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Optimality Conditions for Fuzzy Optimization Problems under Metric Based Derivative

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Abstract. In this paper, we consider fuzzy optimization problems more general than all those that exist in the literature by using the concept of metric-based differentiability of fuzzy-valued functions. We get necessary optimality conditions based on fuzzy stationary point definition, and we prove these conditions are also sufficient under new fuzzy invexity notions. We illustrate these results with numerical examples.

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Keywords and Phrases: Fuzzy-valued functions; fuzzy invexity; GH-differentiability; metric based differentiability; fuzzy stationary point.

1 Introduction

Optimization theory is the most important and oldest classical area, which is of foremost concern in many disciplines. Engineering, Medicine,

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Operation Research, Artificial Intelligence, and many other fields.

In most real-world systems, uncertainty is inherent. There are several theories to describe uncertainty such as probability theory, fuzzy set theory, and possibility theory. Fuzzy mathematics is a powerful tool for processing subjective or vague information in mathematical models, for modeling uncertainty, and for formulating imprecise real-world problems. Zadeh introduced the fuzzy numbers concept [30] and in [4] Chang and Zadeh proposed the fuzzy-valued function notion. Many researchers study the notions of fuzzy calculus.

Fuzzy optimization problems (FOPs) are optimization problems with imprecision/ambiguity that could appear in parameter values or initial conditions. Fuzzy optimization is one area where considerable progress has been made. Farhadinia, [9], studied necessary optimality conditions for fuzzy variational problems using the fuzzy differentiability concept of Buckley and Feuring, see [3], but this work was generalized by Fard et al., [6]; Fard and Salehi, [8]; Fard and Zadeh, [7]. Fard et al. [6], presented the fuzzy Euler-Lagrange condition for fuzzy constrained and unconstrained variational problems under the generalized Hukuhara differentiability of several variables. There is extensive literature establishing necessary and sufficient conditions for fuzzy optimization problems, interested readers also can see [13, 23, 24, 25].

Accurate modeling of many dynamic systems leads to a set of fractional differential equations. Fractional derivatives played a significant role in engineering, science, and mathematics. Most recent advances, and application of this field in science, engineering, and mathematics, can be found in [15]-[16]. Fard and Salehi, [8], investigated fuzzy fractional Euler-Lagrange equations for fuzzy fractional variational problems using their generalized fuzzy fractional Caputo-type derivatives.

As we have known that the concept of stationary point plays a crucial role in classical optimization since it enables us to find the potential candidates to be optimums. Necessary fuzzy optimality conditions are based on a derivative definition. Fuzzy stationary points definitions [14, 23, 24, 25], are very restrictive because of their definition and others because of the derivative definition used.

The fundamental difference between the work that we present here is that other works required H-differentiability, level-wise differentiability, or **G**H-differentiability notions. These differentiability notions are very restrictive, as shown in [12].

So we use a differentiability notion defined in [13], that generalized all definitions of fuzzy stationary points existing in the literature. Optimality conditions that we prove here are simple and equivalent to check that zero belongs to an interval or not. A list of novel contributions of this paper is as follows:

1. Introducing the concepts of maxima and minima for fuzzy-valued functions that generalized the classic ones, and deducing some related results.

2. Defining new convexity notions for fuzzy-valued functions.

3. Proposing the necessary and sufficient optimality conditions for fuzzy optimization problems based on the differentiability concept introduced in [12].

However, distinct efforts have been made in fuzzy optimization to define analogous concepts of convexity and to establish necessary and sufficient optimality conditions (see, [4, 10, 14, 18, 19, 20, 21, 22, 23, 26, 27, 28, 29]). It seems a new idea to derive the necessary and sufficient optimality conditions for fuzzy optimization problems based on the differentiability concept introduced in [12]. First, some basic conceptsare introduced, such as fuzzy differentiability, and then necessary conditions of these functions are derived. We define some new convexity notions that generalize the classic ones. Finally, sufficient conditions are discussed.

2 Preliminaries

R_I is a family of all bounded closed intervals in R, i.e.,

$$\mathsf{R}_I = \{[\mathsf{a}_l, \mathsf{a}_u] : \mathsf{a}_l, \mathsf{a}_u \in \mathsf{R} \text{ and } \mathsf{a}_l \leq \mathsf{a}_u\}.$$

$(\mathsf{R}_I, \boldsymbol{D})$ is a complete metric space [5], where $\boldsymbol{D}$ is defined as

$$\boldsymbol{D}(\mathsf{A}, \mathsf{B}) = \max\{|\mathsf{a}_l - \mathsf{b}_l|, |\mathsf{a}_u - \mathsf{b}_u|\}$$

for all $\mathsf{A} = [\mathsf{a}_l, \mathsf{a}_u]$ and $\mathsf{B} = [\mathsf{b}_l, \mathsf{b}_u]$ belongs to R_I.

A fuzzy set $\widetilde{\mathsf{n}}$ on R is a mapping $\widetilde{\mathsf{n}} : \mathsf{R} \to [0,1]$. a-level sets of each fuzzy set $\widetilde{\mathsf{n}}$ is defined as $[\widetilde{\mathsf{n}}]^{\mathsf{a}} = \{\boldsymbol{v} \in \mathsf{R} : \widetilde{\mathsf{n}}(\boldsymbol{v}) \geq \mathsf{a}\}$ for all $\mathsf{a} \in (0,1]$. Support of fuzzy set $\widetilde{\mathsf{n}}$ (supp $\widetilde{\mathsf{n}}$) is a collection of all real numbers $\boldsymbol{v}$ such that $\widetilde{\mathsf{n}}(\boldsymbol{v})$ is greater than zero, i.e. $\{\boldsymbol{v} \in \mathsf{R} : \widetilde{\mathsf{n}}(\boldsymbol{v}) > 0\}$. $[\widetilde{\mathsf{n}}]^0$ denotes a closure of support $\widetilde{\mathsf{n}}$.

Definition 2.1. *A fuzzy number $\widetilde{\mathsf{n}}$ on R is a fuzzy set with following properties:*

1. $\widetilde{\mathsf{n}}$ is normal (there must exist $\widehat{\boldsymbol{v}} \in \mathsf{R}$ such that $\widetilde{\mathsf{n}}(\widehat{\boldsymbol{v}}) = 1$);

2. $\widetilde{\mathsf{n}}$ is convex ($\widetilde{\mathsf{n}}(\lambda \boldsymbol{v} + (1-\lambda)\boldsymbol{\omega}) \geq \min\{\widetilde{\mathsf{n}}(\boldsymbol{v}), \widetilde{\mathsf{n}}(\boldsymbol{\omega})\}$, for all $\boldsymbol{v}, \boldsymbol{\omega} \in \mathsf{R}$ and $\lambda \in [0,1]$);

3. $\widetilde{\mathsf{n}}$ is an upper semi continuous function;

4. $[\widetilde{\mathsf{n}}]^0$ is compact.

Let $\mathsf{R}_{\widetilde{F}}$ be a family of all fuzzy numbers on R. a-levels of fuzzy number $\widetilde{\mathsf{n}}$ is defined as $[\widetilde{\mathsf{n}}]^{\mathsf{a}} = [\underline{\mathsf{n}}_{\mathsf{a}}, \overline{\mathsf{n}}_{\mathsf{a}}]$, $\underline{\mathsf{n}}_{\mathsf{a}}, \overline{\mathsf{n}}_{\mathsf{a}} \in \mathsf{R}$ for all $a \in [0,1]$. So, for any $\widetilde{\mathsf{n}} \in \mathsf{R}_{\widetilde{F}}$, $[\widetilde{\mathsf{n}}]^{\mathsf{a}} \in \mathsf{R}_I$ for all $a \in [0,1]$. A bounded closed interval $\mathsf{B} = [\mathsf{B}_l, \mathsf{B}_u]$ in R is a special case of fuzzy number encoded as

$$\widetilde{\mathsf{B}}(v) = \begin{cases} 1 & \text{if } v \in [\mathsf{B}_l, \mathsf{B}_u], \\ 0 & \text{if } v \notin [\mathsf{B}_l, \mathsf{B}_u]. \end{cases}$$

Also note that a real number "b" is a special case of fuzzy number encoded as

$$\widetilde{\mathsf{b}}(v) = \begin{cases} 1 & \text{if } v = \mathsf{b}, \\ 0 & \text{if } \underline{v} \neq \mathsf{b}. \end{cases}$$

