is a surjective left $L^1(G)$-module morphism, the result follows by Theorem 2.3 and the proof of [4, IV Proposition1.7]. □

Definition 2.7. Let G be a locally compact group, E a left Banach $L^1(G)$-module and φ_G the augmentation character defined by $\varphi_G(f) = \int_G f d\lambda_G$. Then E is termed augmentation-invariant, if there exists a non-zero $\lambda \in E'$ such that

$$\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G), x \in E).$$

Remark 2.8. Let E be a left Banach $L^1(G)$-module. If E is augmentation-invariant, then E'' will also be augmentation-invariant.
Let $\lambda \in E'$. Then $\langle f \cdot x, \lambda \rangle = \varphi_G(f)\langle x, \lambda \rangle$ for $f \in L^1(G), x \in E$. For each $\varphi \in E''$ there exists $(x_\alpha)_\alpha \in E$ such that $x_\alpha \longrightarrow \varphi$ in $\sigma(E'', E')$-topology. So, if $f \in L^1(G)$ and $\varphi \in E''$, we have

$$\begin{aligned} \langle f \cdot \varphi, \lambda \rangle &= \langle \varphi, \lambda \cdot f \rangle = \lim_\alpha \langle x_\alpha, \lambda \cdot f \rangle = \lim_\alpha \langle f \cdot x_\alpha, \lambda \rangle \\ &= \lim_\alpha \varphi_G(f) \langle x_\alpha, \lambda \rangle = \varphi_G(f) \langle \varphi, \lambda \rangle. \end{aligned}$$

Lemma 2.9. *Let E and F be left Banach $L^1(G)$-modules and $T \in {}_{L^1(G)}B(E, F)$ be an isometric isomorphism of Banach space. If E is augmentation-invariant, then F is augmentation-invariant as well.*

Proof. Let $\lambda \in E'$ such that,

$$\langle f \cdot x, \lambda \rangle = \varphi_G(f) \langle x, \lambda \rangle \quad (f \in L^1(G), x \in E).$$

Since $\lambda \circ T^{-1} \in F'$, for any $y \in F$ we have

$$\langle f \cdot y, \lambda \circ T^{-1} \rangle = \lambda \circ T^{-1}(f \cdot y) = \lambda(f(T^{-1}(y))) = \varphi_G(f) \langle y, \lambda \circ T^{-1} \rangle.$$

□

Corollary 2.10. *Let H be a closed subgroup of G. Then the left $L^1(G)$-module $L^1(G/H)$ is augmentation-invariant.*

Proof. For $f \in \ker T_\rho$ we have

$$0 = \int_{G/H} T_\rho f(xH) d\mu(xH) = \int_G f(x) d\lambda_G(x) = \varphi_G(f).$$

Let $\lambda : L^1(G)/\ker T_\rho \longrightarrow \mathbb{C}$ by $\lambda(f + \ker T_\rho) = \varphi_G(f)$. Then

$$\lambda(f \cdot (g + \ker T_\rho)) \quad = \varphi_G(f)\lambda(g + \ker T_\rho).$$

Thus $L^1(G)/\ker T_\rho$ is augmentation-invariant as a left $L^1(G)$-module, and so $L^1(G/H)$ is augmentation-invariant by Lemma 2.9. □

Definition 2.11. Let A be an algebra, and E be a left A-module. Then E is said to be faithful if for each $x \in E \setminus \{0\}$ there exists $a \in A$ such that $a \cdot x \neq 0$

Lemma 2.12. *Let E and F be left Banach A-modules and $T : E \longrightarrow F$ be an isomorphism of left Banach A-modules. If E is faithful, then F is also faithful.*

Lemma 2.13. *The Banach space $L^1(G)/\ker T_\rho$ is faithful as a left $L^1(G)$-module. Consequently, $L^1(G/H)$ is faithful as a left $L^1(G)$-module.*

Proof. Let $g + \ker T_\rho \neq 0$ and let $\{f_\upsilon\}_\upsilon$ be a bounded approximate identity for $L^1(G)$. Since $f_\upsilon \star g + \ker T_\rho \longrightarrow g + \ker T_\rho \neq 0$, there exists υ such that $f_\upsilon \star g + \ker T_\rho \neq 0$. □

