Journal of Mathematical Extension
Vol. 16, No. 11, (2022) (2)1-15
URL: https://doi.org/10.30495/JME.2022.2272
ISSN: 1735-8299
Original Research Paper

# Frames of Translates by Semiregular Sampling

**F. Valizadeh**
Central Tehran Branch, Islamic Azad University

**H. Rahimi***
Central Tehran Branch, Islamic Azad University

**R. A. Kamyabi Gol**
Ferdowsi University of Mashhad and Center of Excellence in Analysis on Algebraic Structures (CEAAS)

**F. Esmaeelzadeh**
Bojnourd Branch, Islamic Azad University

**Abstract.** In this paper, we consider some kinds of semiregular frames of translates on the Hilbert space $L^2(\mathbb{R}^d)$. More precisely, we investigate the frames of the form $\{T_{\mathcal{A}k}f\}_{k\in\mathbb{Z}^d}$ where $\mathcal{A}$ is a real invertible $d\times d$ matrix, $f\in L^2(\mathbb{R}^d)$ and it is a frame for the closed subspace generated by $\{T_{\mathcal{A}k}f\}_{k\in\mathbb{Z}^d}$.

**AMS Subject Classification:** MSC 43A15; MSC 43A85.
**Keywords and Phrases:** frames; Riesz bases; semi-irregular translates; sampling.

Frames are a generalization of orthonormal bases in Hilbert spaces. In 1946, Gabor [13] formulated a fundamental approach to a signal decomposition in terms of elementary signals. In 1952, Duffin and Schaffer

Received:January 2022; Accepted:February 2022
*Corresponding Author

[9] presented some problems in nonharmonic Fourier series and frames for Hilbert spaces. Later, Daubechies, Grossman and Mayer [8] revived the study of frames and applications. The main property of frames that makes them useful is their redundancy. Many properties of frames make them useful in various applications in mathematics, sciences and engineering. For a nice and comprehensive survey on various types of frames, one may refer to [3, 6, 7, 14] and the references therein.

Frames of translates are an important class of frames that have a special structure. These frames are central in approximation, sampling, Gabor and wavelet theory and they were investigated in the context of general properties of shift invariant spaces in a number of articles, including [10, 15, 16]. Frames of translates are natural examples of frame sequences. Frame sequences are useful in cases where we are interested only in expansions in subspaces. For the literature regarding frames and frame sequences, one may refer to [4, 5, 7, 14]. We fix a real invertible $d \times d$ matrix $\mathcal{A}$ and consider $\mathcal{A}\mathbb{Z}^d$ as an index set. For $f \in L^2(\mathbb{R}^d)$, let $\Phi_f^{\mathcal{A}} := \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |\hat{f}(\mathcal{A}^T)^{-1}(\cdot + k)|^2$ be the periodization of $|\hat{f}|^2$. By using $\Phi_f^{\mathcal{A}}$, among other things, we characterize when the sequence $\tau_{\mathcal{A}}(f) := \{f(\cdot - \mathcal{A}k)\}_{k\in\mathbb{Z}^d}$ is a Bessel sequence, frame of translates, Riesz basis, or orthonormal basis. Finally, we construct an example, in which $\tau_{\mathcal{A}}(f)$ is a Parseval frame of translates, but not a Riesz sequence. This is a generalization of regular frames of translates of the form $\{\lambda_k f\}_{k\in\mathbb{Z}^d}$, where $\{\lambda_k\}_{k\in\mathbb{Z}^d} \subseteq \mathbb{R}^d$ and $f \in L^2(\mathbb{R}^d)$; see [1, 2, 4].

## 1 Preliminaries and Notation

Let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A sequence $\{f_k\}_{k=1}^{\infty}$ is called a *basis* for $\mathcal{H}$ if for every $f \in \mathcal{H}$ there is a unique sequence of scalars $\{c_k\}_{k=1}^{\infty}$ such that $f = \sum_{k=1}^{\infty} c_k f_k$. A sequence $\{e_k\}_{k=1}^{\infty} \subseteq \mathcal{H}$ is an *orthonormal system* if $\langle e_k, e_j \rangle = \delta_{k,j}$. An *orthonormal basis* is an orthonormal system $\{e_k\}_{k=1}^{\infty}$ that is a basis for $\mathcal{H}$. A *Riesz basis* for $\mathcal{H}$ is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \longrightarrow \mathcal{H}$ is a bounded bijective operator. A Riesz basis is actually a basis. In fact, a basis $\{f_k\}_{k=1}^{\infty}$ in $\mathcal{H}$ is a Riesz basis if $\sum_{k=1}^{\infty} c_k f_k$ converges in $\mathcal{H}$ only when $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $\mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leqslant B\|f\|^2, \text{ for all } f \in \mathcal{H}. \tag{1}$$

The numbers $A$ and $B$ are called *frame bounds* which are not unique. A frame with bounds $A = B = 1$ is called *Parseval frame*. A sequence $\{f_k\} \subseteq \mathcal{H}$ is called a *Bessel sequence* if the right-hand side of inequality (1) holds. A frame is called *exact* if it ceases to be a frame whenever any single element is deleted from the sequence and a frame that is not exact is called *overcomplete frame*. We recall that every orthonormal basis is an exact Parseval frame and every exact Parseval frame is an orthonormal basis. A sequence $\{f_k\}_{k=1}^{\infty}$ is called a *Riesz sequence* if there exist $A, B > 0$ such that $A\sum_{k=1}^{\infty}|c_k|^2 \leqslant \|\sum_{k=1}^{\infty} c_k f_k\|^2 \leqslant B\sum_{k=1}^{\infty}|c_k|^2$, for all sequences $\{c_k\}_{k=1}^{\infty} \in l^2$. A Riesz sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for the Hilbert space $\overline{span}\{f_k\}_{k=1}^{\infty}$, which might just be a subspace of $\mathcal{H}$ and a sequence $\{f_k\}_{k=1}^{\infty}$ is a frame sequence if it is a frame for $\overline{span}\{f_k\}_{k=1}^{\infty}$; see [7, 14]. For $\mathcal{A} \in GL_d(\mathbb{R})$ (the set of all $d \times d$ invertible real matrices), let $\Lambda = \mathcal{A}\mathbb{Z}^d$ and $Q_{\mathcal{A}} = \mathcal{A}[0,1)^d$, then $\mathbb{R}^d = \bigcup_{\lambda\in\Lambda}(\lambda + Q_{\mathcal{A}})$ where $(\lambda + Q_{\mathcal{A}})\bigcap(\lambda' + Q_{\mathcal{A}}) = \emptyset$ if and only if $\lambda \neq \lambda'$.

