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Original Research Paper

Semigroup Approach to Global Well-Posedness of the Biharmonic Newell-Whitehead-Segel Equation

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Abstract. The aim of the paper is to establish the global well-posedness of the Newell-Whitehead-Segel Equation driven by the biharmonic operator with Dirichlet boundary conditions through the semigroup method based on the Hille-Yosida Theorem. In particular, using the blow-up criterion we first demonstrate that there exists a unique local maximal classical solution. Next, by showing that the semiflow generated is uniformly bounded in $\mathcal{H}^4$-norm, it has been that the solution is indeed global in time.

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1 Introduction

In this paper, we aim to study the following version of the initial-boundary value problem consisting of the Newell-Whitehead-Segel Equa-

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tion (NWSE) driven by the biharmonic operator with Dirichlet boundary conditions,

$$\begin{aligned} u_t &= -k\Delta^2 u + au - bu^q, \quad (t,x) \in \mathbb{R}^+ \times \mathcal{O} \\ u &= \Delta u = 0 \qquad \text{on} \quad \Gamma = \partial\mathcal{O}, \\ u(0,x) &= u_0(x) \qquad \text{for} \;\; x \in \mathcal{O}. \end{aligned} \tag{1}$$

where a, b, and k be positive real numbers with $b > a$ and q be a positive odd integers, $\mathcal{O} \subset \mathbb{R}^n$, $n \leq 3$, bounded domain with smooth boundary $\Gamma = \partial\mathcal{O}$, $u_0 \in \mathcal{H}_0^2(\mathcal{O})$.

In nature, there are plenty of physical phenomena where stripe patterns emerge, [17], such as the skin of zebra, human fingerprints, visual cortex, ripples in the sand, and stripes of seashells. The existence and dynamics of stripe patterns are better explained by a family of evolution equations, known as Amplitude Equation (AE). Most fundamental equations in this family of amplitude equations are Newell-Whitehead-Segel equation (NSWE) cf. [20], [22] and Swift-Hohenberg equation (SHE) cf. [13]. The problem (1) of our interest also belongs to the family of Amplitude equation and is closely related to NWSE. The NWSE has been studied in a variety of contexts, in [1] it has been used to describe the process related to Bernard-Rayleigh convection of a mixture of fluid around a bifurcation point. Moreover, variants of NSWE type amplitude equations appear in variety of areas of physical sciences, such as, relativity [5], plasma physics [26], astrophysics [23], biological systems [18]. There is plenty of literature available for the numerical and analytical solution for this equation, but nothing is available on abstract existence, this is what the work makes the novel. We aim to employ the semigroup method to study the well-posedness of NWSE driven by a Biharmonic operator. For a detailed description of the semigroup method and Hille-Yosida Theorem, we refer to Chapter 2 of the [25].

Now we give the outline of this paper. Section 1 is a running introduction. In Section 2, we present some notation, necessary assumptions, and essential abstract results that would be needed in the later sections. Moreover, we have stated the key results (Theorem 2.4 and 2.5) in section 2. In Section 3, we prove Theorem 2.4 which is about the existence and uniqueness of the maximal local solution of Problem 2, whose particular case is our main Problem 1. We argue through Theorem 2.3

based on the semigroup method. Section 4 has been devoted to proving Theorem 2.5, concerning the global existence and uniqueness of solution for Problem 1.

2 Main Results, Assumptions, Framework and Abstract Theory

2.1 Notation and assumptions

Let us assume that $(\mathcal{E}, |\cdot|_{\mathcal{E}})$, $(\mathcal{V}, \|\cdot\|)$ and $(\mathcal{H}, |\cdot|_{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ constitutes a Gelfand triple i.e. nested abstract Hilbert spaces such that $\mathcal{E}$ continuously embedded into $\mathcal{V}$, and $\mathcal{V}$ is continuously embedded into $\mathcal{H}$. In particular, we will consider the following maximal regularity spaces

$$X_T \quad := \quad \mathcal{L}^2\big((0,T);\mathcal{E}\big) \cap C\big([0,T];\mathcal{V}\big),$$

with following norm,

$$\left|\left|u\right|\right|_{X_T}^2 \quad = \quad \sup_{t\in[0,T]} \|u(t)\|^2 + \int_0^T \left|\left|u(t)\right|\right|_{\mathcal{E}}^2 \, dt.$$

It is not difficult to verity that $\left(X_T, \left|\left|\cdot\right|\right|_{X_T}\right)$ is Banach space.