Remark 2.14. Since $L^1(G/H)$ is a left Banach $L^1(G)$-module, $L^\infty(G/H, \mu) = L^1(G/H, \mu)'$ and $C_0(G/H)$ become right Banach $L^1(G)$-modules. For $\psi \in C_0(G/H)$ and $f \in L^1(G)$ consider $g_\varphi \in L^1(G)$ with $T_\rho(g_\varphi) = \varphi$. Then

$$\begin{aligned}
\langle \varphi, \psi \cdot f\rangle &= \langle f \cdot \varphi, \psi\rangle \\
&= \int_{G/H} f \cdot \varphi(xH)\psi(xH)d\mu(xH) \\
&= \int_G \int_{G/H} \int_H \frac{f(y)g_\varphi(y^{-1}x\xi)}{\rho(x\xi)}\psi(xH)\frac{\rho(y^{-1}x\xi)}{\rho(y^{-1}x\xi)} d\lambda_H(\xi)d\lambda_G(y)d\mu(xH) \\
&= \int_{G/H} \int_G f(y)\psi(xH)\varphi(y^{-1}xH)\theta(y^{-1}, xH)d\lambda_G(y)d\mu(xH) \\
&= \int_{G/H} \int_G f(y)\psi(yxH)\varphi(xH)d\lambda_G(y)d\mu(xH).
\end{aligned}$$

Thus $\psi \cdot f(xH) = \int_G f(y)R_x(\psi \circ q)(y)d\lambda_G(y)$ and so $\psi \cdot f \in C_0(G/H)$.

Theorem 2.15. *Let G/H be a discrete space. Then $\ell^1(G/H)$ is injective as a left Banach $L^1(G)$-module if and if G is amenable.*

Proof. According to Lemma 2.13, Corollary 2.10 and Remark 2.14, the result is obtained by [1, Proposition4.6]. ☐

3 The Modules $C_0(G/H)$ and $L^\infty(G/H)$

Let G be a locally compact group and H be a compact subgroup of G with a normalized Haar measure. The surjective linear map $T_\infty : L^\infty(G) \longrightarrow L^\infty(G/H)$ with $T_\infty f(xH) = \int_H f(x\xi)d\lambda_H(\xi)$ for $x \in G$ and $f \in L^\infty(G)$ was defined in [5]. It has been proved that $\|T_\infty\| \leq 1$ and $T_\infty(C_c(G)) \subset C_c(G/H)$ [5, Theorem 3.4] and [3, Proposition 2.48]. Moreover, for $\varphi \in L^\infty(G/H)$,

$$\|\varphi\|_\infty = \inf\{\|f\|_\infty : f \in L^\infty(G), \varphi = T_\infty(f)\}.$$

Let $L^\infty(G : H) = \{f \in L^\infty(G) : R_\xi f = f, \xi \in H\}$. Then the restriction of T_∞ on $L^\infty(G : H)$ is an isometric isomorphism. The left module action of $L^1(G)$ on $L^\infty(G/H)$ is defined by

$$f \cdot \varphi = T_\infty(f \star g) \quad (\varphi \in L^\infty(G/H), f \in L^1(G), g \in L^\infty(G)),$$

where $T_\infty(g) = \varphi$. The space $C_0(G/H)$ is a closed $L^1(G)$-submodule of the left Banach $L^1(G)$-module $L^\infty(G/H)$, and $C_0(G/H)$ is essential.

Remark 3.1. Since $L^\infty(G/H)$ is a left $L^1(G)$-module, $L^\infty(G/H)'$ will be a right $L^1(G)$-module. We show that $L^1(G/H)$ is also a right $L^1(G)$-module.

For $f \in L^1(G)$, $\varphi \in L^1(G/H)$, $\psi \in L^\infty(G/H)$ and $g_\psi \in L^\infty(G)$ such that $T_\infty(g_\psi) = \psi$ we have

$$\begin{aligned}
\langle \psi, \varphi \cdot f \rangle &= \langle f \cdot \psi, \varphi \rangle \\
&= \int_{G/H} \int_H \int_G f(y) g_\psi(y^{-1}x\xi)\varphi(xH) d\lambda_H(\xi) d\mu(xH) d\lambda_G(y) \\
&= \int_{G/H} \int_G f(y)\psi(xH)\varphi(yxH)\theta(y, xH) d\lambda_G(y) d\mu(xH).
\end{aligned}$$