A function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is called $\Lambda$-*periodic* if $f(x + \lambda) = f(x)$ for all $x \in \mathbb{R}^d$ and $\lambda \in \Lambda$. It is easy to check that $f$ is $\Lambda$-periodic if and only if $fo\mathcal{A}$ is periodic. For simplicity, we denote $fo\mathcal{A}$ by $f_{\mathcal{A}}$. Recall that for $f \in L^1(\mathbb{R}^d)$, the *Fourier transform* of $f$ is $\mathcal{F}f(\xi) := \hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi\cdot x}dx$, where $\xi \in \mathbb{R}^d$ and the *inversion Fourier transform* is $\check{f}(x) = \hat{f}(-x)$. As usual, the definition of Fourier transform extends to a unitary operator $f \longmapsto \hat{f}$ on $L^2(\mathbb{R}^d)$, known as the Plancharel theorem. Also $(\hat{f_{\mathcal{A}}})(\xi) = \frac{1}{|\det\mathcal{A}|}\hat{f}((\mathcal{A}^T)^{-1}\xi)$ [12] and the Fourier series of $f_{\mathcal{A}}$ is

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d} \hat{f}_{\mathcal{A}}(k)e^{-2\pi ik\cdot x} &= \frac{1}{|\det\mathcal{A}|}\sum_{k\in\mathbb{Z}^d}\hat{f}((\mathcal{A}^T)^{-1}k)e^{-2\pi ik\cdot x}\\ &= \frac{1}{|\det\mathcal{A}|}\sum_{\lambda\in\Lambda^{\perp}}\hat{f}(\lambda)e^{-2\pi i\mathcal{A}^T\lambda x},\end{aligned}$$

where $\Lambda^{\perp} = (\mathcal{A}^T)^{-1}\mathbb{Z}^d$. For $\psi \in L^2(Q_{\mathcal{A}})$, put $\varepsilon_{\mathcal{A}}(\psi) = \{(E_{\mathcal{A}k}\psi)(\gamma)\}_{k\in\mathbb{Z}^d}$ where $(E_{\mathcal{A}k}\psi)(x) = e^{2\pi i\mathcal{A}k\cdot x}\psi(x)$, $(x \in Q_{\mathcal{A}})$ and set $N_{\psi}^{\mathcal{A}} = \{x \in Q_{\mathcal{A}}$ :

$\psi(x) = 0\}$. For any function $f$ on $\mathbb{R}^d$ and $k \in \mathbb{Z}^d$, the *translation of $f$ by $k$* is $T_{\mathcal{A}k}f(x) = f(x - \mathcal{A}k)$ and the set of all translations of $f$ is denoted by $\tau_{\mathcal{A}}(f) = \{T_{\mathcal{A}k}f\}_{k\in\mathbb{Z}^d}$, which is called a *system of translates* of $f$. Moreover, $\mathcal{F}(\tau_{\mathcal{A}}f) := \{(T_{\mathcal{A}k}f)^\wedge\}_{k\in\mathbb{Z}^d} = \{E_{-\mathcal{A}k}\hat{f}\}_{k\in\mathbb{Z}^d}$ for $f \in L^2(\mathbb{R}^d)$ and $\mathcal{A} \in GL_d(\mathbb{R})$. For $f \in L^2(\mathbb{R}^d)$, we define

$$\Phi_f^{\mathcal{A}}(\gamma) := \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2, \quad \gamma \in \mathbb{R}^d.$$

We call $\Phi_f^{\mathcal{A}}$ the *periodization of* $|\hat{f}|^2$. It is worthwhile to mention for $f \in L^1(\mathbb{R}^d)$, the series $\sum_{k\in\mathbb{Z}^d} T_{\mathcal{A}k}f$ converges point-wise almost everywhere and it converges in $L^1(Q_{\mathcal{A}})$ to a function $Pf$ such that $\|Pf\|_{L^1(Q_{\mathcal{A}})} \leq \|f\|_{L^1(\mathbb{R}^d)}$. Moreover, for $\gamma \in \mathbb{Z}^d$, it follows that $(Pf)^\frown(\gamma)$ (Fourier transform on $Q_{\mathcal{A}}$) equals $\hat{f}(\gamma)$ (Fourier transform on $\mathbb{R}^d$); see [11]. For $f \in L^2(\mathbb{R}^d)$, one can check easily that

$$\|\Phi_f^{\mathcal{A}}\|_{L^1(Q_{\mathcal{A}})} = \frac{1}{|\det \mathcal{A}|}\|f\|^2_{L^2(\mathbb{R}^d)}.$$

Consequently,

$$(\Phi_f^{\mathcal{A}})^{\frac{1}{2}}(\gamma) = \frac{1}{\sqrt{|\det \mathcal{A}|}}\Big(\sum_{k\in\mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2\Big)^{\frac{1}{2}}$$

belongs to $L^2(Q_{\mathcal{A}})$ and $\|(\Phi_f^{\mathcal{A}})^{\frac{1}{2}}\|_{L^2(Q_{\mathcal{A}})} = \frac{1}{\sqrt{|\det \mathcal{A}|}}\|f\|_{L^2(\mathbb{R}^d)}$. Therefore the map from $L^2(\mathbb{R}^d)$ into $L^2(Q_{\mathcal{A}})$ defined by $f \mapsto (\Phi_f^{\mathcal{A}})^{\frac{1}{2}}$ is a norm-preserving map. In this paper, among other things, we use $\Phi_f^{\mathcal{A}}$ for characterization when $\tau_{\mathcal{A}}(f)$ is a Bessel sequence, frame of translates, Riesz basis, or an orthonormal basis.