Assumption 2.1. Let $(\mathcal{E}, |\cdot|_{\mathcal{E}})$, $(\mathcal{V}, \|\cdot\|)$ and $(\mathcal{H}, |\cdot|_{\mathcal{H}}, \langle\cdot,\cdot\rangle)$ be as described above. Let $\{T(t) : t \geq 0\}$ be an analytic C_0 semigroup of bounded operators on $\mathcal{H}$.

i) There exists a constant $C_1 > 0$ such that for each $T > 0$ and $f \in \mathcal{L}^2(0,T;\mathcal{H})$, then for any $t \in [0,T]$ we have

$$\left|\left|\int_0^t T(t-s) f(s)\, ds\right|\right|_{X_T} \leq C_1 \left|\left|f\right|\right|_{\mathcal{L}^2(0,T;\mathcal{H})}.$$

ii) There exists a constant $C_2 > 0$ such that for each $T > 0$ and every $u_0 \in \mathcal{V}$, then for any $t \in [0,T]$, we have following estimate

$$\left|\left|T(t)u_0\right|\right|_{X_T} \leq C_2 \left\|u_0\right\|_{\mathcal{V}}.$$

Remark 2.2. In order to deal with Problem 1, we will consider following concrete spaces:

$$\mathcal{H} := \mathcal{L}^2(\mathcal{O}), \quad \mathcal{V} := D\left(A^{\frac{1}{2}}\right) = \mathcal{H}_0^{2,2}(\mathcal{O}), \quad \mathcal{E} := \mathcal{D}(A).$$

We assume that $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain and $n \in \mathbb{N}$ is such that

$$\mathcal{H}^{2,2}(\mathcal{O}) \subset \mathcal{L}^{q+1}(\mathcal{O}).$$

where q is any odd number. Further, we take A as the biharmonic operator with the Dirichlet boundary conditions, i.e.

$$\begin{aligned} \mathcal{D}(A) &= \mathcal{H}_0^{2,2}(\mathcal{O}) \cap \mathcal{H}^{2,4}(\mathcal{O}) \\ Au &= -\Delta^2 u, \;\; u \in \mathcal{D}(A). \end{aligned}$$

Indeed, $A \geq 0$ is a self-adjoint in $\mathcal{H}$ and $\mathcal{V} = \mathcal{D}\left(A^{\frac{1}{2}}\right)$ with following energy norm,

$$\|u\|^2 = \left|A^{\frac{1}{2}}u\right|_{\mathcal{H}}^2 = \int_{\mathcal{O}} |\Delta u(x)|^2 \, dx.$$

Further, the following embeddings are continuous,

$$\mathcal{E} \hookrightarrow \mathcal{V} \hookrightarrow \mathcal{H}.$$

2.2 Some useful abstract results

Following results gives the existence of local maximal solution and provides standard alternatives between the global existence and The proof of the following result can be found in the chapter 2 of [25].

Theorem 2.3. *[25] Let $A : X \to X$ be an abstract m-accretive operator with dense domain, where X is a Banach Space. Assume that $F : \mathcal{D}(A) \to \mathcal{D}(A)$ be nonlinear locally Lipchitz operator. For each $u_0 \in \mathcal{D}(A)$, there exists a unique local maximal solution $u \in \mathcal{C}^1([0, T^*), X) \cap \mathcal{C}([0, T^*), \mathcal{D}(A))$*

$$\begin{aligned} \frac{du}{dt} + Au &= F(u) \\ u(0, x) &= u_0(x), \;\; \textit{for all } x \in \mathcal{O}, \end{aligned}$$

such that exactly one of following is true:
(i) unique global solution exists i.e. $T^* = +\infty$.
(ii) solution blows up in finite time i.e. $T^* < +\infty$, *and*

$$\lim_{t \to T^*-0} ||u(t)||_{\mathcal{D}(A)} = +\infty.$$

2.3 Main result

In this subsection, we present the main results of the paper. Throught the paper, we suppose that we are in Assumptions 2.1, and follow the notation and framework introduced in Remark 2.2. Following abstract result is going to be pivotal for studying the blow-up criterion and the existence of the local maximal solution of the main problem 1.

