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Original Research Paper

On the Relative n-Tensor Nilpotent Degree of Finite Groups

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Abstract. In this paper, we generalize the concepts of the relative commutativity degree $d(G, N)$ of a subgroup N of a finite group G and also the tensor degree of a finite group. We introduce the relative n-tensor nilpotent degree of a finite group G with respect to a subgroup H of G and some bounds on this topic are given.

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1 Introduction and Preliminaries

All groups considered in this paper are finite. Let G be a group with a normal subgroup N. Then (G, N) is said to be a pair of groups. Let G and N act on each other and on themselves by conjugation. The nonabelian tensor product $G \otimes N$ is the group generated by the symbols $g \otimes n$ subject to the relations

$$gg' \otimes n = ({}^{g}g' \otimes {}^{g}n)(g \otimes n)$$

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$$g \otimes nn' = (g \otimes n)({}^{n}g \otimes \ {}^{n}n')$$

for all g, g' in G and n, n' in N, where for example ${}^{g}g'$ is the conjugate of g' by g. For an element $x \in G$, we consider the tensor centralizer of x as

$$C_G^{\otimes}(x) = \{a \in G : a \otimes x = 1_{G \otimes G}\}$$

which is a subgroup of G. The intersection of all tensor centralizers of elements of G is called the tensor center of G and it is denoted by $Z^{\otimes}(G)$. The commutator map $\kappa : G \otimes N \longrightarrow [G, N]$ which is given by $g \otimes n \longrightarrow [g, n]$ for all $g \in G$ and $n \in N$, is an epimorphism of groups and we denote $\ker \kappa$ by $J_2(G)$. We define the tensor upper central series of G as $Z_1^{\otimes}(G) = Z^{\otimes}(G)$ and

$$Z_n^{\otimes}(G) = \{a \in G : [a, x_1, \ldots, x_{n-1}] \otimes x_n = 1; \text{ for all } x_1, \ldots, x_n \in G\}$$

for all $n \geq 2$. In fact $Z_n^{\otimes}(G)/Z^{\otimes}(G) = Z_{n-1}(G/Z^{\otimes}(G))$ for all $n \geqslant 1$. Hence we have the ascending tensor central series as

$$1 \leq Z_1^{\otimes}(G) = Z^{\otimes}(G) \leq Z_2^{\otimes}(G) \leq Z_3^{\otimes}(G) \leq \cdots .$$

In [1], the concept of the tensor degree of a group is introduced as

$$d^{\otimes}(G) = \frac{|\{(x, y) \in G \times G : x \otimes y = 1_{\otimes}\}|}{|G|^2}$$

which may be considered as the distance of G from being equal to $Z^{\otimes}(G)$, because $d^{\otimes}(G) = 1$ if and only if $Z^{\otimes}(G) = G$. On the other hand, one may easily check that $d^{\otimes}(G) = 1$ if and only if G is abelian.

One of the most important concepts in probabilistic group theory is the commutativity degree $d(G)$ of a finite group G (See [4]). Erfanian et. al. in [3] generalized the notation of $d(G)$ by defining the relative commutativity degree of a pair of groups (G, N). Let N be a subgroup of G. The relative commutativity degree $d(G, N)$ is the probability of commuting an element of N with an element of G. It is obviously seen that $d(G) = d(G, G)$ and $d(G, N) = 1$ if and only if N is contained in the center of G. They also proved the following theorem:

Theorem 1.1. *(See [3] Theorem 3.9) Let H and N be two subgroups of G such that $N \trianglelefteq G$ and $N \subseteq H$. Then*

$$d(H,G) \leq d(H/N, G/N)d(N),$$

equality holds if $N \cap [H,G] = 1$.

Theorem 1.2. *[7] Let G be a group and p be the smallest prime divisor of the order of G and $d = d(G)$. Then*

$$\begin{aligned} \frac{d}{|J_2(G)|} + \frac{|Z^{\otimes}(G)|}{|G|}(1 - \frac{1}{|J_2(G)|}) &\leq d^{\otimes}(G) \\ &\leq d - \frac{(p-1)(|Z(G)| - |Z^{\otimes}(G)|)}{p|G|} \end{aligned}$$

The special case when $Z^{\otimes}(G) = 1$ is described by the next result and has analogies with Theorem 2.8 in [6]. There are analogous to the commutativity degree of groups in [1, 2, 7, 8].