For fuzzy numbers $\widetilde{\mathsf{n}}, \widetilde{\mathsf{v}} \in \mathsf{R}_{\widetilde{\mathbf{F}}}$, with a-levels representation $[\underline{\mathsf{n}}_{\mathsf{a}}, \overline{\mathsf{n}}_{\mathsf{a}}]$ and $[\underline{\mathsf{v}}_{\mathsf{a}}, \overline{\mathsf{v}}_{\mathsf{a}}]$, respectively, and for real number λ, the addition $\widetilde{\mathsf{n}} + \widetilde{\mathsf{v}}$ and scalar multiplication $\lambda\widetilde{\mathsf{n}}$ are defined as follows:

$$(\widetilde{\mathsf{n}} + \widetilde{\mathsf{v}})(\boldsymbol{v}) = \sup_{\boldsymbol{\omega}+\sigma=\boldsymbol{v}} \min\{\widetilde{\mathsf{n}}(\boldsymbol{\omega}), \widetilde{\mathsf{v}}(\sigma)\}$$

and

$$(\lambda\widetilde{\mathsf{n}})(\boldsymbol{v}) = \begin{cases} \widetilde{\mathsf{n}}\left(\frac{\boldsymbol{v}}{\lambda}\right) & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0. \end{cases}$$

For every $a \in [0,1]$

$$[\widetilde{\mathsf{n}} + \widetilde{\mathsf{v}}]^{\mathsf{a}} = [\underline{\mathsf{u}}_{\mathsf{a}} + \underline{\mathsf{v}}_{\mathsf{a}}, \overline{\mathsf{u}}_{\mathsf{a}} + \overline{\mathsf{v}}_{\mathsf{a}}],$$

and

$$[\lambda \widetilde{\mathsf{n}}]^{\mathsf{a}} = [\min\{\lambda \underline{\mathsf{n}}_{\mathsf{a}}, \lambda \overline{\mathsf{n}}_{\mathsf{a}}\}, \max\{\lambda \underline{\mathsf{n}}_{\mathsf{a}}, \lambda \overline{\mathsf{n}}_{\mathsf{a}}\}].$$

The Pompeiu-Housdorff metric on $\mathsf{R}_{\widetilde{F}}$ is defined as

$$\boldsymbol{D}(\widetilde{\mathsf{n}},\widetilde{\mathsf{v}}) = \sup_{\mathsf{a}\in[0,1]} \max\{|\underline{\mathsf{n}}_{\mathsf{a}} - \underline{\mathsf{n}}_{\mathsf{a}}|, |\overline{\mathsf{n}}_{\mathsf{a}} - \overline{\mathsf{n}}_{\mathsf{a}}|\}.$$

$(\mathsf{R}_{\widetilde{\mathbf{F}}}, \boldsymbol{D})$ is a complete metric space [5].

Usual order for fuzzy numbers is defined in [13, 14, 21, 13].

Definition 2.2. *For* $\widetilde{\mathsf{n}}, \widetilde{\mathsf{v}} \in \mathsf{R}_{\widetilde{\mathbf{F}}}$*,it is said that*
$\widetilde{\mathsf{n}} \precapprox \widetilde{\mathsf{v}}$*, if for every* $a \in [0,1]$, $\underline{\mathsf{n}}_{\mathsf{a}} \leq \underline{\mathsf{v}}_{\mathsf{a}}$ *and* $\overline{\mathsf{n}}_{\mathsf{a}} \leq \overline{\mathsf{v}}_{\mathsf{a}}$.
$\widetilde{\mathsf{n}} \preceq \widetilde{\mathsf{v}}$*, if* $\widetilde{\mathsf{n}} \precapprox \widetilde{\mathsf{v}}$ *and* $\exists a_0 \in [0,1]$*, such that* $\underline{\mathsf{n}}_{\mathsf{a}_0} < \underline{\mathsf{v}}_{\mathsf{a}_0}$ *or* $\overline{\mathsf{n}}_{\mathsf{a}_0} < \overline{\mathsf{v}}_{\mathsf{a}_0}$.
$\widetilde{\mathsf{n}} \prec \widetilde{\mathsf{v}}$ *if* $\widetilde{\mathsf{n}} \precapprox \widetilde{\mathsf{v}}$ *and* $\exists a_0 \in [0,1]$*, such that* $\underline{\mathsf{n}}_{\mathsf{a}_0} < \underline{\mathsf{v}}_{\mathsf{a}_0}$ *and* $\overline{\mathsf{n}}_{\mathsf{a}_0} < \overline{\mathsf{v}}_{\mathsf{a}_0}$.

Remark 2.3. *Partial order relation on* $\mathsf{R}_{\widetilde{\mathbf{F}}}$ *is "* $\precapprox$ *". If* $\widetilde{\mathsf{n}} \prec \widetilde{\mathsf{v}}$ *then* $\widetilde{\mathsf{n}} \preceq \widetilde{\mathsf{v}}$ *and then* $\widetilde{\mathsf{n}} \precapprox \widetilde{\mathsf{v}}$.

Definition 2.4. *The Hukuhara difference of two fuzzy numbers* $\widetilde{\mathsf{n}}$ *and* $\widetilde{\mathsf{v}}$ *exists if* $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{F}}$ *such that*

$$\widetilde{\mathsf{n}} \ominus_{\mathsf{H}} \widetilde{\mathsf{v}} = \widetilde{\mathsf{w}} \Longleftrightarrow \widetilde{\mathsf{n}} = \widetilde{\mathsf{v}} + \widetilde{\mathsf{w}}.$$

Definition 2.5. *Generalized Hukuhara difference of two fuzzy numbers* $\widetilde{\mathsf{n}}$ *and* $\widetilde{\mathsf{v}}$ *exists if* $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{F}}$ *such that*

$$\widetilde{\mathsf{n}} \ominus_{g\mathsf{H}} \widetilde{\mathsf{v}} = \widetilde{\mathsf{w}} \Longleftrightarrow \begin{cases} \widetilde{\mathsf{n}} = \widetilde{\mathsf{v}} + \widetilde{\mathsf{w}} \\ \text{or } \widetilde{\mathsf{v}} = \widetilde{\mathsf{n}} + (-1)\,\widetilde{\mathsf{w}}. \end{cases}$$

For any two fuzzy numbers $\widetilde{\mathsf{n}}$ and $\widetilde{\mathsf{v}}$, $\widetilde{\mathsf{n}} \ominus \widetilde{\mathsf{v}}$ is called first Hukuhara difference or H^+-difference of $\widetilde{\mathsf{n}}$ and $\widetilde{\mathsf{v}}$ if there exist $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{F}}$ such that $\widetilde{\mathsf{n}} = \widetilde{\mathsf{v}} + \widetilde{\mathsf{w}}$ and $\widetilde{\mathsf{n}} \boxminus \widetilde{\mathsf{v}}$ is called second Hukuhara difference or H^--difference of $\widetilde{\mathsf{n}}$ and $\widetilde{\mathsf{v}}$ if there exist $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{F}}$ such that $\widetilde{\mathsf{v}} = \widetilde{\mathsf{n}} + (-1)\,\widetilde{\mathsf{w}}$. This means that $\widetilde{\mathsf{n}} \ominus_{g\mathsf{H}} \widetilde{\mathsf{v}} = \widetilde{\mathsf{n}} \ominus \widetilde{\mathsf{v}}$ if $\widetilde{\mathsf{n}} = \widetilde{\mathsf{v}} + \widetilde{\mathsf{w}}$ for some $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{F}}$ and $\widetilde{\mathsf{n}} \ominus_{gH} \widetilde{\mathsf{v}} = \widetilde{\mathsf{n}} \boxminus \widetilde{\mathsf{v}}$ if $\widetilde{\mathsf{v}} = \widetilde{\mathsf{n}} + (-1)\,\widetilde{\mathsf{w}}$ for some $\widetilde{\mathsf{w}} \in \mathsf{R}_{\widetilde{\mathbf{F}}}$.

Definition 2.6. *Let* $\widetilde{\Psi} : [a,b] \longrightarrow \mathsf{R}_{\widetilde{F}}$ *be a fuzzy function.* $\widetilde{\Psi}$ *is* H*-differentiable at* $\underline{\mathrm{t}}_0 \in (\boldsymbol{a}, \boldsymbol{b})$ *if there exists a fuzzy number* $\widetilde{B}$ *such that:*

$$\lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0)}{\underline{\mathrm{h}}} = \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}})}{\underline{\mathrm{h}}} = \widetilde{B}.$$

Then $\widetilde{\Psi}$ *is Hukuhara differentiable at* $\underline{\mathrm{t}}_0$*and* $\widetilde{B}$ *is Hukuhara derivative at of* $\widetilde{\Psi}$ *at* $\underline{\mathrm{t}}_0$.