Theorem 2.4. *Consider following problem*

$$\begin{cases} \frac{du}{dt} + k\Delta^2 u = F(u) \\ u(0,x) = u_0(x), \ for \quad all\ x \in \mathcal{O}. \end{cases} \tag{2}$$

Assume that F *nonlinear map of* $\mathcal{C}^q$*-class satisfying* $F(0) = 0$. *Then for* $n = 1, 2, 3$ *and each* $u_0 \in \mathcal{E}$, *there exists a unique local maximal solution to problem* (2) *with regularity*

$$u \in \mathcal{C}^1\left([0, T^*), \mathcal{H}\right) \cap \mathcal{C}\left([0, T^*), \mathcal{E}\right).$$

Moreover, only one of the following is holds:
i) Unique global solution exists i.e. $T^* = +\infty$.
ii) Solution blows up in finite time $T^* < +\infty$ *and* $\lim_{t \to T^*-0} ||u(t)||_{\mathcal{H}^4} = +\infty$.

The following theorem of that paper about the global well-posedness of Problem main problem 1.

Theorem 2.5. *Let* $\mathcal{O}$ *be bounded domain in* $\mathbb{R}^n$ *with* $n \leq 3$ *and* $\mu(\mathcal{O}) < \infty$, q *be odd positive integer with* $\frac{b}{q+1} \geq a$ *and* $a, b \in \mathbb{R}^+$, *then the following problem admits a unique global classical solution to problem* (1). *Furthermore*

$$u \in \mathcal{C}^1\left([0, \infty), \mathcal{H}\right) \cap \mathcal{C}\left([0, \infty), \mathcal{E}\right).$$

3 Existence of Local Maximal Classical Solution

This section has a twofold purpose. Firstly, to prove the abstract result Theorem 2.4. Secondly, by invoking Theorem 2.4 establish the existence of local maximal classical solution of our main problem (1).
Let us begin by proving the Theorem 2.4.
Proof. We will adopt the following setting: $X = \mathcal{L}^2(\mathcal{O}) = \mathcal{H}$, $A = k\Delta^2$ with Dirichlet boundary conditions $\mathcal{E} = \mathcal{D}(A) = \mathcal{H}^4 \cap \mathcal{H}_0^2$. First we will show that F is a mapping from $\mathcal{E}$ to $\mathcal{E}$.
Let us take $u \in \mathcal{E}$. Moreover, it follows that (cf. [12, 25]) $||\nabla u||_{\mathcal{H}}, ||\Delta u||_{\mathcal{H}}$ and $||\Delta^2 u||_{\mathcal{H}} < +\infty$.
Also since $F \in \mathcal{C}^q(\mathbb{R})$. So, $\sup_{t\in\mathbb{R}} |F^k(u(t))| < +\infty$, for $k = 1, 2, 3, ..., q$. Indeed,

$$\nabla F(u) = F'(u)\nabla u.$$

Therefore

$$\begin{aligned} ||\nabla F(u)||_{\mathcal{H}} &= ||F'(u)\nabla u||_{\mathcal{H}} = |F'(u)|\,||\nabla u||_{\mathcal{H}} \\ &\leq \sup_{t\in\mathbb{R}} |F'(u(t))|\,||\nabla u||_{\mathcal{H}} < +\infty. \end{aligned}$$

Similarly

$$\begin{aligned} \Delta F(u) &= F''(u)\nabla u \nabla u + F'(u)\Delta u \\ ||\Delta F(u)||_{\mathcal{H}} &= ||F''(u)\nabla u \nabla u + F'(u)\Delta u||_{\mathcal{H}} \\ &\leq ||F''(u)\nabla u \nabla u||_{\mathcal{H}} + ||F'(u)\Delta u||_{\mathcal{H}} \\ &= \sup_{t\in\mathbb{R}} |F''(u(t))| \sup_{t\in\mathbb{R}} |\nabla u(t)|\,||\nabla u||_{\mathcal{H}} \\ &\quad + \sup_{t\in\mathbb{R}} |F'(u(t))|\,||\nabla u||_{\mathcal{H}}. \end{aligned}$$