Theorem 1.3. *[7] Let G be a nonabelian group with $Z^{\otimes}(G) = 1$ and p be the smallest prime dividing $|G|$. Then $d^{\otimes}(G) \leq \frac{1}{p}$.*

2 Relative n-Tensor Nilpotent Degree

This section is devoted to define the concept of relative n-tensor nilpotent degree of a finite group G and a subgroup H. Then we obtain some results on this concept.

Definition 2.1. *Let H be a subgroup of a finite group G. We define $d_n^{\otimes}(H,G)$, the relative n-tensor nilpotent degree of H in G, as*

$$\frac{|\{(h_1, \dots, h_n, g) : [h_1, \dots, h_n] \otimes g = 1_{H \otimes G}, h_i \in H, g \in G\}|}{|H|^n |G|}.$$

In the special case when $H = G$, it is called the n-tensor nilpotent degree of G is denoted by $d_n^{\otimes}(G)$.

We begin with two following elementary results.

Lemma 2.2. *Let G be a group, $x \in G$ and $H \leq G$, then* *(i)* $[H : C_G^{\otimes}(x) \cap H] \leq [G : C_G^{\otimes}(x)]$*;*

(ii) Equality holds in (i), if $G = HZ^{\otimes}(G)$*. The converse is not true.*

Proof. (i) Since $C_G^{\otimes}(x) \leq G$, we have $HC_G^{\otimes}(x) \subseteq G$ and hence

$$|HC_G^{\otimes}(x)| = \frac{|H||C_G^{\otimes}(x)|}{|H \cap C_G^{\otimes}(x)|} \leq |G|.$$

Therefore

$$\frac{|H|}{|H \cap C_G^{\otimes}(x)|} \leq \frac{|G|}{|C_G^{\otimes}(x)|}.$$

(ii) We know that $Z^{\otimes}(G) = \cap_{x \in G} C_G^{\otimes}(x)$. So, if $G = HZ^{\otimes}(G)$, then $G = HC_G^{\otimes}(x)$, for all $x \in G$. Thus,

$$|HC_G^{\otimes}(x)| = \frac{|H||C_G^{\otimes}(x)|}{|H \cap C_G^{\otimes}(x)|} = |G|.$$

Therefore

$$[H : C_G^{\otimes}(x) \cap H] = [G : C_G^{\otimes}(x)], \tag{1}$$

as required. For the converse, let equation (1) holds. Then obviously we have

$$|HC_G^{\otimes}(x)| = |G|.$$

This does not require to imply $G = HZ^{\otimes}(G)$. For example, let $G = Q_8 = \langle a, b | b^2 = a^4 = 1, b^{-1}ab = a^{-1} \rangle$ and $H = \langle b \rangle$. By Lemma 4.2. of ([7]) we have $Z^{\otimes}(G) = 1$ and hence $G \neq HZ^{\otimes}(G)$. However, by proof of Theorem 4.3. of ([7]) we have $C_G^{\otimes}(a^2) = \langle a \rangle$ and therefore $G = HC_G^{\otimes}(a^2)$. □

Theorem 2.3. *Let* $H \leq G$*. Then* $d_n^{\otimes}(H, G) \leq [G : H]^{n+1} d_n^{\otimes}(G)$ *for all* $n \geq 1$*.*

Proof. Using Lemma 2.2, we have

$$\begin{aligned} d_n^{\otimes}(H,G) &= \frac{1}{|H|^n|G|}\sum_{x_1\in H}\cdots\sum_{x_n\in H}|C_G^{\otimes}([x_1,\ldots,x_n])| \\ &= \frac{1}{|H|^n}\sum_{x_1\in H}\cdots\sum_{x_n\in H}\frac{|C_G^{\otimes}([x_1,\ldots,x_n])|}{|G|} \\ &\le \frac{1}{|H|^n}\sum_{x_1\in H}\cdots\sum_{x_n\in H}\frac{|C_G^{\otimes}([x_1,\ldots,x_n])\cap H|}{|H|} \\ &\le \frac{1}{|H|^n}\sum_{x_1\in H}\cdots\sum_{x_n\in H}\frac{|C_G^{\otimes}([x_1,\ldots,x_n])|}{|H|} \\ &\le \frac{|G|^{n+1}}{|H|^{n+1}|G|^{n+1}}\sum_{x_1\in G}\cdots\sum_{x_n\in G}|C_G^{\otimes}([x_1,\ldots,x_n])| \\ &= [G:H]^{n+1}d_n^{\otimes}(G) \end{aligned}$$