Thus $\varphi{\cdot}f(xH) = \int_G f(y)\varphi(yxH)\theta(y,xH)d\lambda_G(y)$, and so $\varphi{\cdot}f \in L^1(G/H)$. Furthermore,

$$\begin{aligned}\|\varphi\cdot f\|_1 &\leq \int_{G/H}\int_G |f(y)||\varphi(yxH)|\theta(y,xH)d\lambda_G(y)d\mu(xH)\\ &= \int_{G/H}\int_G |f(y)||\varphi(xH)|\theta(y^{-1},xH)\theta(y,y^{-1}xH)d\lambda_G(y)d\mu(xH)\\ &= \|f\|_1\|\varphi\|_1.\end{aligned}$$

It is known that $L^1(G/H\times G)\cong L^1(G/H)\hat{\otimes}L^1(G)$ is a right $L^1(G)$-module. Indeed, for $F\in L^1(G/H\times G)$ and $f\in L^1(G)$,

$$F.f(tH,s) = \int_G F(tH,sy^{-1})f(y)\Delta_G(y^{-1})d\lambda_G(y).$$

Notice if G is compact and $\rho = 1$, $L^1(G/H)$ is essential as a right $L^1(G)$-module.

Theorem 3.2. *Let G be a compact group and H be a closed subgroup of G. Then $L^1(G/H)$ is projective as a right $L^1(G)$-module and flat as a right $L^1(G)$-module.*

Proof. Let $\Pi : L^1(G/H)\hat{\otimes}L^1(G) \longrightarrow L^1(G/H)$ be given by

$$\Pi(\varphi\otimes f) = \varphi\cdot f \quad (\varphi\in L^1(G/H), f\in L^1(G)),$$

where

$$\varphi\cdot f(xH) = \int_G f(y)\varphi(yxH)\theta(y,xH)d\lambda_G(y).$$

Define $\rho : L^1(G/H) \longrightarrow L^1(G/H)\hat{\otimes}L^1(G)\cong L^1(G/H\times G)$ by

$$\rho(\varphi)(tH,s)) = \varphi(s^{-1}tH)\theta(s^{-1},tH).$$

Then

$$\begin{aligned}\Pi(\rho)(\varphi)(xH) &= \int_G \varphi(xH)\theta(y^{-1},yxH)\theta(y,xH)d\lambda_G(y)\\ &= \lambda_G(G)\varphi(xH) = \varphi(xH),\end{aligned}$$

and

$$\begin{aligned}\rho(\varphi\cdot f)(tH,s) &= \int_G f(y)\varphi(ys^{-1}tH)\theta(y,s^{-1}tH)\theta(s^{-1},tH)d\lambda_{G(y)}\\ &= (\rho(\varphi)\cdot f)(tH,s).\end{aligned}$$

□

The following theorem is proved with a similar argument as in [1, Theorem 3.1].

Theorem 3.3. *Let G be a locally compact group, H a compact subgroup of G and E a closed, left $L^1(G)$-submodule of $L^\infty(G/H)$ such that $C_c(G/H) \subset E \subset C^b(G/H)$. If E is projective, then G/H is compact.*

Proof. Suppose that G/H is not compact. Then clearly G is not compact. So, there exists compact and symmetric neighborhoods V, W of e_G such that $V^2 \subset W$ and there exists $0 \le f_1 \le 1, f_1 \in C_c(G)$ with $f_1(e_G) = 1$, supp$f_1 \subset V$ and $\|f_1\|_1 \le 1$. Then supp$(f_1 \star f_1) \subset V^2 \subset W$, $f_1 \star f_1(e_G) \neq 0$ and $f_1 \star f_1 \ge 0$. Set $T_\rho(f_1) = f$. Since f_1 is continuous and $f_1 \ge 0$, $f_1 \cdot f = T_\rho(f_1 \star f_1) \neq 0$ and $T_\rho(f_1 \star f_1) \in C_c(G/H)$. Set $A = L^1(G)$. Because E is projective, there exists $T \in {}_AB(E, A^\flat)$ such that $T(f_1 \cdot f) \neq 0$ by [1, Proposition1.2].
Without loss of generality we may suppose that $T(E) \subset A$. Let $n \in \mathbb{N}$ and F be a compact subset of G. Since G is not compact, there exist $s_1, \ldots, s_n \in G$ such that $s_iF \cap s_jF = \emptyset$ for $i \neq j$.
Set $\eta = \frac{\|f_1 \star T(f)\|_1}{2} > 0$ and pick $k \in \mathbb{N}$ with $\frac{1}{k} < \eta$. There exists $g \in C_c(G)$ such that $\|T(f) - g\|_1 < \dfrac{1}{k}$, and so

$$\|f_1 \star (T(f) - g)\|_1 \le \|f_1\|_1 \|T(f) - g\|_1 \le \frac{1}{k} < \eta.$$