From Sobolev embedding $\mathcal{H}^2 \hookrightarrow C(\overline{\mathcal{O}})$ ([25, 12]), so $\sup_{t\in\mathbb{R}} |\nabla u(t)| < C||\nabla u||_{\mathcal{H}^2} < \infty$. It follows that,

$$||\Delta F(u)||_{\mathcal{H}} < \infty.$$

On same lines we will have $||\Delta^2 F(u)||_{\mathcal{H}} < \infty$. This concludes

$$\begin{aligned} ||\nabla F(u)||_{\mathcal{H}} &< +\infty \\ ||\Delta F(u)||_{\mathcal{H}} &< +\infty \\ ||\Delta^2 F(u)||_{\mathcal{H}} &< +\infty. \end{aligned}$$

This pushes $F(u)$ to be in $\mathcal{E} = \mathcal{H}^4(\mathcal{O}) \cap \mathcal{H}_0^2(\mathcal{O})$. In this way F is a non-linear map from $\mathcal{E}$ to $\mathcal{E}$. Since $F \in \mathcal{C}^q(\mathbb{R})$ so does $F \in C^3(\mathbb{R})$. Therefore, F is locally Lipschitz [25, 10]. Hence, we are in assumptions of Theorem 2.3 and thus we have the required result. □

As a consequence, we have the following immediate corollary.

Corollary 3.1. *For each $u_0 \in \mathcal{H}^4(\mathcal{O}) \cap \mathcal{H}_0^2(\mathcal{O})$, Problem 1 admits a unique local maximal classical solution u such that*

$$u \in \mathcal{C}^1([0, T^*), \mathcal{H}) \cap \mathcal{C}([0, T^*), \mathcal{E}).$$

Proof. Let us begin by setting

$$F(u) = au(t) - bu^q(t).$$

Then F is a q-th degree polynomial so $F \in \mathcal{C}^q(\mathbb{R})$ and hence we can apply Theorem 2.4 to conclude that we have unique maximal classical solution u to Problem (1) such that

$$u \in \mathcal{C}^1([0, T^*), \mathcal{H}) \cap \mathcal{C}([0, T^*), \mathcal{E}).$$

□

4 Global well-Posedness of the Solution

In this section, we aim to provide a detailed proof of the main result Theorem 2.5.

Proof. We already know from Corollary 3.1 that there exists a unique maximal classical solution u to problem (1) such that

$$u \in \mathcal{C}^1([0, T^*), \mathcal{H}) \cap \mathcal{C}([0, T^*), \mathcal{E}).$$

Therefore, for the global existence it is enough to prove that $\|u\,(t)\|_{\mathcal{H}^4}$ is uniformly bounded. This leads us to the conclusion that $\mathcal{H}_4$ -norm of the solution does not blow up in finite time i.e.

$$\lim_{t \to T^* - 0} \|u\,(t)\|_{\mathcal{H}^4} < +\infty.$$

As in this way condition ii) in Theorem (2.4) will fail to hold and condition i) would be true, and therefore the solution will be the unique global classical solution.
Let us beging by showing the uniform boundedness of $\|u\,(t)\|_{\mathcal{H}^4}$.
For $t \in [0, T^*]$ cosider following

$$u_t(t) + k\Delta^2 u(t) - au(t) + bu^q(t) = 0. \tag{3}$$

Taking inner product of (3) with $u_t(t)$, we get

$$\langle u_t(t) + k\Delta^2 u(t) - au(t) + bu^q(t), u_t(t)\rangle = 0.$$

i.e.

$$\langle u_t(t), u_t(t)\rangle + k\langle \Delta^2 u(t), u_t(t)\rangle - \langle au(t), u_t(t)\rangle + b\langle u^q(t), u_t(t)\rangle = 0. \tag{4}$$

Let us simplify the each of the term in above equation (4).

$$\frac{1}{2}\frac{d}{dt}||\Delta u(t)||^2 = \langle \Delta u(t), \Delta u_t(t)\rangle = \langle \Delta^2 u(t), u_t(t)\rangle. \tag{5}$$