Definition 2.7. *Let* $\widetilde{\Psi} : [a,b] \longrightarrow \mathsf{R}_{\widetilde{F}}$ *be a fuzzy function.* $\widetilde{\Psi}$ *is* $\mathbf{G}\mathsf{H}$-*differentiable at* $\underline{\mathrm{t}}_0 \in (a,b)$ *if there exists a fuzzy number* $\widetilde{B}$ *such that:*

$$
\begin{aligned}
i)\ & \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0+\underline{\mathrm{h}}) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0)}{\underline{\mathrm{h}}} &=& \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0-\underline{\mathrm{h}})}{\underline{\mathrm{h}}} = \widetilde{B} \text{ or}\\
ii)\ & \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0+\underline{\mathrm{h}})}{-\underline{\mathrm{h}}} &=& \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0-\underline{\mathrm{h}}) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0)}{-\underline{\mathrm{h}}} = \widetilde{B} \text{ or}\\
iii)\ & \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0+\underline{\mathrm{h}}) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0)}{\underline{\mathrm{h}}} &=& \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0-\underline{\mathrm{h}}) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0)}{-\underline{\mathrm{h}}} = \widetilde{B} \text{ or}\\
iv)\ & \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0+\underline{\mathrm{h}})}{-\underline{\mathrm{h}}} &=& \lim_{\underline{\mathrm{h}}\to 0^+} \frac{\widetilde{\Psi}(\underline{\mathrm{t}}_0) \ominus_{\mathsf{H}} \widetilde{\Psi}(\underline{\mathrm{t}}_0-\underline{\mathrm{h}})}{\underline{\mathrm{h}}} = \widetilde{B}\ .
\end{aligned}
$$

Then $\widetilde{\Psi}$ is Hukuhara differentiable at $\underline{\mathrm{t}}_0$and $\widetilde{B}$ is strictly generalized Hukuhara derivative at of $\widetilde{\Psi}$ at $\underline{\mathrm{t}}_0$.

Note that the **G**H-differentiability in the first form (i) of the Definition [2] coincides with the H-differentiability Thus, **G**H-differentiability is a more general than the H-differentiability notion for fuzzy mappings.

H-derivative is one of the first derivatives for fuzzy-valued functions, based on the Hukuhara difference of intervals [11]. There are some drawbacks of H-derivative:

It exists only under very restrictive conditions. If a fuzzy function Ψ is H-differentiable, then it presents a non-decreasing diameter, and the function has non-decreasing fuzziness.

To overcome this difficulty, Bede et al. [1] introduced the concept of a strongly generalized Hukuhara derivative (**G**H-derivative). The class of **G**H-differentiable fuzzy functions is more general than that of H-differentiable fuzzy functions. Recently, Bede et al. rigorously study the strongly generalized Hukuhara differentiable (**G**H-differentiable) fuzzy functions [2], and obtain sufficient conditions for the **G**H-differentiability fuzzy functions.

However, some fuzzy functions are not **G**H-differentiable. This restriction is because of the use of the H-difference in the definition of the **G**H-derivative. A more general concept of differentiability is obtained if we use a less restrictive difference definition. Luplescu et al. introduce a new derivative of fuzzy-valued functions [12], being more general than the H-derivative and $\mathbf{G}\mathsf{H}$-derivative, in the sense that the latter is strictly contained in the former.

Definition 2.8. *Let $\widetilde{\Psi} : [a,b] \longrightarrow \mathsf{R}_{\widetilde{F}}$ be a fuzzy function. $\widetilde{\Psi}$ is left differentiable at $\underline{\mathrm{t}}_0 \in (a,b]$ if there exists a fuzzy number $\widetilde{B}$ such that:*

$$\lim_{\underline{\mathrm{h}} \longrightarrow 0^+} \frac{1}{\underline{\mathrm{h}}} \mathrm{H}(\widetilde{\Psi}(\underline{\mathrm{t}}_0), \widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) + \underline{\mathrm{h}}\widetilde{B}) = 0,$$

or

$$\lim_{\underline{\mathrm{h}} \longrightarrow 0^+} \frac{1}{\underline{\mathrm{h}}} \mathrm{H}(\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}), \widetilde{\Psi}(\underline{\mathrm{t}}_0) - \underline{\mathrm{h}}\widetilde{B}) = 0.$$

$\widetilde{B}$ is called left derivative of function $\widetilde{\Psi}$ at $\underline{\mathrm{t}}_0$ and it is denoted by $\widetilde{\Psi}'_-(\underline{\mathrm{t}}_0)$. $\widetilde{\Psi}$ is said to be left differentiable on $(a,b]$, if $\widetilde{\Psi}$ is left differentiable at each $\underline{\mathrm{t}}_0 \in (a,b]$.

Definition 2.9. *Let $\widetilde{\Psi} : [a,b] \longrightarrow \mathsf{R}_{\widetilde{F}}$ be a fuzzy function. $\widetilde{\Psi}$ is right differentiable at $\underline{\mathrm{t}}_0 \in [a,b)$ if there exists a fuzzy number $\widetilde{B}$ such that:*

$$\lim_{\underline{\mathrm{h}} \longrightarrow 0^+} \frac{1}{\underline{\mathrm{h}}} \mathrm{H}(\widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}), \widetilde{\Psi}(\underline{\mathrm{t}}_0) + \underline{\mathrm{h}}\widetilde{B}) = 0,$$

or

$$\lim_{\underline{\mathrm{h}} \longrightarrow 0^+} \frac{1}{\underline{\mathrm{h}}} \mathrm{H}(\widetilde{\Psi}(\underline{\mathrm{t}}_0), \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) - \underline{\mathrm{h}}\widetilde{B}) = 0.$$

$\widetilde{B}$ is called right derivative of function $\widetilde{\Psi}$ at $\underline{\mathrm{t}}_0$ and it is denoted by $\widetilde{\Psi}'_+(\underline{\mathrm{t}}_0)$. $\widetilde{\Psi}$ is said to be right differentiable on $[a,b)$, if $\widetilde{\Psi}$ is right differentiable at each $\underline{\mathrm{t}}_0 \in [a,b)$.

Definition 2.10. *$\widetilde{\Psi}$ is differentiable at $\underline{\mathrm{t}}_0 \in [a,b]$ if $\widetilde{\Psi}$ is left and right differetiable at $\underline{\mathrm{t}}_0$ and $\widetilde{\Psi}'_-(\underline{\mathrm{t}}_0) = \widetilde{\Psi}'_+(\underline{\mathrm{t}}_0)$.*

Theorem 2.11. *If $\widetilde{\Psi} : [a,b] \longrightarrow \mathsf{R}_{\widetilde{F}}$ is H-differentiable at $\underline{\mathrm{t}}_0 \in (a,b)$, then $\widetilde{\Psi}$ is differentiable at $\underline{\mathrm{t}}_0$ and $\boldsymbol{D}_{\mathsf{H}}\widetilde{\Psi}(\underline{\mathrm{t}}_0) = \widetilde{\Psi}'(\underline{\mathrm{t}}_0)$.*

Note that converse of above theorem is not true in general.

3 Optimality Conditions

3.1 Necessity

Hereafter consider $\boldsymbol{D}$ be an open subset of R and $\widetilde{\Psi} : \boldsymbol{D} \subseteq \mathsf{R} \longrightarrow \mathsf{R}_{\widetilde{\mathcal{F}}}$ be a fuzzy-valued function and suppose that it is differentible on $\boldsymbol{D}$. We introduce the following definitions for $\widetilde{\Psi}$.

Definition 3.1. *Let* $\underline{\mathrm{t}}_0 \in \boldsymbol{D}$.

1. $\underline{\mathrm{t}}_0$ *is local weak minima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{\mathrm{h}} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succapprox 0,\\ \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succapprox 0,\end{aligned}$$

or

$$\begin{aligned}\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) \boxminus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succapprox 0,\\ \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \boxminus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succapprox 0.\end{aligned}$$

$\widetilde{\Psi}(\underline{\mathrm{t}}_0)$ *is called local weak minimum of fuzzy function* $\widetilde{\Psi}$.

2. $\underline{\mathrm{t}}_0$ *is local minima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{\mathrm{h}} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succeq 0,\\ \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succeq 0,\end{aligned}$$

or

$$\begin{aligned}\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) \boxminus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succeq 0,\\ \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \boxminus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succeq 0.\end{aligned}$$

$\widetilde{\Psi}(\underline{\mathrm{t}}_0)$ *is called local minimum of fuzzy function* $\widetilde{\Psi}$.