Therefore, we have

$$\begin{aligned}\|f_1 \star g\|_1 &= \|f_1 \star g - f_1 \star T(f) + f_1 \star T(f)\|_1\\ &\ge \|f_1 \star T(f)\|_1 - \|f_1 \star g - f_1 \star T(f)\|_1\\ &> 2\eta - \eta = \eta.\end{aligned}$$

Set $K = \text{supp}(g)$ and observe that $(K \cup V)^2$ is compact. There exist $s_1, \ldots, s_k \in G$ such that the sets $s_i(K \cup V)^2$ are pairwise disjoint for

$i = 1, \ldots, k$. Set $f_{1j} = s_j \star f_1 = L_{s_j} f$. Then $\text{supp} f_{1j} \subset s_j V$, $\text{supp}(f_{1j} \star f) \subset s_j V.V$ and $|\sum_{j=1}^k f_{1j}|_G = 1$.
Therefore, $\| \sum_1^k f_{1j} \star g\|_1 = \sum_1^k \int_{s_j(K \cup V)^2} |f_{1j} \star g| = k\|f_1 \star g\|_1$.
Now set $\lambda = \sum_{j=1}^k (f_{1j} \cdot f)$. Since $s_j V.V \cap s_i V.V = \emptyset$ for $i \neq j$ and $f_{1j} \star f_1 \geq 0$, we have

$$\begin{aligned} |\lambda|_{G/H} &= \sup_{t \in G} |\lambda(tH)| \\ &= \sup_{t \in G} |\sum_1^k T_\rho(f_{1j} \star f_1)(tH)| \\ &= M|f_1 \star f_1|_G, \end{aligned}$$

in which $M = \sup(\frac{1}{\rho})$ on $\cup_{j=1}^k (s_j V.V)$. Since

$$\|f_{1j} \star (T(f) - g)\|_1 \leq \|f_{1j}\|_1 \|T(f) - g\|_1 \leq \frac{1}{k},$$

we have

$$\begin{aligned} \|T(\lambda)\|_1 = \| \sum_1^k f_{1j} \star T(f)\|_1 \geq \| \sum_1^k f_{1j} \star g\|_1 - 1 = k\|f_1 \star g\|_1 - 1 \\ \geq k\eta - 1. \end{aligned}$$

Therefore,

$$k\eta - 1 \leq \|T(\lambda)\|_1 \leq \|T\| |\lambda|_{G/H} \leq |f_1 \star f_1|_G M \|T\|.$$

This holds for any $k \in \mathbb{N}$, which is a contradiction with boundedness of T. $\square$

Theorem 3.4. *Let H be a compact subgroup of G and G/H be a compact space. Then $C(G/H)$ is projective as a left $L^1(G)$-module.*

Proof. Since H and G/H are compact, G is compact and so $L^1(G)$ is biprojective. If $C_0(G : H) = \{\varphi \in C_0(G) : \varphi(x\xi) = \varphi(x),\ x \in G, \xi \in H\}$, then $C_0(G : H)$ is a left $L^1(G)$-module by the module action defined by

$$f \cdot \varphi(x) = f \star \varphi(x) = \int_G f(y)\varphi(y^{-1}x) d\lambda_G(y),\ (f \in L^1(G), \varphi \in C_0(G : H)).$$

Let $(f_\alpha)_\alpha$ be a bounded approximate identity for $L^1(G)$. Then $f_\alpha \cdot \varphi = f_\alpha \star \varphi \longrightarrow \varphi$ and so $L^1(G) \cdot C_0(G:H) = L^1(G) \star C_0(G:H) = C_0(G:H)$. By [4, IV, Proposition 5.3] $C_0(G:H)$ is projective as a left $L^1(G)$-module. Since T_∞ is an isometric isomorphism of $C_0(G:H)$ onto $C_0(G/H) = C(G/H)$ as a left $L^1(G)$-module, $C(G/H)$ is projective. □

Corollary 3.5. *Let H be a compact subgroup of G. Then $C_0(G/H)$ is projective as a left $L^1(G)$-module ($M(G)$-module) if and only if G/H is a compact space.*

Remark 3.6. If G is a locally compact group, and if H is a compact subgroup of G, then by Theorem 2.3, $L^\infty(G/H)$ is an injective right $L^1(G)$-module. If G/H is finite, by Theorem 3.4 and the proof of [4, IV Proposition1.7], $L^\infty(G/H)$ is projective as a left $L^1(G/H)$-module.