Similarly, the second term in equation (4) can be simplified as following

$$\begin{aligned} b\langle u^q(t), u_t(t)\rangle &= b\int_{\mathcal{O}} \frac{d}{dt} u^{q+1}(t)dx = b(q+1)\int_{\mathcal{O}} u^q(t)u_t(t)dx \\ &= \frac{b}{q+1}\frac{d}{dt}\int_{\mathcal{O}} u^{q+1}(t)dx = \frac{b}{q+1}\frac{d}{dt}||u(t)||^{q+1}_{\mathcal{L}^{q+1}}. \end{aligned} \tag{6}$$

Further third term in equation (4) can be rewritten as

$$a\langle u, u_t(t)\rangle = \frac{a}{2}\frac{d}{dt}||u(t)||^2_{\mathcal{H}}. \tag{7}$$

Using equations (5), (6) and (7) in equation (4) it follows that

$$||u_t(t)||^2_{\mathcal{H}} + \frac{k}{2}\frac{d}{dt}||\Delta u(t)||^2_{\mathcal{H}} - \frac{a}{2}\frac{d}{dt}||u(t)||^2_{\mathcal{H}} + \frac{b}{q+1}\frac{d}{dt}||u(t)||^{q+1}_{\mathcal{L}^{q+1}} = 0.$$

On integrating from zero to t we get

$$\begin{aligned}
&\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} - \frac{a}{2}||u(t)||^2_{\mathcal{H}} + \frac{b}{q+1}||u(t)||^{q+1}_{\mathcal{L}^{q+1}} \\
= \ & \frac{k}{2}||\Delta u(0)||^2_{\mathcal{H}} - \frac{a}{2}||u(0)||^2_{\mathcal{H}} + \frac{b}{q+1}||u(0)||^{q+1}_{\mathcal{L}^{q+1}} \\
&\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} - \frac{a}{2}||u(t)||^2_{\mathcal{H}} + \frac{b}{q+1}||u(t)||^{q+1}_{\mathcal{L}^{q+1}} \\
\leq \ & +\frac{k}{2}||\Delta u_0||^2_{\mathcal{H}} + \frac{a}{2}||u_0||^2_{\mathcal{H}} + \frac{b}{q+1}||u_0||^{q+1}_{\mathcal{L}^{q+1}}. \qquad (8)
\end{aligned}$$

As we know that $u_0 \in \mathcal{E} = \mathcal{H}^4(\mathcal{O}) \cap \mathcal{H}^2_0(\mathcal{O})$ therefore $||\Delta u_0||^2_{\mathcal{H}} < +\infty$. Further, using the continuity of embedding $\mathcal{H}^2_0(\mathcal{O}) \hookrightarrow \mathcal{L}^{q+1}(\mathcal{O})$ it follows that

$$\frac{b}{q+1}||u_0||^{q+1}_{\mathcal{L}^{q+1}} < +\infty. \qquad (9)$$

From continuity of embedding $\mathcal{H}^2(\mathcal{O}) \hookrightarrow \mathcal{L}^2(\mathcal{O})$ (see [12, 25]) we have,

$$\frac{a}{2}||u_0||^2_{\mathcal{H}} < +\infty. \qquad (10)$$

Set

$$C := \frac{k}{2}||\Delta u_0||^2_{\mathcal{H}} + \frac{a}{2}||u_0||^2_{\mathcal{H}} + \frac{b}{q+1}||u_0||^{q+1}_{\mathcal{L}^{q+1}}. \qquad (11)$$

Using inferences (9), (10) and (11) in estimate (8) it follows that

$$\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} - \frac{a}{2}||u(t)||^2_{\mathcal{H}} + \frac{b}{q+1}||u(t)||^{q+1}_{\mathcal{L}^{q+1}} \leq C$$

$$\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} + \frac{b}{q+1}||u(t)||^{q+1}_{\mathcal{L}^{q+1}} \leq \frac{a}{2}||u(t)||^2_{\mathcal{H}} + C. \qquad (12)$$