## 2 MAIN RESULTS

It is known that for $f \in L^2(\mathbb{R}^d)$, if $\tau_{\mathcal{A}}(f)$ is a Bessel sequence then for any $\{c_k\}_{k\in\mathbb{Z}^d} \in l^2(\mathbb{Z}^d)$, $\sum_{k\in\mathbb{Z}^d} c_k T_{\mathcal{A}k}(f)$ converges in $L^2(\mathbb{R}^d)$ and $\sum_{k\in\mathbb{Z}^d} c_k E_{-\mathcal{A}k}$ converges in $L^2[0,1)^d$, where $E_{-\mathcal{A}k}(x) = e^{-2\pi i \mathcal{A}k\cdot x}$ , $x \in$

$\mathbb{R}^d$. Now for $f \in L^1(\mathbb{R}^d)$, let $\psi_f^{\mathcal{A}}(\gamma) = \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k)$ be the $\Lambda$-periodization of $f$. We show that $\psi_f^{\mathcal{A}}$ is absolutely convergent in $L^1(Q_{\mathcal{A}})$ and that $\|\psi_f^{\mathcal{A}}\|_{L^1(Q_{\mathcal{A}})} = \|f\|_{L^1(\mathbb{R}^d)}$.

**Lemma 2.1.** *Let $f \in L^1(\mathbb{R}^d)$. With notations as above, the series $\psi_f^{\mathcal{A}}$ converges absolutely in $L^1(Q_{\mathcal{A}})$ and $\int_{Q_{\mathcal{A}}} \psi_f^{\mathcal{A}}(\gamma)d\gamma = \int_{\mathbb{R}^d} f(\gamma)d\gamma$.*

**Proof.** Let $f \in L^1(\mathbb{R}^d)$, $\mathcal{A} \in GL_d(\mathbb{R})$ and put $\psi_f^{\mathcal{A}}(\gamma) = \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k)$. Using the fact that $\psi_f^{\mathcal{A}}$ is $\Lambda$-periodic, we have

$$\begin{aligned}\int_{Q_{\mathcal{A}}} |\psi_f^{\mathcal{A}}(\gamma)|d\gamma \leq \quad & \int_{Q_{\mathcal{A}}} \sum_{k\in\mathbb{Z}^d} |f(\gamma + \mathcal{A}k)|d\gamma \\ &= \sum_{k\in\mathbb{Z}^d} \int_{Q_{\mathcal{A}}} |f(\gamma + \mathcal{A}k)|d\gamma \\ &= \int_{\bigcup_{k\in\mathbb{Z}^d} Q_{\mathcal{A}}+\mathcal{A}k} |f(\gamma + \mathcal{A}k)|d\gamma \\ &= \int_{\mathbb{R}^d} |f(\gamma + \mathcal{A}k)|d\gamma = \|f\|_1 < \infty,\end{aligned}$$

which is finite. This implies that $\psi_f^{\mathcal{A}}(\gamma) = \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k) \in L^1(Q_{\mathcal{A}})$ and $\|\psi_f^{\mathcal{A}}\|_{L^1(Q_{\mathcal{A}})} \leqslant \|f\|_1$. Using the dominated convergence theorem, we have

$$\psi_N(\gamma) = \sum_{|k|\leq N} f(\gamma + \mathcal{A}k) \to \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k) \quad \text{as } N \to \infty$$

where $|k| = \sum_1^n k_j$, and

$$|\psi_N(\gamma)| \leq \sum_{k\in\mathbb{Z}^d} |f(\gamma + \mathcal{A}k)| \in L^1(Q_{\mathcal{A}}).$$

Then

$$\int_{Q_{\mathcal{A}}} \psi_N(\gamma)d\gamma \to \int_{Q_{\mathcal{A}}} \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k)d\gamma.$$

Moreover,

$$\begin{aligned}\int_{Q_{\mathcal{A}}} \sum_{k\in\mathbb{Z}^d} f(\gamma + \mathcal{A}k)d\gamma \quad &= \sum_{k\in\mathbb{Z}^d} \int_{Q_{\mathcal{A}}} f(\gamma + \mathcal{A}k)d\gamma \\ &= \int_{\bigcup_{k\in\mathbb{Z}^d} Q_{\mathcal{A}}+\mathcal{A}k} \psi(\gamma)d\gamma \\ &= \int_{\mathbb{R}^d} f(\gamma)d\gamma.\end{aligned}$$

□

It is known that, $\varepsilon_{\mathcal{A}}(\psi)$ is a Bessel sequence in $L^2(Q_{\mathcal{A}})$ if and only if $\psi_{\mathcal{A}} \in L^\infty(Q_{\mathcal{A}})$. In this case, $|\psi(\gamma)|^2 \leqslant B$ a.e., where $B$ is a Bessel bound.