□

The inequalities that we have here, are obtained with a series of inequalities. The equalities do not need to be true in general. Even in special cases, we cannot obtain a good condition. For example, in Theorem 2.3, if we want the equality to hold, we need to have
1. $[G:C_G^{\otimes}([x_1,\ldots,x_n])]=[H:C_G^{\otimes}([x_1,\ldots,x_n])\cap H]$ and
2. $C_G^{\otimes}([x_1,\ldots,x_n])\subseteq H$ and
3. If $x_i\in G-H$ for some $1\le i\le n$, then $C_G^{\otimes}([x_1,\ldots,x_n])=\emptyset$.

Theorem 2.4. *Let $H\le G$. Then*

$$d_{n+1}^{\otimes}(H,G)\le\frac{1}{2}(1+d_n^{\otimes}(\frac{H}{H\cap Z^{\otimes}(G)})).$$

Proof. Put $\overline{H}$ for $\frac{H}{H\cap Z^{\otimes}(G)}$ and for each $x\in H$ let $\overline{x}$ stands for $x(H\cap Z^{\otimes}(G))$ as an element of $\overline{H}$. We know that

$$d_{n+1}^{\otimes}(H,G)=\frac{|\{(x_1,\ldots,x_{n+1},y):[x_1,\ldots,x_{n+1}]\otimes y=1_{H\otimes G},x_i\in H,y\in G\}|}{|H|^{n+1}|G|}.$$

Therefore

$$
\begin{aligned}
&|H|^{n+1}|G|d^{\otimes}_{n+1}(H,G)\\
=&|\{(x_1,\dots,x_{n+1},y):[x_1,\dots,x_{n+1}]\otimes y=1, x_i\in H, y\in G\}|\\
=&\sum_{x_1\in H}\cdots\sum_{x_{n+1}\in H}|C^{\otimes}_G([x_1,\dots,x_{n+1}])|\\
=&\sum_{x_1\in H}\cdots\sum_{x_{n+1}\in H,\ [x_1,\dots,x_{n+1}]\in H\cap Z^{\otimes}(G)}|C^{\otimes}_G([x_1,\dots,x_{n+1}])|\\
+&\sum_{x_1\in H}\cdots\sum_{x_{n+1}\in H,\ [x_1,\dots,x_{n+1}]\notin H\cap Z^{\otimes}(G)}|C^{\otimes}_G([x_1,\dots,x_{n+1}])|.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&d^{\otimes}_n(\overline{H})=\\
&\left(\frac{1}{|\overline{H}|}\right)^{n+1}|\{(\overline{x_1},\dots,\overline{x_{n+1}}):[\overline{x_1},\dots,\overline{x_n}]\otimes\overline{x_{n+1}}=1,\overline{x_i}\in\overline{H}\}|=\\
&\left(\frac{|H\cap Z^{\otimes}(G)|}{|H|}\right)^{n+1}\times\\
&\frac{|\{(x_1,\dots,x_{n+1}):[x_1,\dots,x_{n+1}]\in H\cap Z^{\otimes}(G), x_i\in H\}|}{|H\cap Z^{\otimes}(G)|^{n+1}}
\end{aligned}
$$

and we have

$$
\begin{aligned}
&\sum_{x_1\in H}\cdots\sum_{x_{n+1}\in H,\ [x_1,\dots,x_{n+1}]\in H\cap Z^{\otimes}(G)}|C^{\otimes}_G([x_1,\dots,x_{n+1}])|=\\
&|H|^{n+1}d^{\otimes}_n(\overline{H})|G|.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&|H|^{n+1}|G|d^{\otimes}_{n+1}(H,G)\le\\
&|H|^{n+1}d^{\otimes}_n(\overline{H})|G|+(|H|^{n+1}-|H|^{n+1}d^{\otimes}_n(\overline{H}))\frac{|G|}{2}
\end{aligned}
$$

and

$$
\frac{|H|^{n+1}|G|}{2}d^{\otimes}_n(\overline{H})+\frac{|H|^{n+1}|G|}{2} \quad = \quad \frac{|H|^{n+1}|G|}{2}(1+d^{\otimes}_n(\overline{H})).
$$

Hence we have

$$d^{\otimes}_{n+1}(H,G) \leq \frac{1}{2}(1 + d^{\otimes}_{n}(\overline{H})).$$

□

3 Tensor Nilpotent Groups

We are ready to define the concept of tensor nilpotency of a group.