3. $\underline{\mathrm{t}}_0$ *is local strict minima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{\mathrm{h}} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{\mathrm{t}}_0 - \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succ 0,\\ \widetilde{\Psi}(\underline{\mathrm{t}}_0 + \underline{\mathrm{h}}) \ominus \widetilde{\Psi}(\underline{\mathrm{t}}_0) &\succ 0,\end{aligned}$$

or

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0-\underline{h})\boxminus\widetilde{\Psi}(\underline{t}_0) &\succ 0,\\ \widetilde{\Psi}(\underline{t}_0+\underline{h})\boxminus\widetilde{\Psi}(\underline{t}_0) &\succ 0.\end{aligned}$$

$\widetilde{\Psi}(\underline{t}_0)$ *is called local strict minimum of fuzzy function* $\widetilde{\Psi}$.

Definition 3.2. *Let* $\underline{t}_0 \in \boldsymbol{D}$.

1. $\underline{t}_0$ *is local weak maxima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{h} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0-\underline{h}) &\gtrapprox 0,\\ \widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0+\underline{h}) &\gtrapprox 0,\end{aligned}$$

or

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0)\boxminus\widetilde{\Psi}(\underline{t}_0-\underline{h}) &\gtrapprox 0,\\ \widetilde{\Psi}(\underline{t}_0)\boxminus\widetilde{\Psi}(\underline{t}_0+\underline{h}) &\gtrapprox 0.\end{aligned}$$

$\widetilde{\Psi}(\underline{t}_0)$ *is called local weak maximum of fuzzy function* $\widetilde{\Psi}$.

2. $\underline{t}_0$ *is local maxima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{h} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0-\underline{h}) &\succeq 0,\\ \widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0+\underline{h}) &\succeq 0,\end{aligned}$$

or

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0)\boxminus\widetilde{\Psi}(\underline{t}_0-\underline{h}) &\succeq 0,\\ \widetilde{\Psi}(\underline{t}_0)\boxminus\widetilde{\Psi}(\underline{t}_0+\underline{h}) &\succeq 0.\end{aligned}$$

$\widetilde{\Psi}(\underline{t}_0)$ *is called local maximum of fuzzy function* $\widetilde{\Psi}$.

3. $\underline{t}_0$ *is local strict maxima for* $\widetilde{\Psi}$ *if there exist* $\delta > 0$ *such that for all* $0 < \underline{h} < \delta$, *we have*

$$\begin{aligned}\widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0-\underline{h}) &\succ 0,\\ \widetilde{\Psi}(\underline{t}_0)\ominus\widetilde{\Psi}(\underline{t}_0+\underline{h}) &\succ 0,\end{aligned}$$

or

$$\widetilde{\Psi}(\underline{t}_0) \boxminus \widetilde{\Psi}(\underline{t}_0 - \underline{h}) \succ 0,$$
$$\widetilde{\Psi}(\underline{t}_0) \boxminus \widetilde{\Psi}(\underline{t}_0 + \underline{h}) \succ 0.$$

$\widetilde{\Psi}(\underline{t}_0)$ *is called local stricat maximum of fuzzy function* $\widetilde{\Psi}$.

Lemma 3.3. *If* $\underline{t}_0$ *is a local strict minima for a fuzzy-valued function* $\widetilde{\Psi}$, *then* $\underline{t}_0$ *is local minima for* $\widetilde{\Psi}$, *and* $\underline{t}_0$ *is also local weak minimum for* $\widetilde{\Psi}$.

Proof. Let $\underline{t}_0$ be a local strict minima for a fuzzy-valued function $\widetilde{\Psi}$. By Remark 2.3, $\underline{t}_0$ is local minima for $\widetilde{\Psi}$, and $\underline{t}_0$ is a local weak minimum for $\widetilde{\Psi}$.
□

Note that the above definitions of maxima and minima for fuzzy-valued function generalized the concepts of maxima and minima for a real-valued function.

Proposition 3.4. *Let* $\digamma : \boldsymbol{D} \subseteq \mathsf{R} \longrightarrow \mathsf{R}$ *be real-valued function and consider the fuzzy-valued function* $\widetilde{\Psi} : \boldsymbol{D} \subseteq \mathsf{R} \longrightarrow \mathsf{R}_{\widetilde{\mathbf{F}}}$, *defined by* $\widetilde{\Psi}(\boldsymbol{v}) = \digamma(\boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{D}$. *Then* $\underline{t}_0$ *is a local strict minima for* $\digamma$ *iff* $\underline{t}_0$ *is a local strict minima for* $\widetilde{\Psi}$.

Proof. For all $a \in [0,1]$, we have $\left[\widetilde{\Psi}(\boldsymbol{v})\right]^{\mathsf{a}} = \{\digamma(\boldsymbol{v})\}$, singleton. Then result is immediate. □

Remark 3.5. *If* $\digamma : \boldsymbol{D} \subseteq \mathsf{R} \longrightarrow \mathsf{R}$ *is a real-valued function and the fuzzy-valued function* $\widetilde{\Psi} : \boldsymbol{D} \subseteq \mathsf{R} \longrightarrow \mathsf{R}_{\widetilde{\mathbf{F}}}$, *defined by* $\widetilde{\Psi}(\boldsymbol{v}) = \digamma(\boldsymbol{v})$ *for all* $\boldsymbol{v} \in \boldsymbol{D}$. *Then:*

1. $\underline{t}_0$ *is local minima for* $\widetilde{\Psi}$ *iff* $\underline{t}_0$ *is local minima for* $\digamma(\boldsymbol{v})$.
2. $\underline{t}_0$ *is local weak minima for* $\widetilde{\Psi}$ *iff* $\underline{t}_0$ *is local weak minima for* $\digamma(\boldsymbol{v})$.

Theorem 3.6. *Suppose that fuzzy-valued function* $\widetilde{\Psi} : \boldsymbol{D} \subseteq \mathbf{R} \longrightarrow \mathsf{R}_{\widetilde{\digamma}}$ *has a local weak minimum at an interior point* $\underline{t}_0 \in \boldsymbol{D}$ *and* $\widetilde{\Psi}$ *is differentiable at* $\underline{t}_0$. *Then deravetive of* $\widetilde{\Psi}$ *at* $\underline{t}_0$ *is zero.*

Proof. Let fuzzy function $\widetilde{\Psi}$ has a local weak minimum at an interior point $\underline{t}_0 \in \boldsymbol{D}$ and $\widetilde{\Psi}$ is differentiable at $\underline{t}_0$. This implies there exist $\delta > 0$ such that for all $0 < \underline{h} < \delta$, we have

$$\begin{aligned} \widetilde{\Psi}(\underline{t}_0 - \underline{h}) \ominus \widetilde{\Psi}(\underline{t}_0) &\succapprox 0, \\ \widetilde{\Psi}(\underline{t}_0 + \underline{h}) \ominus \widetilde{\Psi}(\underline{t}_0) &\succapprox 0, \end{aligned}$$

or

$$\begin{aligned} \widetilde{\Psi}(\underline{t}_0 - \underline{h}) \boxminus \widetilde{\Psi}(\underline{t}_0) &\succapprox 0, \\ \widetilde{\Psi}(\underline{t}_0 + \underline{h}) \boxminus \widetilde{\Psi}(\underline{t}_0) &\succapprox 0. \end{aligned}$$

Then for all $\alpha \in [0,1]$, we have

$$\begin{aligned} \widetilde{\Psi}^-_\alpha(\underline{t}_0 - \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{t}_0) &\geq 0, \\ \widetilde{\Psi}^+_\alpha(\underline{t}_0 - \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{t}_0) &\geq 0, \\ \widetilde{\Psi}^-_\alpha(\underline{t}_0 + \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{t}_0) &\geq 0, \\ \widetilde{\Psi}^+_\alpha(\underline{t}_0 + \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{t}_0) &\geq 0. \end{aligned}$$