**Proposition 2.2.** *Let $f \in L^2(\mathbb{R}^d)$ and let $B > 0$. Then $\tau_{\mathcal{A}}(f)$ is a Bessel sequence with bound $B$ if and only if $\Phi_f^{\mathcal{A}}(\gamma) \leqslant B$ for a.e. $\gamma \in Q_{\mathcal{A}}$.*

**Proof.** Given a finite sequence $\{c_k\}_{k\in\mathbb{Z}^d} \subseteq \mathbb{C}$, we put

$$\psi(\gamma) = \sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i(\mathcal{A}^T)^{-1}k\cdot\gamma}.$$

Since $\psi \in L^2(Q_{(\mathcal{A}^T)^{-1}})$, we have

$$\begin{aligned}
\|T\{c_k\}_{k\in\mathbb{Z}^d}\|^2 &= \|\sum_{k\in\mathbb{Z}^d} c_k T_{(\mathcal{A}^T)^{-1}k} f\|^2 \\
&= \Big\|\mathcal{F}(\sum_{k\in\mathbb{Z}^d} c_k T_{(\mathcal{A}^T)^{-1}k} f)\Big\|^2 \\
&= \|\sum_{k\in\mathbb{Z}^d} c_k E_{-(\mathcal{A}^T)^{-1}k} \hat{f}\|^2 \\
&= \int_{\mathbb{R}^d} \Big|\sum_{k\in\mathbb{Z}^d} c_k e^{-2\pi i(\mathcal{A}^T)^{-1}k\cdot\gamma} \hat{f}(\gamma)\Big|^2 d\gamma \\
&= \int_{\hat{\mathbb{R}^d}} |\psi(\gamma)|^2 |\hat{f}(\gamma)|^2 d\gamma \\
&= \frac{1}{|\det\mathcal{A}|} \int_{[0,1]^d} |\psi(\gamma)|^2 \sum_{k\in\mathbb{Z}^d} |\hat{f}(\mathcal{A}^T)^{-1}(\gamma+k)|^2 d\gamma \\
&= \int_{[0,1]^d} |\psi(\gamma)|^2 \Phi_f^{\mathcal{A}}(\gamma) d\gamma.
\end{aligned}$$

Conversely, if $\Phi_f^{\mathcal{A}}(\gamma) \leqslant B$ for a.e. $\gamma \in \mathbb{R}^d$, then

$$\|T\{c_k\}_{k\in\mathbb{Z}^d}\|^2 \leqslant B \int_{[0,1]^d} |\psi(\gamma)|^2 d\gamma = B \sum_{k\in\mathbb{Z}^d} |c_k|^2.$$

□

**Proposition 2.3.** *Let $f \in L^2(\mathbb{R}^d)$, then $\tau_{\mathcal{A}}(f)$ is an orthonormal sequence if and only if $\Phi_f^{\mathcal{A}}(\gamma) = 1$ for a.e. $\gamma \in Q_{\mathcal{A}}$.*

**Proof.** We have to show that $\langle T_{\mathcal{A}k_1}f, T_{\mathcal{A}k_2}f\rangle = \delta_{\mathcal{A}k_1,\mathcal{A}k_2}$ (for $k_1, k_2 \in \Lambda$) if and only if $\Phi_f^{\mathcal{A}}(\gamma) = 1$ for a.e. $\gamma \in \mathbb{R}^d$. Using the Plancharel theorem and Weil's formula, we have

$$\begin{aligned}
\langle T_{\mathcal{A}k_1}f, T_{\mathcal{A}k_2}f\rangle &= \langle f, T_{\mathcal{A}(k_1-k_2)}f\rangle \\
&= \langle \hat{f}, \widehat{T_{\mathcal{A}(k_1-k_2)}f}\rangle \\
&= \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 \cdot e^{-2\pi i \mathcal{A}(k_1-k_2)\cdot\gamma} d\gamma \\
&= \frac{1}{|\det \mathcal{A}|} \int_{Q_{\mathcal{A}}} \sum_{\lambda\in\mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma+\lambda)\big)|^2 e^{-2\pi i (\mathcal{A}^T)^{-1}(k_1-k_2)\cdot\gamma} d\gamma \\
&= \int_{Q_{\mathcal{A}}} \Phi_f^{\mathcal{A}}(\gamma) \cdot e^{-2\pi i (\mathcal{A}^T)^{-1}k\cdot\gamma} d\gamma. \qquad (2)
\end{aligned}$$

The Pontryagin duality theorem and [11, Proposition 4.3] imply that $\Lambda$ is an orthonormal basis for $L^2(Q_{\mathcal{A}})$ and (2) completes the proof. □

**Lemma 2.4.** *Let $\psi \in L^2(Q_{\mathcal{A}})$ be bounded. Then*

$$\overline{span}\,\varepsilon_{\mathcal{A}}(\psi) = \{f \in L^2(Q_{\mathcal{A}}) : f = 0 \text{ a.e. on } N_\psi^{\mathcal{A}}\}.$$

**Proof.** It is easy to see that $\{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$ is a closed subspace and $span(\varepsilon_{\mathcal{A}}(\psi)) \subseteq \{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$. To show that $\overline{span}\,\varepsilon_{\mathcal{A}}(\psi) = \{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$, suppose that $f \in \{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$ satisfies $\langle f, \psi e_k\rangle = 0$ for every $k \in \mathbb{Z}^d$. Since $\psi$ is bounded, $f\overline{\psi} \in L^2(Q_{\mathcal{A}})$ and $Q_{(\mathcal{A}^T)^{-1}}$ is an orthonormal basis for $L^2(Q_{\mathcal{A}})$ (see [11]). We have $\langle f\overline{\psi}, e_k\rangle = \langle f, \psi e_k\rangle = 0$ for $k \in \mathbb{Z}^d$. Therefore $f\overline{\psi} = 0$ a.e. and since $f \in \{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$, $f = 0$ a.e. Hence $\varepsilon_{\mathcal{A}}(\psi)$ is complete in the set $\{f \in L^2(Q_{\mathcal{A}}) : f = 0$ a.e. on $N_\psi^{\mathcal{A}}\}$. □

**Theorem 2.5.** *Suppose that $\psi \in L^2(Q_{\mathcal{A}})$, then $\varepsilon_{\mathcal{A}}(\psi)$ is a frame sequence in $L^2(Q_{\mathcal{A}})$ if and only if there exist $A, B > 0$ such that $A \leq |\psi(\gamma)|^2 \leq B$ for a.e. $\gamma \notin N_\psi^{\mathcal{A}}$. In this case, the closed span of $\varepsilon_{\mathcal{A}}(\psi)$ is*

$$H_\psi^{\mathcal{A}} = \{f \in L^2(Q_{\mathcal{A}}) : \ f = 0 \text{ a.e. on } N_\psi^{\mathcal{A}}\},$$

