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Original Research Paper

Coding Theorem Based on λ-Norm Entropy for Partitions in Product MV-Algebras

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Abstract. In the present paper Shannon and λ-norm mean code word length is defined in Product MV-algebra. Two new measures $L^f(P)$ and $L^f_\lambda(P)$ called average code word lengths with respect to entropies of finite partitions in product MV-algebras are given and its relationship with the Shannon type information measure and λ-norm type information measures of finite partitions in product MV-algebras has been examined. Some coding theorems using Kraft inequality has been eastablished in this structure.

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1 Introduction

The concept of MV-algebras were given by Chang [4]. Various reseachers have investigated the notion MV-algebras. In this structure there are several results regarding information measure; as an example, reader can

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consult the references ([8], [21] and [12]). With this notion probability theory was studied on by authors [22]. Riečan [20] independently studied the concept of Product MV-algebra in concern with probability and and Montagna [15] studied ths concept with mathematical logic. For this concept reader can also see references [7] and [9]. Some families of fuzzy sets have also been generalized by this notion [25].

Further, Petrovičová [16] (see also [17]), developed the concept of Shannon measure for a finite partition in product MV-algebra and derived a number of basic results for this measure. Recently, Markechová et.al [12] studied the logical entropy, the logical divergence and the logical mutual information of partitions in this structure. Dagmar Markechova and Abolfazl Ebrahimzadeh [14] have generalized the results related to the Shannon measure and Kullback-Leiber divergence in this notion. Markechová and Reičan [11], Zarenezhad and Jamaljadeh [26] have generalized Tsallis entropy, R-norm measure and R-norm divergence using this notion and derived its basic properties.

Shannon's entropy [23] for the discrete probability distribution $E = \{e_1, e_2, \ldots, e_k\}$, $e_j \geq 0$, $\sum_{j=1}^{k} e_j = 1$, is given by

$$\chi(E) = -\sum_{j=1}^{k} (e_j) \log(e_j),$$

where the base of the logarithm is in general arbitrary. There is an important link between entropy and noiseless coding. If $Y = (y_1, y_2, \ldots, y_k)$ represents an information source with k messages and input probabilities $e_1, e_2, \ldots, e_k$, as given above, that is encoded into words of lengths $M = (m_1, m_2, \ldots, m_k)$ forming an instantaneous code, then

$$\sum_{j=1}^{k} S^{-m_j} \leq 1.$$

where S is the size of the alphabet to be coded. Also, if

$L = \sum_{j=1}^{k} e_j m_j$ is the mean codeword length, then

$$\sum_{j=1}^{k} e_j m_j \geq -\sum_{j=1}^{k} e_j \log_S e_j, \tag{1}$$

in (1) equality holds if and only if

$$m_j = -\log_S(e_j).$$

Taking appropriate encoded words of large sequences, we can make the mean length approximate close to $\chi(E)$, (see [5]). This result is known as Shannon's noiseless coding theorem.

Campbell [3] proved certain results similar to (1) in coding theory corresponding to Renyi's measure (see [18]), and established the bounds for mean code word length in terms of Renyi's measure.

A class rule was developed by Kieffer [10] to encode two source of sequence of length M, for $M \to \infty$ with expected cost and eastablished that Renyi's entropy is the best alternative out of these, for which it is supposed that the encoding cost of a message depends upon size only. Also Jelinek [6] eastablished that to minimize the problem of buffer overflow, Campbell's mean length [3] is more appropriate.

The issue of storage and transmission is investigated by several researchers in terms of coding theorem by taking various information measure with the condition of unique decipherability concerning discrete memoryless sources possessing an additional parameters, reader can see ([5], [24]).

In this article, we introduced an appropriate measure called the Shannon and λ-norm average length of code words for partitions in product MV-algebras and proved some noiseless coding theorems in finite partitions scheme. It is proven that the proposed average code word length is bounded below by the entropy defined for this scheme. This is illustrated with a numerical example.

2 Preliminaries

Here some primary terms and definitions are given which will be used in this paper.

Definition 2.1. ([21]) *An algebraic system* $(V, \oplus, *, 0, w)$ *which elate* *(i) there exists a commutative lattice ordered group* $(D, +, \leq)$ *such that* $V = [0, w] = \{\eta \in D : 0 \leq \eta \leq w\}$*, where 0 is the neutral element of* $(D, +)$ *and* $w \in D$ *such that* $w > 0$ *and to each* $\eta \in D$ *there exist a positive integer* k *such that* $\eta \leq kw$*;*

(ii) $\oplus, *$ *are binary operation on* V *such that* $\eta \oplus \xi = (\eta + \xi) \wedge w$*,* $\eta * \xi = (\eta + \xi - w) \vee 0$*.*

is known as MV-Algebra.