Next, using the Young's inequality with $a = u^2$, $b = 1$, $p = \frac{q+1}{2}$, and $q := \frac{q+1}{q-1}$, we have

$$\begin{aligned}
\frac{a}{2}||u(t)||^2 &= \frac{a}{2}\int_{\mathcal{O}} u^2 dx \leq \frac{a}{2}\int_{\mathcal{O}} \left[\frac{2u^{q+1}}{q+1} + \frac{q-1}{q+1}\right] dx \\
&= \frac{a}{q+1}\int_{\mathcal{O}} u^{q+1} dx + C_1. \qquad (13)
\end{aligned}$$

where $C_1 = a\frac{q-1}{q+1}\mu(\mathcal{O})$. As $q+1 \leq 1$ so $\frac{a}{q+1} \geq a$. Using this into (13) we infer that

$$\frac{a}{2}||u(t)||^2 \leq a||u(t)||^{q+1}_{\mathcal{L}^{q+1}} + C_1.$$

Set $C_2 := C + C_1$. This way (12) yields

$$\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} + \left(\frac{b}{q+1} - a\right)||u(t)||^{q+1}_{\mathcal{L}^{q+1}} \leq C_2.$$

Since the third term in the last inequality is non-negative so we can drop it i.e.

$$\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds + \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} \leq C_2.$$

This leads to following two estimates

$$\int_0^t ||u_s(s)||^2_{\mathcal{H}} ds \leq C_2$$

$$\frac{k}{2}||u(t)||^2_{\mathcal{H}^2} = \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} \leq C_2.$$

for all $t \in [0, T^*)$.
Next set

$$v(t) := u(t+h)$$

As $u \in \mathcal{C}^1([0,T^*), \mathcal{H}) \cap \mathcal{C}([0,T^*), \mathcal{E})$ so we can deduce that

$$v \in \mathcal{C}^1([0,T^*-h), \mathcal{H}) \cap \mathcal{C}([0,T^*-h), \mathcal{E}).$$

And v also satisfies

$$\begin{aligned} v_t + k\Delta^2 v &= -bv^q + av(t) \\ v|_\Gamma &= 0 \\ v|_{t=0} &= u(x,h). \end{aligned} \tag{14}$$

Suppose $w := v(t) - u(t) = u(t+h) - u(t)$. Taking difference of (14) and (1), we infer that

$$\begin{aligned} &(v_t - u_t) + k\Delta^2(v-u) + b(v^q - u^q) - a(v-u) = 0 \\ &v|_\Gamma - u|_\Gamma = 0 \\ &v(0) - u(0) = u(x,h) - u_0(x). \end{aligned}$$

That is

$$\begin{aligned} & w_t + k\Delta^2 w + bw\left(\sum_{i=1}^{q} u^{q-1-i}v^i\right) - aw = 0 \\ & w|_{\Gamma} = 0 \\ & w|_{t=0} = u(x,h) - u_0(x). \end{aligned} \tag{15}$$

Taking inner product of w and (15)

$$\begin{aligned} \langle w_t, w\rangle + \langle k\Delta^2 w, w\rangle + b\left\langle w\left(\sum_{i=1}^{q} u^{q-1-i}v^i\right), w\right\rangle - a\langle w, w\rangle &= 0 \\ \frac{1}{2}\frac{d}{dt}||w||^2_{\mathcal{H}} + k||\Delta w||^2_{\mathcal{H}} + b\int_{\mathcal{O}} w^2\left(\sum_{i=1}^{q} u^{q-1-i}v^i\right)dx &= a||w||^2_{\mathcal{H}}. \end{aligned} \tag{16}$$

As q is odd natural so we can deduce inductively following

$$\left(\sum_{i=1}^{q} u^{q-1-i}v^i\right) \geq 0.$$

From the last equation it follows that

$$b\int_{\mathcal{O}} w^2\left(\sum_{i=1}^{q} u^{q-1-i}v^i\right)dx \geq 0. \tag{17}$$

Further,

$$k||\Delta w||^2_{\mathcal{H}} \geq 0. \tag{18}$$

From the non-negativity in (17) and (18), we can infer that we can drop the second and third term in (16). Hence

$$\frac{1}{2}\frac{d}{dt}||w(t)||^2_{\mathcal{H}} \leq a||w(t)||^2_{\mathcal{H}}.$$

On integrating from zero to t we infer that

$$||w(t)||^2_{\mathcal{H}} \leq ||w(0)||^2_{\mathcal{H}} + 2a\int_0^t ||w(s)||^2_{\mathcal{H}}ds.$$