*and $A$ and $B$ are frame bounds for $\varepsilon_{\mathcal{A}}(\psi)$ as a frame for $H_\psi^{\mathcal{A}}$.*

**Proof.** Let $A$ and $B$ be frame bounds for $\overline{span}\varepsilon_{\mathcal{A}}(\psi)$. Since $\varepsilon_{\mathcal{A}}(\psi)$ is a Bessel sequence, $|\psi|^2 \leq B$ a.e. Fix any $f \in H_\psi^{\mathcal{A}}$. Then $f\overline{\psi} \in L^2(Q_{\mathcal{A}})$. Since $\psi$ is bounded, $\varepsilon_{\mathcal{A}}(\psi)$ is a frame for $H_\psi^{\mathcal{A}}$. Using the fact that $e_k(\gamma) := e^{-2\pi i(\mathcal{A}^T)^{-1}k\cdot\gamma}$ is an orthonormal basis for $L^2(Q_{\mathcal{A}})$, we have

$$\begin{aligned} A\int_{Q_{\mathcal{A}}} |f(\gamma)|^2 d\gamma &= A\|f\|_{L^2}^2 \\ &\leq \sum_{k\in\mathbb{Z}^d} |\langle f, \psi e_k\rangle|^2 \\ &= \sum_{k\in\mathbb{Z}^d} |\langle f\overline{\psi}, e_k\rangle|^2 \\ &= \|f\overline{\psi}\|_{L^2}^2 \\ &= \int_{Q_{\mathcal{A}}} |f(\gamma)|^2 |\psi(\gamma)|^2 d\gamma. \end{aligned}$$

Since $f$ and $\psi$ both vanish on $N_\psi^{\mathcal{A}}$,

$$\int_{Q_{\mathcal{A}}\setminus N_\psi^{\mathcal{A}}} |f(\gamma)|^2 (|\psi(\gamma)|^2 - A) d\gamma \geq 0. \tag{3}$$

If $|\psi(\gamma)|^2 < A$ on any set $D \subseteq Q_{\mathcal{A}} \setminus N_\psi^{\mathcal{A}}$ of positive measure, then taking $f = \chi_D$ in inequality (3) leads to a contradiction. Hence we have $|\psi(\gamma)|^2 \geq A$ for a.e. $\gamma \notin N_\psi^{\mathcal{A}}$. □

The next Theorem shows the condition under which, $\tau_{\mathcal{A}}(f)$ is a frame sequence.

**Theorem 2.6.** *Let $f \in L^2(\mathbb{R}^d)$ and $A, B > 0$. Then $\tau_{\mathcal{A}}(f)$ is a frame sequence with bounds $A$ and $B$ if and only if $A \leqslant \Phi_f^{\mathcal{A}}(\gamma) \leqslant B$ for a.e. $\gamma \notin N_{\Phi_f^{\mathcal{A}}}$, where $N_{\Phi_f^{\mathcal{A}}} = \{f \in Q_{\mathcal{A}} : \Phi_f^{\mathcal{A}}(\gamma) = 0\}$.*

**Proof.** Suppose that $f \in L^2(\mathbb{R}^d)$ and define $V(f) := \overline{span}\{\tau_{\mathcal{A}}(f)\}$. We have to show that $\tau_{\mathcal{A}}(f)$ is a frame for $V(f)$ if and only if, for $\gamma \in \mathbb{R}^d$, $\{(E_{-\mathcal{A}k}|\hat{f}|)(\gamma)\}_{k\in\mathbb{Z}^d}$ is a frame for the set $\{F \in L^2(Q_{\mathcal{A}}) : F = 0$ a.e. on $N_{\Phi_f^{\mathcal{A}}}\}$. This happens if and only if $\{(E_{-\mathcal{A}k}|\hat{f}|)(\gamma)\}_{k\in\mathbb{Z}^d}$ is a frame sequence in $L^2(Q_{\mathcal{A}})$. Now the result follows from Theorem 2.5. □

**Remark 2.7.** *With the assumption in Theorem 2.6, we call the frame generated by $\tau_{\mathcal{A}}(f)$, the frame determined by $\Phi_f^{\mathcal{A}}$.*

The following corollary characterize the members of the subspace generated by a frame sequence of translates.

**Corollary 2.8.** *Suppose that $f \in L^2(\mathbb{R}^d)$ and $\tau_{\mathcal{A}}(f)$ is a frame sequence then a function $\psi \in L^2(\mathbb{R}^d)$ belongs to $\overline{span}\{\tau_{\mathcal{A}}(f)\}$ if and only if there exists a $\Lambda$-periodic function $F$ whose restriction to $Q_{\mathcal{A}}$, belongs to $L^2(Q_{\mathcal{A}})$ such that $\hat{\psi} = F\hat{f}$.*

**Proof.** Let $\psi \in \overline{span}\{\tau_{\mathcal{A}}(f)\}$. Then there exists $\{c_k\}_{k\in\mathbb{Z}^d} \subseteq l^2(Q_{\mathcal{A}})$ such that $\psi = \sum_{k\in\mathbb{Z}^d} c_k T_{\mathcal{A}k} f$. Hence $\hat{\psi} = \sum_{k\in\mathbb{Z}^d} c_k E_{-\mathcal{A}k}\hat{f}$. Thus $\hat{\psi} = F\cdot\hat{f}$ for $F = \sum_{k\in\mathbb{Z}^d} c_k E_{-\mathcal{A}k}$.