As $\widetilde{\Psi}$ is differentiable at $\underline{t}_0$. This implies

$$\lim_{\underline{h} \longrightarrow 0^+} \frac{1}{\underline{h}} H(\widetilde{\Psi}(\underline{t}_0 + \underline{h}), \widetilde{\Psi}(\underline{t}_0) + \underline{h}\widetilde{B}) = \lim_{\underline{h} \longrightarrow 0^+} \frac{1}{\underline{h}} H(\widetilde{\Psi}(\underline{t}_0), \widetilde{\Psi}(\underline{t}_0 - \underline{h}) + \underline{h}\widetilde{B}) = 0, \tag{1}$$

or

$$\lim_{\underline{h} \longrightarrow 0^+} \frac{1}{\underline{h}} H(\widetilde{\Psi}(\underline{t}_0), \widetilde{\Psi}(\underline{t}_0 + \underline{h}) - \underline{h}\widetilde{B}) = \lim_{\underline{h} \longrightarrow 0^+} \frac{1}{\underline{h}} H(\widetilde{\Psi}(\underline{t}_0 - \underline{h}), \widetilde{\Psi}(\underline{t}_0) - \underline{h}\widetilde{B}) = 0. \tag{2}$$

If equation (1) holds. Then Pompeiu-Housdorff metric on $\mathsf{R}_{\widetilde{F}}$ gives

$$\begin{aligned} &\lim_{\underline{h} \longrightarrow 0^+} \tfrac{1}{\underline{h}} \sup_{\alpha \in [0,1]} \max \Big\{ \Big| \widetilde{\Psi}^-_\alpha(\underline{t}_0 + \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{t}_0) - \underline{h}\widetilde{B}^-_\alpha \Big|, \\ &\qquad \Big| \widetilde{\Psi}^+_\alpha(\underline{t}_0 + \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{t}_0) - \underline{h}\widetilde{B}^+_\alpha \Big| = 0, \\ &\lim_{\underline{h} \longrightarrow 0^+} \tfrac{1}{\underline{h}} \sup_{\alpha \in [0,1]} \max \Big\{ \Big| \widetilde{\Psi}^-_\alpha(\underline{t}_0) - \widetilde{\Psi}^-_\alpha(\underline{t}_0 - \underline{h}) - \underline{h}\widetilde{B}^-_\alpha \Big|, \\ &\qquad \Big| \widetilde{\Psi}^+_\alpha(\underline{t}_0) - \widetilde{\Psi}^+_\alpha(\underline{t}_0 - \underline{h}) - \underline{h}\widetilde{B}^+_\alpha \Big| = 0. \end{aligned} \tag{3}$$

Since for all $\alpha \in [0,1]$, we have

$$\begin{aligned} \widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0) &\geq 0, \\ \widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0) &\geq 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} \widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{-} &\geq -\underline{h}\widetilde{B}_{\alpha}^{-}, \\ \widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{+} &\geq -\underline{h}\widetilde{B}_{\alpha}^{+}. \end{aligned} \tag{4}$$

Equation (3) and relation (4) implies:

$$\begin{aligned} 0 &= \lim_{\underline{h}\longrightarrow 0^{+}} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\left|\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{-}\right|, \\ &\left|\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{+}\right|\} \\ &\geq \lim_{\underline{h}\longrightarrow 0^{+}} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{-}, \\ &\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{+}\} \\ &\geq \lim_{\underline{h}\longrightarrow 0^{+}} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{-}, \\ &\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0+\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)-\underline{h}\widetilde{B}_{\alpha}^{+}\} \\ &\geq \lim_{\underline{h}\longrightarrow 0^{+}} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\left\{-\underline{h}\widetilde{B}_{\alpha}^{-}, -\underline{h}\widetilde{B}_{\alpha}^{+}\right\} \\ &= \sup_{\alpha\in[0,1]} \max\left\{-\widetilde{B}_{\alpha}^{-}, -\widetilde{B}_{\alpha}^{+}\right\}. \end{aligned}$$

This implies $-\widetilde{B}_{\alpha}^{-} \leq 0$ and $-\widetilde{B}_{\alpha}^{+} \leq 0$ for all $\alpha \in [0,1]$. It shows $\widetilde{B}_{\alpha}^{-} \geq 0$ and $\widetilde{B}_{\alpha}^{+} \geq 0$ for all $\alpha \in [0,1]$.

Now since for all $\alpha \in [0,1]$, we have

$$\begin{aligned} \widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0) &\geq 0, \\ \widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0) &\geq 0. \end{aligned}$$

Equivalently,

$$\begin{aligned} \widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{-} &\geq \underline{h}\widetilde{B}_{\alpha}^{-}, \\ \widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{+} &\geq \underline{h}\widetilde{B}_{\alpha}^{+}. \end{aligned} \tag{5}$$

Equation (3) and (5) implies:

$$\begin{aligned}
0 &= \lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\{\left|\widetilde{\Psi}^-_\alpha(\underline{t}_0)-\widetilde{\Psi}^-_\alpha(\underline{t}_0-\underline{h})-\underline{h}\widetilde{B}^-_\alpha\right|,\\
&\quad\left|\widetilde{\Psi}^+_\alpha(\underline{t}_0)-\widetilde{\Psi}^+_\alpha(\underline{t}_0-\underline{h})-\underline{h}\widetilde{B}^+_\alpha\right|\}\\
&= \lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\{\left|\widetilde{\Psi}^-_\alpha(\underline{t}_0-\underline{h})-\widetilde{\Psi}^-_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^-_\alpha\right|,\\
&\quad\left|\widetilde{\Psi}^+_\alpha(\underline{t}_0-\underline{h})-\widetilde{\Psi}^+_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^+_\alpha\right|\}\\
&\geq \lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\{\widetilde{\Psi}^-_\alpha(\underline{t}_0-\underline{h})-\widetilde{\Psi}^-_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^-_\alpha,\\
&\quad\widetilde{\Psi}^+_\alpha(\underline{t}_0+\underline{h})-\widetilde{\Psi}^+_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^+_\alpha\}\\
&\geq \lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\left\{\underline{h}\widetilde{B}^-_\alpha,\underline{h}\widetilde{B}^+_\alpha\right\}\\
&= \sup_{\alpha\in[0,1]}\max\left\{\widetilde{B}^-_\alpha,\widetilde{B}^+_\alpha\right\}.
\end{aligned}$$

This implies $\widetilde{B}^-_\alpha\leq 0$ and $\widetilde{B}^+_\alpha\leq 0$ for all $\alpha\in[0,1]$. Hence $\widetilde{B}^-_\alpha=0$ and $\widetilde{B}^+_\alpha=0$ for all $\alpha\in[0,1]$. Equivalently $\widetilde{B}=0$.

If equation (2) holds. Then Pompeiu-Housdorff metric on $\mathsf{R}_{\widetilde{F}}$ gives

$$\begin{aligned}
&\lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\{\left|\widetilde{\Psi}^-_\alpha(\underline{t}_0)-\widetilde{\Psi}^-_\alpha(\underline{t}_0+\underline{h})+\underline{h}\widetilde{B}^-_\alpha\right|,\\
&\qquad\left|\widetilde{\Psi}^+_\alpha(\underline{t}_0)-\widetilde{\Psi}^+_\alpha(\underline{t}_0+\underline{h})+\underline{h}\widetilde{B}^+_\alpha\right|\}=0,\\
&\lim_{\underline{h}\longrightarrow 0^+}\frac{1}{\underline{h}}\sup_{\alpha\in[0,1]}\max\{\left|\widetilde{\Psi}^-_\alpha(\underline{t}_0-\underline{h})-\widetilde{\Psi}^-_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^-_\alpha\right|,\\
&\qquad\left|\widetilde{\Psi}^+_\alpha(\underline{t}_0-\underline{h})-\widetilde{\Psi}^+_\alpha(\underline{t}_0)+\underline{h}\widetilde{B}^+_\alpha\right|\}=0. \qquad (6)
\end{aligned}$$

Since for all $\alpha\in[0,1]$, we have

$$\widetilde{\Psi}^-_\alpha(\underline{t}_0+\underline{h})-\widetilde{\Psi}^-_\alpha(\underline{t}_0)\geq 0 \text{ and } \widetilde{\Psi}^+_\alpha(\underline{t}_0+\underline{h})-\widetilde{\Psi}^+_\alpha(\underline{t}_0)\geq 0.$$

Equivalently,

$$\begin{aligned}
\widetilde{\Psi}^-_\alpha(\underline{t}_0+\underline{h})-\widetilde{\Psi}^-_\alpha(\underline{t}_0)-\underline{h}\widetilde{B}^-_\alpha &\geq -\underline{h}\widetilde{B}^-_\alpha, \qquad (7)\\
\widetilde{\Psi}^+_\alpha(\underline{t}_0+\underline{h})-\widetilde{\Psi}^+_\alpha(\underline{t}_0)-\underline{h}\widetilde{B}^+_\alpha &\geq -\underline{h}\widetilde{B}^+_\alpha.
\end{aligned}$$