Application of Gronwall's inequality gives

$$||w(t)||^2_{\mathcal{H}} \leq ||w(0)||^2_{\mathcal{H}} \exp(2at) \leq ||w(0)||^2_{\mathcal{H}} \exp(2aT).$$

Set $C_T := \exp(2aT)$. So

$$||w(t)||^2_{\mathcal{H}} \leq C_T||w(0)||^2_{\mathcal{H}} = C_T||u(x,h) - u(x)||^2_{\mathcal{H}}. \tag{19}$$

Dividing both sides of (19) by $\mathcal{H}^2$ we obtain

$$\begin{aligned}
\frac{||w(t)||^2_{\mathcal{H}}}{\mathcal{H}^2} &\leq C_T \left\| \frac{u(x,h) - u(x)}{h} \right\|^2_{\mathcal{H}} \\
\frac{\|u(t+h) - u(t)\|^2_{\mathcal{H}}}{\mathcal{H}^2} &\leq C_T \left\| \frac{u(x,h) - u(x)}{h} \right\|_{h^2} \\
\left\| \frac{u(t+h) - u(t)}{h} \right\|^2_{\mathcal{H}} &\leq C_T \left\| \frac{u(x,h) - u(x)}{h} \right\|^2_{\mathcal{H}}.
\end{aligned}$$

Applying $t \to 0$ we get

$$||u_t(t)||^2_{\mathcal{H}} \leq C_T||u'(0)||^2_{\mathcal{H}}. \tag{20}$$

Recall from Problem 1

$$u_t(t) = -k\Delta^2 u(t) + au(t) - bu^q(t).$$

Therefore

$$u_t(0) = -k\Delta^2 u(0) + au(0) - bu^q(0) = -k\Delta^2 u_0 + au_0 - bu_0^q.$$

In this way (20) yields

$$\begin{aligned}
||u_t(t)||^2_{\mathcal{H}} &\leq C_T|| - k\Delta^2 u_0 + au_0 - bu_0^q||^2_{\mathcal{H}} \\
||u_t(t)||^2_{\mathcal{H}} &\leq C_T \left(k||\Delta^2 u_0||^2_{\mathcal{H}} + a^2||u_0||^2_{\mathcal{H}} + b^2||u_0^q||^2_{\mathcal{H}}\right).
\end{aligned} \tag{21}$$

Taking $a := u_0^{2q}, b = 1, p := \frac{1}{q}, q := 1-q$ and applying Young's inequality, it follows that

$$||u_0^q||^2_{\mathcal{H}} = \int_{\mathcal{O}} u_0^{2q} dx \leq \int_{\mathcal{O}} \left(\frac{(u_0^{2q})^{\frac{1}{q}}}{\frac{1}{q}} + \frac{1}{1-q} \right) dx \leq ||u_0||^2_{\mathcal{H}} + \frac{\mu(\mathcal{O})}{1-q}. \tag{22}$$

Inequalities (21) and (22) together imply

$$||u_t(t)||^2_{\mathcal{H}} \leq C_T \left(k||\Delta^2 u_0||^2_{\mathcal{H}} + a^2||u_0||^2_{\mathcal{H}} + b^2 q||u_0||^2_{\mathcal{H}} + \frac{\mu(\mathcal{O})}{1-q} \right). \quad (23)$$

Using fact that $u_0 \in \mathcal{E}$ and the continuity of embedding $\mathcal{E} \hookrightarrow \mathcal{H}$ (cf. [12, 6]), it follows that

$$||u_0||^2_{\mathcal{H}} \leq Ck||\Delta^2 u_0||^2_{\mathcal{H}} < +\infty. \quad (24)$$

From inequalities (23) and (24) we may conclude existence of constant $C(T, u_0, k, a, b)$ such that

$$||u_t(t)||_{\mathcal{H}} \leq C(T, u_0, k, a, b) < +\infty. \quad (25)$$

This may be also interpreted as that u_t is uniformly bounded in $\mathcal{L}^\infty(0, T, \mathcal{H})$. Recall (14), along with (25) concludes

$$\begin{aligned} \int_0^t ||u_s(s)||^2_{\mathcal{H}} ds &\leq C_2 \\ \frac{k}{2}||u(t)||^2_{\mathcal{H}^2} = \frac{k}{2}||\Delta u(t)||^2_{\mathcal{H}} &\leq C_2. \end{aligned}$$