Conversely, suppose $\hat{\psi} = F\cdot\hat{f}$. Then $\psi = \hat{F}\cdot f$ and $\psi\big|_{Q_{\mathcal{A}}}$ is $\Lambda$-periodic. Hence $F \in L^2(Q_{\mathcal{A}})$ and $F = \sum_{k\in\mathbb{Z}^d} c_k E_{-\mathcal{A}k}$. $\square$

At this point, we recall the facts that a sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$ is a Riesz basis for $\mathcal{H}$ if and only if $\{f_k\}_{k=1}^{\infty}$ is a bounded unconditional basis for $\mathcal{H}$, and that the functions $e_k(\gamma) := e^{-2\pi i(\mathcal{A}^T)^{-1}k\cdot\gamma}$ are an orthonormal basis for $L^2(Q_{\mathcal{A}})$. These facts are applied in the following lemma to characterize when $\varepsilon_{\mathcal{A}}(\psi)$ is a Riesz basis for the Hilbert space $L^2(Q_{\mathcal{A}})$.

**Lemma 2.9.** *Given $\psi \in L^2(Q_{\mathcal{A}})$, it holds that $\varepsilon_{\mathcal{A}}(\psi)$ is an unconditional basis for $L^2(Q_{\mathcal{A}})$ if and only if there exist $A, B > 0$ such that $A \leq |\psi(\gamma)| \leq B$ for a.e. $\gamma$. In this case $\varepsilon_{\mathcal{A}}(\psi)$ is a Riesz basis for $L^2(Q_{\mathcal{A}})$.*

**Proof.** Suppose that $\varepsilon_{\mathcal{A}}(\psi)$ is a bounded unconditional basis for $L^2(Q_{\mathcal{A}})$. Since $\|\psi e_k\|_{L^2} = \|\psi\|_{L^2}$ for every $k$, it is a bounded unconditional basis and so is a Riesz basis. Since every Riesz basis is an exact frame, by Theorem 2.5, we have $A \leq |\psi(\gamma)|^2 \leq B$ a.e. $\square$

Using Lemma 2.9, we are able to show that $\tau_{\mathcal{A}}(f)$ is a Riesz sequence if and only if $A \leqslant \Phi_f^{\mathcal{A}} \leqslant B$ a.e. for some $0 < A \leq B < \infty$.

**Proposition 2.10.** *Let $f \in L^2(\mathbb{R}^d)$ and $A, B > 0$. Then $\tau_{\mathcal{A}}(f)$ is a Riesz sequence with bounds $A$ and $B$ if and only if $A \leqslant \Phi_f^{\mathcal{A}}(\gamma) \leqslant B$ for a.e. $\gamma \in Q_{\mathcal{A}}$.*

**Proof.** Suppose that $\tau_{\mathcal{A}}(f)$ is a bounded unconditional basis for $L^2(\mathbb{R}^d)$ and that $\|\Phi_f^{\mathcal{A}} e_k\|_{L^2} = \|\Phi_f^{\mathcal{A}}\|_{L^2}$ for every $k$. By Lemma 2.9, it is a bounded unconditional basis and so is a Riesz basis with $A \leq \Phi_f^{\mathcal{A}}(\gamma) \leq B$ a.e. □

In the next proposition, we determine the Fourier coefficients of the periodic function $\Phi_f^{\mathcal{A}}$.

**Proposition 2.11.** *For $f \in L^2(\mathbb{R}^d)$, the Fourier coefficients of $\Phi_f^{\mathcal{A}}$ are*

$$c_k = \int_{\mathbb{R}^d} f(\lambda)\overline{f(\lambda + k)}d\lambda, \quad k \in \mathbb{Z}^d.$$

**Proof.** Let $f \in L^2(\mathbb{R}^d)$. Then using the $\mathbb{Z}^d$-periodicity of $\Phi_f^{\mathcal{A}}$, the Fourier coefficients $c_n$ are

$$\begin{aligned}
c_n &= \int_{[0,1]^d} \Phi_f^{\mathcal{A}}(\gamma) \cdot e^{-2\pi i \gamma \cdot n} d\gamma \\
&= \frac{1}{|\det \mathcal{A}|} \int_{[0,1]^d} \sum_{k \in \mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \cdot e^{-2\pi i \gamma \cdot n} d\gamma \\
&= \frac{1}{|\det \mathcal{A}|} \int_{\mathbb{R}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma)\big)|^2 \cdot e^{-2\pi i \gamma \cdot n} d\gamma \\
&= \int_{\mathbb{R}^d} |\hat{f}(\gamma)|^2 \cdot e^{-2\pi i (\mathcal{A}^T)^{-1}\gamma \cdot n} d\gamma \\
&= \int_{\mathbb{R}^d} \hat{f}(\gamma) \cdot \overline{(E_{\mathcal{A}k}\hat{f})}(\gamma) d\gamma \\
&= \langle \hat{f}, T_{-\mathcal{A}k}\hat{f} \rangle \\
&= \int_{\mathbb{R}^d} f(\lambda)\overline{T_{-\mathcal{A}k} f(\lambda)} d\lambda \\
&= \int_{\mathbb{R}^d} f(\lambda)\overline{f(\lambda + k)} d\lambda.
\end{aligned}$$

□

**Remark 2.12.** *If $f$ has compact support, then Proposition 2.11, implies that only finitely many of the Fourier coefficients of $\Phi_f^{\mathcal{A}}$ are nonzero and therefore $\Phi_f^{\mathcal{A}}$ is continuous.*

In the following theorem, we determine conditions under which for $f \in L^2(\mathbb{R}^d)$ with compact support, $\tau_{\mathcal{A}}(f)$ is a Bessel sequence that cannot be an overcomplete frame sequence. This determines under which conditions for such $f$, $\tau_{\mathcal{A}}(f)$ is a Riesz sequence.