Definition 3.1. *Let G be a group. Then G is called tensor nilpotent if $Z^{\otimes}_{n}(G) = G$ for some $n \geq 0$. For a tensor nilpotent group G, the smallest $c \geq 0$ in which $Z^{\otimes}_{c}(G) = G$ is called the tensor nilpotency class or briefly the tensor class of G.*

Theorem 3.2. *For a finite group G, we have*

$$d^{\otimes}_{n+1}(G) \leq \frac{1}{2^n}(2^n - 1 + d^{\otimes}(\frac{G}{Z^{\otimes}_{n}(G)}))$$

for all $n \geq 1$.

Proof. We khow that

$$Z^{\otimes}_{n}(G)/Z^{\otimes}(G) = Z_{n-1}(G/Z^{\otimes}(G))$$

for all $n \geq 1$. We proceed by induction on n. For $n = 1$, by using Theorem 2.4, we have

$$\begin{aligned} d^{\otimes}_{2}(G) &\leq \frac{1}{2}(1 + d(\frac{G}{G \cap Z^{\otimes}(G)})) \\ &= \frac{1}{2}(1 + d(\frac{G}{Z^{\otimes}(G)})). \end{aligned}$$

Using Theorem 2.4 and the induction, we have

$$\begin{aligned} d^{\otimes}_{n+1}(G) &\leq \frac{1}{2}(1 + d^{\otimes}_{n}(\frac{G}{G \cap Z^{\otimes}(G)})) \\ &\leq \frac{1}{2}(1 + \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^{\otimes}(\frac{\frac{G}{Z^{\otimes}(G)}}{Z_{n-1}(\frac{G}{Z^{\otimes}(G)})})) \\ &= \frac{1}{2}(1 + \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^{\otimes}(\frac{G}{Z^{\otimes}_{n}(G)})) \\ &= \frac{1}{2}(\frac{1}{2^{n-1}}(2^{n-1} + 2^{n-1} - 1 + d^{\otimes}(\frac{G}{Z^{\otimes}_{n}(G)}))) \\ &= \frac{1}{2^{n}}(2^{n} - 1 + d^{\otimes}(\frac{G}{Z^{\otimes}_{n}(G)})), \end{aligned}$$

as required. □

Theorem 3.3. *If G is not a tensor nilpotent group of class at most n, then*

$$d^{\otimes}_{n}(G) \leq \frac{2^{n+2} - 3}{2^{n+2}}.$$

Proof. Since G is not a tensor nilpotent group of class at most n, $Z^{\otimes}_{n}(G) \neq G$ and $G/Z^{\otimes}_{n-1}(G)$ is a nonabelian group. We khow that $d^{\otimes}(G) \leq d(G)$, therefore using Theorem 2.2 in [3] implies $d^{\otimes}(\frac{G}{Z^{\otimes}_{n-1}(G)}) \leq \frac{5}{8}$. So we have

$$\begin{aligned} d^{\otimes}_{n}(G) &\leq \frac{1}{2^{n-1}}(2^{n-1} - 1 + d^{\otimes}(\frac{G}{Z^{\otimes}_{n-1}(G)})) \\ &\leq \frac{1}{2^{n-1}}(2^{n-1} - 1 + \frac{5}{8}) \\ &= \frac{1}{2^{n-1}}(2^{n-1} - \frac{3}{2^{3}}) \\ &= \frac{2^{n+2} - 3}{2^{n+2}}, \end{aligned}$$

as required. □

Lemma 3.4. *If G is tensor nilpotent of class at most n, then G is nilpotent of class n.*

Proof. We khow that $Z_n^{\otimes}(G) \leq Z_n(G)$, so the result followes. $\square$