Therefore

$$||u_t(t)|| < +\infty.$$

Using Sobolev embedding theorem[25], there exists constant C_3 such that

$$||u(t)||_{\mathcal{H}^4} \leq C_3||\Delta^2 u||_{\mathcal{H}}. \quad (26)$$

From equation (1)

$$\frac{-u_t(t) + au(t) - bu^q(t)}{k} = \Delta^2 u(t). \quad (27)$$

Inequality (26) in the view of (27) produces

$$\begin{aligned} \|u(t)\|_{\mathcal{H}^4} &\leq C_3 \left\| \frac{-u_t(t) + au(t) - bu^q(t)}{k} \right\|_{\mathcal{H}} \\ &\leq \frac{C_3}{k} \left(||u_t(t)||_{\mathcal{H}} + |a|||u(t)||_{\mathcal{H}} + |b|||u^q(t)||_{\mathcal{H}} \right). \quad (28) \end{aligned}$$

We now going to show that each term in right hand side of inequality (28) is finite and bounded. We already know from the (25) that first term is bounded. Let us deal with other terms.
For $t \in [0, T^*]$, as local maximal solution $u \in \mathcal{C}^1([0, T^*), \mathcal{H}) \cap \mathcal{C}([0, T^*), \mathcal{E})$. Therefore there exists a constant C_4 depending on T^* such that

$$||u(t)||_{\mathcal{H}} \leq C_4 < +\infty. \tag{29}$$

Taking $a := u(t)^{2q}, b = 1, p := \frac{1}{q}, q := 1 - q$ and applying Young's inequality, consider following,

$$\begin{aligned} ||u(t)^q||^2_{\mathcal{H}} &= \int_{\mathcal{O}} u(t)^{2q} dx \leq \int_{\mathcal{O}} \left(\frac{(u(t)^{2q})^{\frac{1}{q}}}{\frac{1}{q}} + \frac{1}{1-q} \right) dx \\ &\leq ||u(t)||^2_{\mathcal{H}} + \frac{\mu(\mathcal{O})}{1-q}. \end{aligned} \tag{30}$$

In the view of (30), (25), and (29). We conclude that each term on right hand side of (28) is finite so we can conclude that

$$||u(t)||_{\mathcal{H}^4} < +\infty.$$

This way condition (ii) in Theorem 2.4 fails so condition (i) holds and we have $T^* = +\infty$. Thus, the unique local maximal classical solution u to Problem 1 (from Theorem 2.4) is also global, such that

$$u \in \mathcal{C}^1([0, \infty), \mathcal{H}) \cap \mathcal{C}([0, \infty), \mathcal{E}).$$

This concludes the proof of Theorem 2.5. $\square$

4.1 Numerical example:

Now we present a simple numerical example of the one dimensional problem on $\mathcal{O} = (0, 1)$. By making a choice of $k = 1, a = 1, b = 2$ and $q = 1$, the evolution equation in Problem 1 reduces to following

$$u_t = -\Delta^2 u - u.$$

Making a choice of initial condition $u(x, 0) = u_0(x) = x^3 \in \mathcal{E} = \mathcal{H}^4 \cap \mathcal{H}^2_0$. Corresponding to this initial condition there exists a unique solution $u(x, t) = x^3 e^{-t}$. Its not difficult to see that this solution indeed belongs to $\mathcal{C}^1([0, \infty), \mathcal{H}) \cap \mathcal{C}([0, \infty), \mathcal{E})$.

5 Conclusion

In this paper we have done the analysis of an initial value problem, comprising of a biharmonic amplitude equation of NWS type subject to Dirichlet boundary conditions. Such evolution equations are a key tool to understand the stripe patterns appearing in nature. In particular, we have investigated global well-posedness. Using a standard result from the semigroup theory of Hille-Yosida operators, we demonstrated there exists a unique local maximal classical solution to the problem that is locally maximal and classical in suitable function spaces. Finally, it was shown that $\mathcal{H}^4$-norm is uniformly bounded and a solution does not blow up in finite, which leads to the conclusion that the solution of the problem under consideration is global.

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