Equation (6) and relation (7) implies:

$$\begin{aligned}
0 &= \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\left|\widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0) - \widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) + \underline{h}\widetilde{B}^-_\alpha\right|, \\
&\left|\widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0) - \widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) + \underline{h}\widetilde{B}^+_\alpha\right|\} \\
&= \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\left|\widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0) - \underline{h}\widetilde{B}^-_\alpha\right|, \\
&\left|\widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0) - \underline{h}\widetilde{B}^+_\alpha\right|\} \\
&\geq \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0) - \underline{h}\widetilde{B}^-_\alpha, \\
&\widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0 + \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0) - \underline{h}\widetilde{B}^+_\alpha\} \\
&\geq \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\left\{-\underline{h}\widetilde{B}^-_\alpha, -\underline{h}\widetilde{B}^+_\alpha\right\} \\
&= \sup_{\alpha\in[0,1]} \max\left\{-\widetilde{B}^-_\alpha, -\widetilde{B}^+_\alpha\right\}.
\end{aligned}$$

This implies $-\widetilde{B}^-_\alpha \leq 0$ and $-\widetilde{B}^+_\alpha \leq 0$ for all $\alpha \in [0,1]$. It indicates $\widetilde{B}^-_\alpha \geq 0$ and $\widetilde{B}^+_\alpha \geq 0$ for all $\alpha \in [0,1]$.

Now since for all $\alpha \in [0,1]$, we have

$$\begin{aligned}
\widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0 - \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0) &\geq 0, \\
\widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0 - \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0) &\geq 0.
\end{aligned}$$

Equivalently,

$$\begin{aligned}
\widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0 - \underline{h}) - \widetilde{\Psi}^-_\alpha(\underline{\mathrm{t}}_0) + \underline{h}\widetilde{B}^-_\alpha &\geq \underline{h}\widetilde{B}^-_\alpha, \\
\widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0 - \underline{h}) - \widetilde{\Psi}^+_\alpha(\underline{\mathrm{t}}_0) + \underline{h}\widetilde{B}^+_\alpha &\geq \underline{h}\widetilde{B}^+_\alpha.
\end{aligned} \tag{8}$$

Equation (6) and relation (8) implies

$$\begin{aligned} 0 &= \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\left|\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{-}\right|, \\ &\quad \left|\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{+}\right|\} \\ &\geq \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\{\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{-}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{-}, \\ &\quad \widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0-\underline{h})-\widetilde{\Psi}_{\alpha}^{+}(\underline{t}_0)+\underline{h}\widetilde{B}_{\alpha}^{+}\} \\ &\geq \lim_{\underline{h}\longrightarrow 0^+} \frac{1}{\underline{h}} \sup_{\alpha\in[0,1]} \max\left\{\underline{h}\widetilde{B}_{\alpha}^{-},\underline{h}\widetilde{B}_{\alpha}^{+}\right\} \\ &= \sup_{\alpha\in[0,1]} \max\left\{\widetilde{B}_{\alpha}^{-},\widetilde{B}_{\alpha}^{+}\right\}. \end{aligned}$$

This implies $\widetilde{B}_{\alpha}^{-} \leq 0$ and $\widetilde{B}_{\alpha}^{+} \leq 0$ for all $\alpha \in [0,1]$ .Hence $\widetilde{B}_{\alpha}^{-} = 0$ and $\widetilde{B}_{\alpha}^{+} = 0$ for all $\alpha \in [0,1]$. Equivalently, $\widetilde{B} = 0$. □

Theorem 3.7. *Suppose that fuzzy function* $\widetilde{\Psi} : \boldsymbol{D} \subseteq R \longrightarrow \mathsf{R}_{\widetilde{F}}$ *has a local maximum at an interior point* $\underline{t}_0 \in \boldsymbol{D}$ *and* $\widetilde{\Psi}$ *is differentiable at* $\underline{t}_0$. *Then deravetive of* $\widetilde{\Psi}$ *at* $\underline{t}_0$ *is zero.*

Proof. It is same as proof of Theorem 3.6. □

Definition 3.8. *Let* $\widetilde{\Psi} : \boldsymbol{D} \subseteq R \longrightarrow \mathsf{R}_{\widetilde{F}}$ *be a fuzzy-valued function and suppose that* $\widetilde{\Psi}$ *is differentiable on* $\boldsymbol{D}$, *an open subset of* R. *If* $0 \in \left[\widetilde{\Psi}'(\underline{t}_0)\right]^{\alpha}$ *for some* $\alpha \in [0,1]$ *and* $\underline{t}_0 \in \boldsymbol{D}$, *then* $\underline{t}_0$ *is called fuzzy stationary point for* $\widetilde{\Psi}$.

Theorem 3.9. *Let* $\widetilde{\Psi} : \boldsymbol{D} \subseteq R \longrightarrow \mathsf{R}_{\widetilde{F}}$ *be a differentiable fuzzy-valued function. If* $\underline{t}_0 \in \boldsymbol{D}$ *is a local weak minima of* $\widetilde{\Psi}$ *then* $0 \in \left[\widetilde{\Psi}'(\underline{t}_0)\right]^{\alpha}$ *for all* $\alpha \in [0,1]$.

Proof. Let $\widetilde{\Psi} : \boldsymbol{D} \subseteq R \longrightarrow \mathsf{R}_{\widetilde{F}}$ be a differentiable fuzzy-valued function and $\underline{t}_0 \in \boldsymbol{D}$ is a local weak minima of $\widetilde{\Psi}$ then by Theorem 3.6, it implies that $\widetilde{\Psi}'(\underline{t}_0) = 0$.Hence $0 \in \left[\widetilde{\Psi}'(\underline{t}_0)\right]^{\alpha}$ for all $\alpha \in [0,1]$. □

Corollary 3.10. *Let* $\widetilde{\Psi} : \boldsymbol{D} \subseteq R \longrightarrow \mathsf{R}_{\widetilde{\mathcal{F}}}$ *be a differentiable fuzzy-valued function. Then* $\underline{\mathrm{t}}_0 \in \boldsymbol{D}$ *is a fuzzy stationary point for* $\widetilde{\Psi}$ *if* $\underline{\mathrm{t}}_0$ *is a local weak minima of* $\widetilde{\Psi}$.

Remark 3.11. *Theorem 3.9 and Corollary 3.10 are also holds for local weak maxima of differentiable fuzzy-valued function.*

Theorem 3.12. $\underline{\mathrm{t}}_0 \in \boldsymbol{D}$ *is a fuzzy stationary point of differentiable fuzzy-valued function* $\widetilde{\Psi}$ *iff* $0 \in \left[\widetilde{\Psi}'(\underline{\mathrm{t}}_0)\right]^0$.

Proof. if $\underline{\mathrm{t}}_0 \in \boldsymbol{D}$ is a fuzzy stationary point of $\widetilde{\Psi}$ then by definition of fuzzy stationary point, there must exists $\widetilde{\mathsf{a}} \in [0,1]$ such that $0 \in \left[\widetilde{\Psi}'(\underline{\mathrm{t}}_0)\right]^{\widetilde{\mathsf{a}}}$, but $\widetilde{\Psi}'(\underline{\mathrm{t}}_0) \in \mathsf{R}_{\widetilde{\mathcal{F}}}$ then $0 \in \left[\widetilde{\Psi}'(\underline{\mathrm{t}}_0)\right]^{\mathsf{a}}$ for all $a < \widetilde{\mathsf{a}}$. Inparticular $0 \in \left[\widetilde{\Psi}'(\underline{\mathrm{t}}_0)\right]^0$. Conversely, If $0 \in \left[\widetilde{\Psi}'(\underline{\mathrm{t}}_0)\right]^0$, then it varifies the definition of stationary point. □

3.1.1 Sufficient

Now we introduce invexity notations for fuzzy mappings, to prove that all stationary points are point of minimum.