**Theorem 2.13.** *Assume that $f \in L^2(\mathbb{R}^d)$ has compact support. Then the following assertions hold:*

*i) $\tau_{\mathcal{A}}(f)$ is a Bessel sequence.*

*ii) $\tau_{\mathcal{A}}(f)$ cannot be an overcomplete frame sequence.*

**Proof.** i) By Remark 2.12, $\Phi_f^{\mathcal{A}}$ is continuous. Since $(\mathcal{A}^T)^{-1}[0,1]^d$ is compact, $\Phi_f^{\mathcal{A}}$ is bounded. Therefore there exists $B > 0$ such that $|\Phi_f^{\mathcal{A}}(\gamma)| \leqslant B$ a.e.. Consequently, by Proposition 2.2, $\tau_{\mathcal{A}}(f)$ is a Bessel sequence.
For proving $(ii)$, by Theorem 2.6, $\tau_{\mathcal{A}}(f)$ is a frame sequence. Also $\Phi_f^{\mathcal{A}}$ is continuous. Hence Proposition 2.10 implies that it is a Riesz sequence; that is, it is a Riesz basis for $\overline{span}\{\tau_{\mathcal{A}}(f)\}$. Thus their members are linearly independent and therefore it is not overcomplete. □

**Corollary 2.14.** *Assume that $f \in L^2(\mathbb{R}^d)$ is compactly supported. Then $\tau_{\mathcal{A}}(f)$ is a Riesz sequence if and only if for every $\gamma \in \mathbb{R}^d$, there exists a $k \in \mathbb{Z}^d$ such that $\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big) \neq 0$.*

**Proof.** By Proposition 2.10, $\Phi_f^{\mathcal{A}}$ is continuous. Suppose that there exists $\gamma \in \mathbb{R}^d$ such that for every $k \in \mathbb{Z}^d$, $\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma+k)\big) = 0$. Then $\Phi_f^{\mathcal{A}} = 0$, which contradicts Proposition 2.10. Thus $\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big) \neq 0$.
For the converse, assume that for any $\gamma \in \mathbb{R}^d$, $\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big) \neq 0$. This implies that $\tau_{\mathcal{A}}(f)$ is a Riesz sequence. Now, $\Phi_f^{\mathcal{A}}(\gamma) \neq 0$ implies that $\Phi_f^{\mathcal{A}}$ is continuous and bounded. Thus $\Phi_f^{\mathcal{A}}(\gamma) > 0$; that is, it takes its minimum, which is strictly positive. Therefore, it is a Riesz sequence bounded from below. □

In what follows, using Lemma 2.15 and Proposition 2.16, we determine conditions under which $\{T_{\mathcal{A}k}f + T_{\mathcal{A}k+n}f\}_{k\in\mathbb{Z}^d}$ is a frame for $\overline{span}\{\tau_{\mathcal{A}}(f)\}$.

**Lemma 2.15.** *Let $\{\psi_k\}_{k\in\mathbb{Z}^d}$ be a Bessel sequence and complete in a Hilbert space $\mathcal{H}$ and let $n_0 \in \mathbb{Z}^d$ be given. Then $\{\psi_k + \psi_{k+n_0}\}_{k\in\mathbb{Z}^d}$ is complete.*

**Proof.** We prove the lemma for the case $d = 1$ and $n_0 = 1$. The general case is almost the same. Suppose that $\psi \in \mathcal{H}$ is arbitrary such that for every $k$, $\langle \psi, \psi_k + \psi_{k+1}\rangle = 0$. We show that $\psi = 0$. Indeed $\langle \psi, \psi_k + \psi_{k+1}\rangle = 0$ implies that $\langle \psi, \psi_k\rangle = -\langle \psi, \psi_{k+1}\rangle$. Then for every $k$, $|\langle \psi, \psi_k\rangle| = |\langle \psi, \psi_{k+1}\rangle|$. Hence $|\langle \psi, \psi_k\rangle|$ is constant. On the other hand, by our assumption, $\{\psi_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence. Therefore,

$$\sum_{k\in\mathbb{Z}} |\langle \psi, \psi_k\rangle|^2 \leqslant B\|\psi\|^2,$$

for some $B > 0$. Hence $\langle \psi, \psi_k\rangle = 0$, for all $k$. Thus $\psi = 0$ (because $\{\psi_k\}_{k\in\mathbb{Z}}$ is complete). □

**Proposition 2.16.** *For $f \in L^2(\mathbb{R}^d)$, let $\tilde{f} = f + T_{\mathcal{A}n} f$. Then $\Phi^{\mathcal{A}}_{\tilde{f}}(\gamma) = |1 + e^{-2\pi i(\mathcal{A}^T)^{-1}n\cdot\gamma}|^2 \Phi^{\mathcal{A}}_f(\gamma)$ and $\Phi^{\mathcal{A}}_{\tilde{f}}(\gamma)$ determines a frame for $V(f) := \overline{span}\{\tau_{\mathcal{A}}(f)\}$ if and only if $\Phi^{\mathcal{A}}_f(\gamma)$ determines a frame for $V(f)$ and $e^{-2\pi i(\mathcal{A}^T)^{-1}n\cdot\gamma} \neq -1$.*

**Proof.** By the definition of $\Phi^{\mathcal{A}}_{\tilde{f}}$, we have

$$\begin{aligned}
\Phi^{\mathcal{A}}_{\tilde{f}}(\gamma) &= \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |\mathcal{F}\tilde{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \\
&= \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |(\mathcal{F}f + \mathcal{F}T_{\mathcal{A}n} f)\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \\
&= \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big) + E_{-\mathcal{A}n}\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \\
&= \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big) \\
&\quad + e^{-2\pi i(\mathcal{A}^T)^{-1}n\cdot(\gamma+k)}\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \\
&= \frac{1}{|\det \mathcal{A}|} \sum_{k\in\mathbb{Z}^d} |1 + e^{-2\pi i(\mathcal{A}^T)^{-1}n\cdot(\gamma+k)}|^2 \cdot |\hat{f}\big((\mathcal{A}^T)^{-1}(\gamma + k)\big)|^2 \\
&= |1 + e^{-2\pi i n\cdot(\mathcal{A}^T)^{-1}\gamma}|^2 \Phi^{\mathcal{A}}_f(\gamma).
\end{aligned}$$