Theorem 3.5. *If G is a nontrivial group and $Z(G) = 1$, then*

$$d_n^{\otimes}(G) \leq \frac{2^n - 1}{2^n}$$

Proof. We proceed by induction on n. Let $n = 1$ since $Z(G) = 1$, G is not nilpotent and Theorem 3 in [5] implies that $d(G) \leq \frac{1}{2}$. We know that $d^{\otimes}(G) \leq d(G) \leq \frac{1}{2}$. Therefore

$$\begin{aligned} d_{n+1}^{\otimes}(G) &\leq \frac{1}{2^n}(2^n - 1 + d^{\otimes}(G)) \\ &\leq \frac{1}{2^n}(2^n - 1 + \frac{1}{2}) \\ &= \frac{1}{2^n}(2^n - \frac{1}{2}) \\ &= \frac{2^{n+1} - 1}{2^{n+1}}. \end{aligned}$$

$\square$

Theorem 3.6. *Let H be a proper subgroup of G. Then for all $n \geq 1$, we have*

(i) If $H \subseteq Z_n^{\otimes}(G)$, then $d_n^{\otimes}(H, G) = 1$.

(ii) If $H \nsubseteq Z_n^{\otimes}(G)$ and $H/H \cap Z^{\otimes}(G)$ is tensor nilpotent of class at most $n - 1$, then $d_n^{\otimes}(H, G) = 1$.

(iii) If $H \nsubseteq Z_n^{\otimes}(G)$ and $H/H \cap Z^{\otimes}(G)$ is not tensor nilpotent of class at most $n - 1$, then $d_n^{\otimes}(H, G) \leq \frac{2^{n+2}-3}{2^{n+2}}$.

Proof. (i) If $H \subseteq Z_n^{\otimes}(G)$, then $[h_1, ..., h_n] \otimes x = 1$ for all $h_1, \ldots, h_n \in H$ and $x \in G$. So

$$d_n^{\otimes}(H, G) = \frac{1}{|H|^n |G|} \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_G^{\otimes}([h_1, \ldots, h_n])| = 1.$$

(ii) Since $H/H \cap Z^{\otimes}(G)$ is a tensor nilpotent group of class at most $n-1$, for all $\overline{h_1}, \ldots, \overline{h_n}$ in $H/H \cap Z^{\otimes}(G)$ where $\overline{h_i} = h_i H/H \cap Z^{\otimes}(G)$,

$h_i \in H$, $i = 1, \ldots, n$ and $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} = 1$. We know that there exists homomorphism $H/H \cap Z^{\otimes}(G) \otimes H/H \cap Z^{\otimes}(G) \longrightarrow (H/H \cap Z^{\otimes}(G))'$ given $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} \longrightarrow [\overline{h_1}, \ldots, \overline{h_n}]$. Since $[\overline{h_1}, \ldots, \overline{h_{n-1}}] \otimes \overline{h_n} = 1$, then $[\overline{h_1}, \ldots, \overline{h_n}] = 1$. Therefore there exist $h_1, \ldots, h_n$ in H such that $[h_1, \ldots, h_n]H \cap Z^{\otimes}(G) = H \cap Z^{\otimes}(G)$, so $[h_1, \ldots, h_n] \in H \cap Z^{\otimes}(G)$. Therefore for all x in G, we have $[h_1, \ldots, h_n] \otimes x = 1$. Hence, $C_G^{\otimes}([h_1, \ldots, h_n]) = G$. Thus,

$$\begin{aligned}
& |H^n||G|d_n^{\otimes}(H,G) \\
= & |\{(h_1, \ldots, h_n, x) \in H^n \times G : [h_1, \ldots, h_n] \otimes x = 1\}| \\
= & \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_G^{\otimes}([h_1, \ldots, h_n])| \\
= & |H|^n|G|,
\end{aligned}$$

and so, $d_n^{\otimes}(H,G) = 1$.
(iii) Since $H/H \cap Z^{\otimes}(G)$ is not tensor nilpotent of class at most $n-1$, by Theorem 3.3 we have

$$d_{n-1}^{\otimes}(\frac{H}{H \cap Z^{\otimes}(G)}) \leq \frac{2^{n+1}-3}{2^{n+1}}.$$