Let G be a locally compact group and H be a compact subgroup of G. According to [7], we can define a norm decreasing linear map $\tilde{T}: M(G) \longrightarrow M(G/H)$ by $\tilde{T}m(E) = m(q^{-1}(E))$, where E is a Borel subset of G/H, that satisfies

$$\int_{G/H} \varphi(xH) d\tilde{T}m(xH) = \int_G \varphi(xH) dm(x) \quad (\varphi \in C_0(G/H)).$$

Set $M(G:H) = \{m \in M(G) : m(Ah) = m(A),\ A \in B_G\ ,\ h \in H\}$, where B_G is the σ-algebra of Borel sets. Then $M(G:H)$ is a closed left ideal of $M(G)$ and $\tilde{T}: M(G:H) \longrightarrow M(G/H)$ is an isometric isomorphism of Banach spaces ([2]).

For $m \in M(G)$ and $\nu \in M(G/H)$ define a left $M(G)$-module action on $M(G/H)$ by

$$m \cdot \nu(\varphi) = \int_{G/H} \int_G \varphi(yxH) dm(y) d\nu(xH) \quad (\varphi \in C_0(G/H).$$

Then $M(G/H)$ is a left Banach $L^1(G)$-module and $\tilde{T}(m_1 \star m_2) = m_1 \cdot \tilde{T}(m_2)$ for all m_1, $m_2 \in M(G)$. Also, if $\omega, \nu \in M(G/H)$, then $\omega \star \nu = \tilde{T}(\omega_v \star \nu_v)$, where $\omega_v, \nu_v \in M(G:H)$ such that $\tilde{T}(\omega_v) = \omega$, $\tilde{T}(\nu_v) = \nu$. Moreover, $M(G/H)$ with this convolution is a Banach algebra and $L^1(G/H)$ is an ideal of $M(G/H)$ (see [2] for more details).

Remark 3.7. If G/H is discrete, then $M(G/H) = \ell^1(G/H)$ is projective as a left $L^1(G)$-module ($M(G)$ module, $L^1(G/H)$-module).

Lemma 3.8. *Let G be a locally compact group and H be a compact subgroup of G. Then $M(G/H)$ is faithful and augmentation-invariant as a left $L^1(G)$-module.*

Proof. Since $M(G : H)$ is a submodule of the faithful Banach module $M(G)$ as a left $L^1(G)$-module, $M(G/H)$ is faithful. Since $M(G)$ is augmentation-invariant, there exists $\lambda \in M(G)'$ such that $\langle f \cdot m, \lambda \rangle = \varphi_G(f)\langle m, \lambda \rangle$ for $f \in L^1(G)$ and $m \in M(G)$. The map $\lambda \circ \iota \circ \tilde{T}^{-1}$ is an element of $M(G/H)'$ that satisfies definition of augmentation-invariance for $M(G/H)$, where $\iota : M(G : H) \longrightarrow M(G)$ is the inclusion map. □

Theorem 3.9. *Let G be a locally compact group and H be a compact subgroup of G. The following conditions are equivalent.*

(a) G is amenable;

(b) $M(G/H)$ is injective as a left Banach $L^1(G)$-module;

(c) $L^\infty(G/H)$ is flat as a right Banach $L^1(G)$-module;

(d) $C_0(G/H)$ is flat as a right Banach $L^1(G)$-module.

Proof. According to Lemma 3.8, $M(G/H)$ is augmentation-invariant as a left Banach $L^1(G)$-module, and also based on Remark 2.8, $(L^\infty(G/H))'$ is augmentation-invariant as a left Banach $L^1(G)$-module. Therefore, according to [6, Theorem 3.4.2], the implications $(d) \Leftrightarrow (a) \Leftrightarrow (b) \Leftrightarrow (c)$ follow. □

Acknowledgements
The authors express their sincere thanks to the reviewer for the careful reading of the manuscript and very helpful suggestions that improved the manuscript.

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