Therefore using Theorem 2.5, there exist $A, B > 0$ such that $A \leqslant \Phi^{\mathcal{A}}_{\tilde{f}}(\gamma) \leqslant B$ for a.e. $\gamma \in Q_{\mathcal{A}} \setminus N$, where $N = \{\gamma \in Q_{\mathcal{A}} : \Phi^{\mathcal{A}}_{\tilde{f}}(\gamma) = 0\}$.
□

**Corollary 2.17.** *Let $f \in L^2(\mathbb{R}^d)$, $\tau_{\mathcal{A}}(f)$ be a frame for $V := \overline{span}\{\tau_{\mathcal{A}}(f)\}$ and $n \in \mathbb{Z}^d$. Then $\{T_{\mathcal{A}k}f + T_{\mathcal{A}k+n}f\}_{k\in\mathbb{Z}^d}$ is a frame for $V$ if and only if $2(\mathcal{A}^T)^{-1}\gamma \cdot n$ is not an odd number, for $\gamma \in [0,1]^d$.*

**Proof.** It is an immediate consequence of Lemmas 2.15 and 2.16. □

In the following we give an example in which $\tau_{\mathcal{A}}(f)$ is a Parseval frame, but it is not a Riesz sequence.

**Example 2.18.** Let $f \in L^2(\mathbb{R}^d)$ and let $\hat{f}(\gamma) = \chi_{(\mathcal{A}^T)^{-1}}\big[-\frac{1}{3},\frac{1}{3}\big]^d(\gamma)$ for $\gamma \in \mathbb{R}^d$. Then

$$\Phi_f^{\mathcal{A}}(\gamma) = \frac{1}{|\det \mathcal{A}|}\sum_{k\in\mathbb{Z}^d}\left|\chi_{(\mathcal{A}^T)^{-1}[-\frac{1}{3},\frac{1}{3}]^d}\big((\mathcal{A}^T)^{-1}(\gamma+k)\big)\right|^2,$$

$$2 = \chi_{(\mathcal{A}^T)^{-1}}\Big[-\frac{1}{3},\frac{1}{3}\Big]^d.$$

By Theorem 2.6, $\tau_{\mathcal{A}}(f)$ is a frame sequence with frame bounds $A = B = 1$ but does not form a Riesz sequence.

**Acknowledgements**
The authors sincerely thank the referee's for valuable comments, who suggested several improvements.

## References

[1] A. Aldroubi and C. Cabrelli and U. Molter, Wavelets on irregular grids with arbitrary dilation matrices, and frame atoms for $L^2(\mathbb{R}^d)$, *Appl. Comput. Harmon. Anal*, 17, (2004), 119-140.

[2] P. Balazs, C. Cabrelli, S. Heinken and U. Molter, Frames by multiplication, *Current Development in Theory and Applications of Wavelets*, 5 (2-3), (2011), 165-186.

[3] P. Balazs and H. Harbrecht, Frames for the solution of operator equations in Hilbert spaces with fixed dual pairing, *Numerical Functional Analysis and Optimization*, 40, (1), (2019), 65-84.

[4] P. Balazs and S. Heineken, An operator based approach to irregular frames of translates, *MDPI*, 7, no 5, (2019) Article Number 449.

[5] M. Bownik and E. Weber, Affine frames, GMRA'S and the canonical dual, *Studia Math*, 159, no 3, (2003), 453-479.

[6] P. G. Casazza, The art of frame theory, *Taiwanese J. Math*, 4, no 2,(2000), 129-201.

[7] O. Christensen, *An Introduction to Frames and Riesz Basis*, Birkhäuser. (2016).

[8] I. Daubechies, A. Grossmann and Y. Mayer, Painless non-orthogonal expansions, *J. Math. Physics*, 27, (1986), 1271-1283.

[9] R. J. Duffin and A. C. Schaeffer, A class of non-harmonic Fourier series, *Trans. Amer. Math. Soc*, 72, (1952), 341-366.

[10] S. J. Favier and R. A. Zalik, On stability of frames and Riesz bases, *Appl. Comp. Harm. Anal*, 2, (1995), 160-173.

[11] G. B. Folland, *A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics*, CRC Press, Boca Raton, FL, (1995).

[12] G. B. Folland, *Real Analysis Modern Techniques and Their Applications*, 2nd ed, (1999).

[13] D. Gabor, Theory of communications, *J. IEEE (London)*, 93, no. 3, (1946), 429-457.

[14] C. Heil,*A basis theory primer, Expanded edition (Applied and Numerical Harmonic Analysis)*, Birkhäuser/Springer, New York, (2011).

[15] C. E. Heil and D. F. Walnut, Continuous and discrete wavelet transform, *SSIAM Rev*, 31, no 4, (1989), 628-666.

[16] E. Matusiak, Frames of translates for model sets, *Applied and Computational Harmonic*, 57, (2021), 27-57.

**Fahimeh Valizadeh**
Department of Mathematics,
Central Tehran Branch, Islamic Azad University,
Tehran, Iran

E-mail: fahimevalizade@yahoo.com

**Hamidreza Rahimi**
Department of Mathematics
Professor of Mathematics
Central Tehran Branch, Islamic Azad University,
Tehran, Iran.
E-mail: rahimi@iauctb.ac.ir

**Rajab Ali Kamyabi Gol**
Department of Mathematics
Professor of Mathematics
Ferdowsi University of Mashhad, and Center of Excellence in Analysis on Algebraic Structures (CEAAS),
Mashhad, Iran.
E-mail: kamyabi@um.ac.ir

**Fatemeh Esmaeelzadeh**
Department of Mathematics
Assistant Professor of Mathematics
Bojnourd Branch, Islamic Azad University,
Bojnourd, Iran.
E-mail: esmaeelzadeh@bojnourdiau.ac.ir