Hence

$$\begin{aligned}
d_n^{\otimes}(H,G) & \leq \frac{1}{2}(1 + d_n^{\otimes}(\frac{H}{H \cap Z^{\otimes}(G)})) \\
& \leq \frac{1}{2}(1 + \frac{2^{n+1}-3}{2^{n+1}}) \\
& \leq \frac{1}{2}(\frac{2^{n+1}+2^{n+1}-3}{2^{n+1}}) \\
& = \frac{2^{n+2}-3}{2^{n+2}}.
\end{aligned}$$

□

Theorem 3.7. *Let G be finite group, H and N be subgroups of G such that $N \trianglelefteq G$ and $N \subseteq H$. Then*

$$d_n^{\otimes}(H,G) \leq d_n^{\otimes}(H/N, G/N).$$

Proof. We have

$$
\begin{aligned}
&|H|^n|G|d_n^{\otimes}(H,G)\\
=&|\{(h_1,\ldots,h_n,y):[h_1,\ldots,h_n]\otimes y=1, h_i\in H, y\in G\}|\\
=&\sum_{h_1\in H}\cdots\sum_{h_n\in H}|C_G^{\otimes}([h_1,\ldots,h_n])|\\
=&\sum_{h_1\in H}\cdots\sum_{h_n\in H}\frac{|C_G^{\otimes}([h_1,\ldots,h_n])N||C_N^{\otimes}([h_1,\ldots,h_n])|}{|N|}\\
\leq&\sum_{h_1\in H}\cdots\sum_{h_n\in H}|C_{G/N}^{\otimes}([h_1N,\ldots,h_nN])||C_N^{\otimes}([h_1,\ldots,h_n])|\\
=&\sum_{t_1\in H/N}\sum_{h_1\in H}\cdots\sum_{t_n\in H/N}\sum_{h_n\in H}|C_{G/N}^{\otimes}([t_1,\ldots,t_n])||C_N^{\otimes}([h_1,\ldots,h_n])|\\
=&\sum_{t_1\in H/N}\cdots\sum_{t_n\in H/N}|C_{G/N}^{\otimes}([t_1,\ldots,t_n])|\sum_{h_1\in H}\cdots\sum_{h_n\in H}|C_N^{\otimes}([h_1,\ldots,h_n])|\\
\leq&|N|^{n+1}\sum_{t_1\in H/N}\cdots\sum_{t_n\in H/N}|C_{G/N}^{\otimes}([t_1,\ldots,t_n])|\\
=&|H/N|^n|G/N|d_n^{\otimes}(H/N,G/N)|N|^{n+1}\\
=&|H|^n|G|d_n^{\otimes}(H/N,G/N).
\end{aligned}
$$

Therefore

$$d_n^{\otimes}(H,G)\leq d_n^{\otimes}(H/N,G/N).$$

□

Corollary 3.8. *If $N\trianglelefteq G$, then $d_n^{\otimes}(G)\leq d_n^{\otimes}(G/N)$*

Proof. Let $H=G$. Then by using Theorem 3.7, the result follows . □

4 Some Examples

In this section, we compute the relative n-tensor nilpotent degree of some groups.

Example 4.1. Let $G = C_4$ and $H = 2C_4$. Then

$$\begin{aligned} d_2^{\otimes}(2C_4, C_4) &= \frac{1}{|2C_4|^2|C_4|} \sum_{h_1 \in 2C_4} \sum_{h_2 \in 2C_4} |C_{C_4}^{\otimes}([h_1, h_2])| \\ &= \frac{1}{2^2 \times 4} \times 16 = 1. \end{aligned}$$

Example 4.2. Let $G = D_8 =< a, b | a^4 = b^2 = 1, ba = a^{-1}b >$ be the dihedral group of order 8 and H the subgroup generated by $\{a^2, ab\}$. Let $n \geq 3$. Since G is a nilpotent group of class 2, we have $\gamma_n(G) = 1$. Hence

$$\begin{aligned} d_n^{\otimes}(H, D_8) &= \frac{1}{|H|^n|D_8|} \sum_{h_1 \in H} \cdots \sum_{h_n \in H} |C_{D_8}^{\otimes}([h_1, \ldots, h_n])| \\ &= \frac{1}{|H|^n \times 8} \times |H|^n = \frac{1}{8}. \end{aligned}$$

The same is true if $G = Q_8 =< a, b | a^4 = 1, a^2 = b^2, ba = a^{-1}b >$, the quaternion group of order 8, and $H = \{a^2, ab\}$.

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