Definition 3.13. *A subset* K *of* R *is invex if there exists a function* $\eta : R \times R \longrightarrow R$ *such that*

$$\boldsymbol{\upsilon}, \boldsymbol{\omega} \in K, \lambda \in [0,1] \Longrightarrow \boldsymbol{\omega} + \lambda \eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \in K.$$

Definition 3.14. *Let* $\widetilde{\Psi}$ *be a differentiable fuzzy mapping.* $\widetilde{\Psi}$ *is called weak pseudoinvex on* K, *if for all* $\boldsymbol{\upsilon}, \boldsymbol{\omega} \in K$, *there exists* $\eta : K \times K \longrightarrow R$ *such that*

$$\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\, \eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0,$$

and

$$\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\, \eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0.$$

Remark 3.15. *Weak pseudoinvex fuzzy function is called pseudoinvex and strict pseudoinvex if above conditions holds for* $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \preceq 0$, $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \preceq 0$ *and* $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \precapprox 0$, $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \precapprox 0$ *respectively.*

Example 3.16. Let us consider the fuzzy map $\widetilde{\Psi} : R \to \mathsf{R}_{\widetilde{F}}$ defined by

$$\widetilde{\Psi}(\upsilon) = B\upsilon^2,$$

where B is a fuzzy number and its α-cuts are

$$\begin{aligned} [B]^\alpha &= \left[\underline{B}_\alpha, \overline{B}_\alpha\right] \\ &= [1+\alpha, 3-\alpha]\,; \text{ for all } \alpha \in [0,1]\,. \end{aligned}$$

Clearly, $\widetilde{\Psi}$ is differentiable and

$$\left[\,\widetilde{\Psi}'\,[\upsilon]\right]^\alpha = [(1+\alpha)\,2\upsilon, (3-\alpha)\,2\upsilon]\,; \text{ for all } \alpha \in [0,1]\,.$$

Thus, $\widetilde{\Psi}$ is weak pseudoinvex, since there exist $\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) = \boldsymbol{\omega} - \boldsymbol{\upsilon}$, such that $\widetilde{\Psi}'(\boldsymbol{\upsilon})\,\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0$ whenever $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0$ or $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0$.

Definition 3.17. *Let $\widetilde{\Psi}$ be a differentiable fuzzy mapping. $\widetilde{\Psi}$ is known as pseudoinvex on* K *if for all $\boldsymbol{\upsilon}, \boldsymbol{\omega} \in$ K, there exists $\eta : \mathrm{K} \times \mathrm{K} \longrightarrow R$ such that*

$$\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \preceq 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\,\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0,$$

and

$$\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \preceq 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\,\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0.$$

Definition 3.18. *Let $\widetilde{\Psi}$ be a differentiable fuzzy mapping. $\widetilde{\Psi}$ is called strict pseudoinvex on* K *if for all $\boldsymbol{\upsilon}, \boldsymbol{\omega} \in$ K, there exist $\eta : \mathrm{K} \times \mathrm{K} \longrightarrow R$ such that*

$$\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \precapprox 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\,\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0,$$

and

$$\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \precapprox 0 \Longrightarrow \widetilde{\Psi}'(\boldsymbol{\upsilon})\,\eta(\boldsymbol{\upsilon}, \boldsymbol{\omega}) \prec 0.$$

Lemma 3.19. *If $\widetilde{\Psi}$ is a strict pseudoinvex function then it is pseudoinvex function and also weak pseudoinvex function.*

In following example we will show that these new convexity concepts are generalized.

Example 3.20. Let $\widetilde{\Psi}$ be an interval valued function with domain $[-1, 1]$, defined by

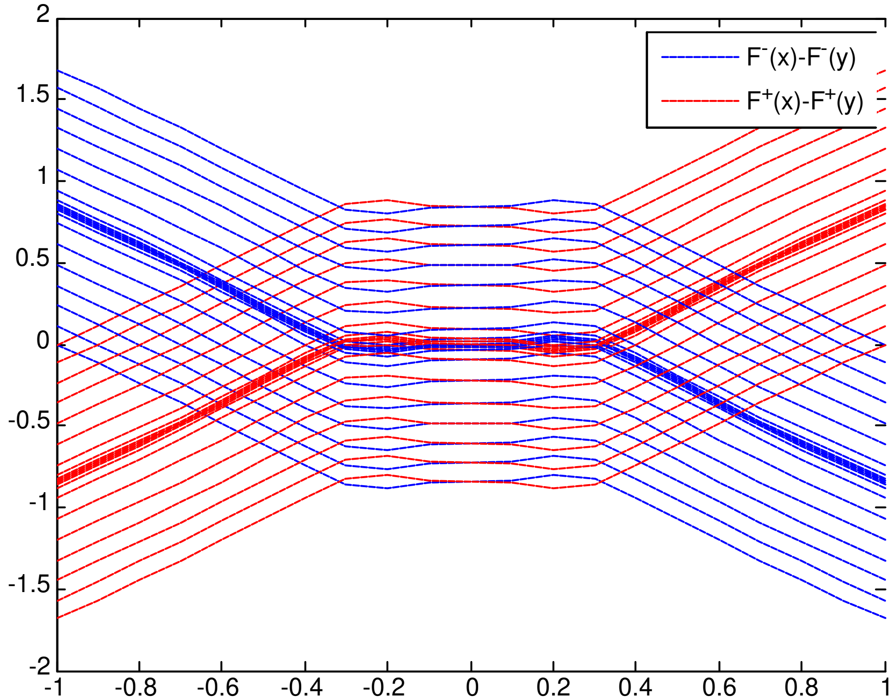

Figure 1

$$\widetilde{\Psi}(\underline{t}) = \begin{cases} \left(1+\underline{t}^2 \sin\left(1/\underline{t}\right)\right).[-1,1] & \text{if } \underline{t} \in [-1,1] \setminus \{0\} \\ [-1,1] & \text{if } \underline{t} = 0. \end{cases}$$

This function is differentiable at $\underline{t}= 0$ and $\widetilde{\Psi}'(0) = \widetilde{0}$, but not GH-differentiable at $\underline{t}= 0$ (see Example 5.37 [12]). It is weak pseudoinvex function since $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0$ and $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon}) \prec 0$ are never true, pictured in Fig. 1.

Now, we relate the above definitions with classical ones [**?**].

Proposition 3.21. *Consider a real valued function* $\mathrm{f} : \mathrm{K} \longrightarrow R$*. The fuzzy function* $\widetilde{\Psi}$ *is defined as* $\widetilde{\Psi}(\boldsymbol{\upsilon}) = \widetilde{\mathrm{f}(\boldsymbol{\upsilon})}$ *for all* $\boldsymbol{\upsilon} \in K$*. Then*

$\widetilde{\Psi}$ *is strict pseudoinvex on* K *iff* f *is strict pseudoinvex on* K.
$\widetilde{\Psi}$ *is pseudoinvex on* K *iff* f *is pseudoinvex on* K.
$\widetilde{\Psi}$ *is weak pseudoinvex on* K *iff* f *is weak pseudoinvex on* K.

Proof. For all $a\in[0,1]$, we have $\left[\widetilde{\Psi}(\boldsymbol{v})\right]^{\mathsf{a}} = \{f(\boldsymbol{v})\}$ a singleton. Then result is immediate. □

Theorem 3.22. *Let* $\widetilde{\Psi}$ *be a weak pseudoinvex function on* K*. Then every stationary point* $\boldsymbol{v}$ *is a weak minimum if* $\widetilde{\Psi}(\boldsymbol{\omega})\ominus\widetilde{\Psi}(\boldsymbol{v})$ *or* $\widetilde{\Psi}(\boldsymbol{\omega})\boxminus\widetilde{\Psi}(\boldsymbol{v})$ *exists for all* $\boldsymbol{\omega}\in$ K.

Proof. Let us consider $\boldsymbol{v}$ is a stationary point but it is not a weak minimum. Then for all $\delta>0$ there exists $0<\underline{h}<\delta$ such that if $\widetilde{\Psi}(\boldsymbol{\omega})\ominus\widetilde{\Psi}(\boldsymbol{v})$ exists for all $\boldsymbol{\omega}\in K$. Then

$$\widetilde{\Psi}(\boldsymbol{v}-\underline{h})\ominus\widetilde{\Psi}(\boldsymbol{v})\prec 0,$$

or

$$\widetilde{\Psi}(\boldsymbol{v}+\underline{h})\ominus\widetilde{\Psi}(\boldsymbol{v})\prec 0.$$

if $\widetilde{\Psi}(\boldsymbol{\omega})\boxminus\widetilde{\Psi}(\boldsymbol{v})$ exists for all $\boldsymbol{\omega}\in K$. Then

$$\widetilde{\Psi}(\boldsymbol{v}-\underline{h})\boxminus\widetilde{\Psi}(\boldsymbol{v})\prec 0,$$

or

$$\widetilde{\Psi}(\boldsymbol{v}+\underline{h})\boxminus\widetilde{\Psi}(\boldsymbol{v})\prec 0.$$

We can say there exists $\boldsymbol{\omega}\in K$ such that

$$\widetilde{\Psi}(\boldsymbol{\omega})\ominus\widetilde{\Psi}(\boldsymbol{v})\prec 0,$$

or

$$\widetilde{\Psi}(\boldsymbol{\omega})\boxminus\widetilde{\Psi}(\boldsymbol{v})\prec 0.$$

This implies

$$\widetilde{\Psi}(\boldsymbol{\omega})\prec\widetilde{\Psi}(\boldsymbol{v}).$$

By hypothesis there exists $\eta(\boldsymbol{v},\boldsymbol{\omega})\neq 0$ such that $\widetilde{\Psi}'(\boldsymbol{v}).\eta(\boldsymbol{v},\boldsymbol{\omega})\prec 0$.

If $\eta(\boldsymbol{v},\boldsymbol{\omega})\succ 0$ then $\left[\widetilde{\Psi}'(\boldsymbol{v})\right]^{\mathsf{a}}\subseteq R^{+}$ for all $a\in[0,1]$ and if $\eta(\boldsymbol{v},\boldsymbol{\omega})\prec 0$ then $\left[\widetilde{\Psi}'(\boldsymbol{v})\right]^{\mathsf{a}}\subseteq R^{-}$ for all $a\in[0,1]$. This implies $0\notin\left[\widetilde{\Psi}'(\boldsymbol{v})\right]^{\mathsf{a}}$ for all $a\in[0,1]$ and so $\boldsymbol{v}$ is not fuzzy stationary point. □

Arguing in the same way the following results can be proved.

Theorem 3.23. *Let* $\widetilde{\Psi}$ *is pseudo-invex on* K. *Then every stationary point* $\boldsymbol{\upsilon}$ *is minimum, if* $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon})$ *or* $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon})$ *exists for all* $\boldsymbol{\omega} \in \mathrm{K}$.

Theorem 3.24. *Let* $\widetilde{\Psi}$ *is pseudo-invex on* K. *Then every stationary point* $\boldsymbol{\upsilon}$ *is strict minimum, if* $\widetilde{\Psi}(\boldsymbol{\omega}) \ominus \widetilde{\Psi}(\boldsymbol{\upsilon})$ *or* $\widetilde{\Psi}(\boldsymbol{\omega}) \boxminus \widetilde{\Psi}(\boldsymbol{\upsilon})$ *exists for all* $\boldsymbol{\omega} \in \mathrm{K}$.

4 Conclusion

In this article, we have considered fuzzy optimization problem based on differentiability concept introduced in [12]. In Theorem 3.6 we have derived necessary optimality conditions. For the sake of sufficient optimality conditions, we have defined new generalized invexity notions for fuzzy-valued functions. Also in Proposition 3.21, we have proved that these invexity concepts generalize the classic ones. Moreover Example 3.20 illustrated that these invexity concepts are more general then all that exist in litrature. Finally, in Theorem 3.22 we have derived sufficient optimality conditions.

In future studies, the authors plan to study new classes of fuzzy optimization problems based on new differentiability concepts and new fuzzy fractional derivatives. Also investigate necessary and sufficient optimality conditions for these new classes of fuzzy optimization problems.

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References

[1] B. Bede, S. G. Gal, Almost periodic fuzzy-number-valued functions, *Fuzzy Sets and Systems*, 147(3)(2004) 385-403.

[2] B. Bede, L. Stefanini, Generalized differentiability of fuzzy-valued functions, *Fuzzy Sets and Systems*, 230(1)(2013) 119-141.

[3] J. J. Buckley, T. Feuring, Introduction to fuzzy partial differential equations, *Fuzzy sets and systems*, 105(2)(1999) 241-248

[4] S. S. Chang, L. A. Zadeh, On fuzzy mapping and control, *In Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh* (1996) 180-184.

[5] P. Diamond, P. Kloeden, Metric spaces of fuzzy sets, *Fuzzy Sets and Systems*, 100 (1999) 63-71.

[6] O. S. Fard, A. H. Borzabadi, M. Heidari, On fuzzy Euler Lagrange equations, *Annals of Fuzzy Mathematics and Informatics*, 7(3) (2014) 447-461.

[7] O. S. Fard, M. S. Zadeh, Note on necessary optimality conditions for fuzzy variational problems, *J. Adv. Res. Dyn. Control Syst*, 4(3)(2012) 1-9.

[8] O. S. Fard, M. Salehi, A survey on fuzzy fractional variational problems, *Journal of Computational and Applied Mathematics*, 271(2014) 71-82.

[9] B. Farhadinia, Necessary optimality conditions for fuzzy variational problems, *Information Sciences*, 181(7)(2011) 1348-1357.

[10] N. Furukawa, Convexity and local Lipschitz continuity of fuzzy-valued mappings, *Fuzzy Sets and Systems*, 93(1)(1998) 113-119.

[11] M. Hukuhara, Integration des applications mesurable dont la valeur est un compact convexe, *Funkcialaj Ekvacioj*, 10(3)(1967) 205-223.

[12] V. Luplescu, D. O'Regan, A new derivative concept for set-valued and fuzzy-valued functions. Differential and integral calculus in quasilinear metric spaces, *Fuzzy Sets and Systems*, 404 (2021): 75-110.

[13] R. Osuna-Gómez, Y. Chalco-Cano, A. án-Lizana, B. Hernández-Jiménez, Necessary and sufficient conditions for fuzzy optimality problems, *Fuzzy Sets and Systems*, 296, (2016) 112-123.

[14] M. Panigrahi, G. Panda, S. Nanda, Convex fuzzy mapping with differentiability and its application in fuzzy optimization, *European Journal of Operational Research*, 185(1)(2008) 47-62.

[15] S. Rashid, M. K. Kaabar, A. Althobaiti, M. S. Alqurashi, Constructing analytical estimates of the fuzzy fractional-order Boussinesq model and their application in oceanography, *Journal of Ocean Engineering and Science*, (2022).

[16] S. Rezapour, F. M. Sakar, S. M. Aydogan, E. Ravash, Some results on a system of multiterm fractional integro-differential equations, *Turkish J. Math*, 44(2020) 2004-2020.

[17] S. Rezapour, A. Imran, A. Hussain, F. Martínez, S. Etemad, M. K. Kaabar, Condensing functions and approximate endpoint criterion for the existence analysis of quantum integro-difference FBVPs,*Symmetry*, 13(3) (2021) 469.

[18] A. Rufián-Lizana, R. Osuna-Gómez, Y. Chalco-Cano, H. Román-Flores, Some remarks on optimality conditions for fuzzy optimization problems, *Investigación Operacional*, 38(2)(2018) 188-193.

[19] L. Stefanini, M. Arana-Jiménez, Karush–Kuhn–Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability, *Fuzzy Sets and Systems*, 362, (2019) 1-34.

[20] Y. R. Syau, Differentiability and convexity of fuzzy mappings, *Computers & Mathematics with Applications*, 41(1-2) (2001) 73-81.

[21] Y. R. Syau, Some properties of convex fuzzy mappings, *Journal of Fuzzy Mathematics*, 7(1999) 151-160.

[22] G. Wang, C. Wu, Directional derivatives and subdifferential of convex fuzzy mappings and application in convex fuzzy programming, *Fuzzy Sets and Systems*, 138(3)(2003) 559-591.

[23] H. C. Wu, The optimality conditions for optimization problems with convex constraints and multiple fuzzy-valued objective functions, *Fuzzy Optimization and Decision Making*, 8(3) (2009) 295-321.

[24] H. C. Wu, The Karush-Kuhn-Tucker optimality conditions for multi-objective programming problems with fuzzy-valued objective functions, *Fuzzy Optimization and Decision Making*, 8(1)(2009) 1-28.

[25] H. C. Wu, The Karush-Kuhn-Tucker optimality conditions for the optimization problem with fuzzy-valued objective function, *Mathematical Methods of Operations Research*, 66(2)(2007) 203-224.

[26] Z. Wu, J. Xu, Generalized convex fuzzy mappings and fuzzy variational-like inequality, *Fuzzy Sets and Systems*, 160(11)(2009) 1590-1619.

[27] Z. Wu, J. Xu, Nonconvex fuzzy mappings and the fuzzy pre-variational inequality, *Fuzzy Sets and Systems* 159(16)(2008) 2090-2103.

[28] H. Yan, J. Xu, A class of convex fuzzy mappings, *Fuzzy Sets and Systems*, 129(1)(2002) 47-56.

[29] A. Younus, M. Asif, K. Farhad, O. Nisar, Some new variants of interval-valued Gronwall type inequalities on time scales, *Iranian Journal of Fuzzy Systems*, 16(5)(2019) 187-198.

[30] L. A. Zadeh, Fuzzy sets, *Information and control*, 8(3)(1965) 338-353.

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