Definition 2.2. ([22]) *A maping* $f : V \to [0, 1]$ *which satifies*

(i) $f(w) = 1$;

(ii) if $v_1, v_2 \in V$ *such that* $v_1 + v_2 \leq w$*, then* $f(v_1 + v_2) = f(v_1) + f(v_2)$*.*

is known as a state on an MV-algebra $(V, \oplus, *, 0, w)$*.*

Definition 2.3. ([19]) *An MV-algebra* $(V, \oplus, *, 0, w)$ *which satisfies*

(i) For every $v \in V$*,* $w \cdot v = v$*;*

(ii) If $v_1, v_2, v_3 \in V$ *such that* $v_1 + v_2 \leq w$*, then* $v_3 \cdot v_1 + v_3 \cdot v_2 \leq w$ *and* $v_3 \cdot (v_1 + v_2) = v_3 \cdot v_1 + v_3 \cdot v_2$*,*

where . is a binary operation whcch is commutative on V *is known as a product MV-algebra.*

For the sake of simplicity, we will write $(V, \cdot)$ instead of $(V, \oplus, *, 0, w)$. Riečan [20] studied this concept with probability theory. Authors [16], [17] and [21] defined Shannon measure for the notion product MV-algebras. Here we mention the basic idea and some results of these concepts that which will be used in this manuscript.

Any k-tuple $P = (\xi_1, \xi_2, \ldots, \xi_k)$, $\xi_i \in V$ which sayisfies $\xi_1 + \xi_2 + \ldots + \xi_k = w$ is known as partition in a product MV-algebra $(V, \cdot)$.

Definition 2.4. [16] *Let* $P = (\xi_1, \xi_2, \ldots, \xi_k)$ *be a partition in the structure* $(V, \cdot)$ *and let* $f : V \to [0, 1]$ *be state on* $(V, \cdot)$*. Corresponding to Shannon measure for the partition* P *in relation to* f *is defined as*

$$H^f(P) = -\sum_{j=1}^{k} f(\xi_j) \log f(\xi_j). \tag{2}$$

This function satisfies all properties analogous to properties of Shannon's measure of measurable partitions, for more details reader can see Petrovicova [16].

Remark 2.5. *Here logarithmic base can be any positive integer with the convention* $0 \log_B(\frac{0}{\xi}) = 0$ *if* $\xi \geq 0$.

3 Measure of Code Length in Product of MV-algebras

In this section we will prove some coding results analogous to the Shannon and λ-norm entropies of partitions in product MV-algebras. For it consider $P = (x\xi_1, \xi_2, \ldots, \xi_k)$ be a partion in the structure $(V, \cdot)$ and $f : V \to [0, 1]$ be a state. Let $f(\xi_1), f(\xi_2), \ldots, f(\xi_k)$ be the state values of k input symbols $\xi_1, \xi_2, \ldots, \xi_k$ which we wish to encode. We assume that $f(\xi_j) > 0$ for $j = 1, 2, \ldots, k$ such that $\sum_{j=1}^{k} f(\xi_j) = 1$. Let the input symbols are encoded in an alphabet with S symbols. . Let ξ_j, be represented by a string of m_j morphemes from the alphabet. For a unique decipherable code with lengths $m_1, \ldots, m_k$, Feinstein [5] gives the following relation

$$\sum_{j=1}^{k} S^{-m_j} \leq 1. \tag{3}$$

Next we define mean code word length $L^f(P)$ of P with respect to state f corresponding to the definition of Shannon (see [1]).

Definition 3.1. *The mean code word length for partition* $P = (\xi_1, \xi_2, \ldots, \xi_k)$ *in a product MV-algebra* $(V, \cdot)$ *with respect to state* $f : V \to [0, 1]$ *on* $(V, \cdot)$. *is defined as:*

$$L^f(P) = \sum_{j=1}^{k} f(\xi_j) m_j,$$

where $m_j, j = 1, 2, \ldots, k$ *are the lengths of the code words* ξ_j.

Theorem 3.2. *Let $P = (\xi_1, \xi_2, \ldots, \xi_k)$ be a partition in the structure $(V, \cdot)$ and f be state on $(V, \cdot)$. If $m_j, j = 1, 2, \ldots, k$ are the lengths of the codewords ξ_j satisfying (3), then*

$$\frac{H^f(P)}{\log S} \leq L^f(P) < \frac{H^f(P)}{\log S} + 1.$$

In this result equality will holds iff

$$f(\xi_j) = S^{-m_j} \quad j = 1, 2, 3, \ldots, k,$$

where $S > 1$ is a positive integer.

Proof. Proof are simple and omitted. □

Definition 3.3. *The mean code word length $L^f_\lambda(P)$ for a partition $P = (\xi_1, \xi_2, \ldots, \xi_k)$ in the structure $(V, \cdot)$ with respect to state f on $(V, \cdot)$ corresponding to the λ-norm entropy of partitions in this structure is given by*

$$L^f_\lambda(P) = \frac{\lambda}{\lambda - 1}[1 - \sum_{j=1}^{k} f(\xi_j) S^{-(m_j(\lambda-1))/\lambda}],$$

$\lambda \in (0, 1) \cup (1, \infty)$.

Clearly $L^f_\lambda(P)$ will increase for increasing word lengths. For $\lambda \to 1$, $L^f_\lambda(P)$ is analogous to the mean code word length by Shannon type, up to a constant.

In the following theorem we eastalished this relationship.

Theorem 3.4. *Let $P = (\xi_1, \xi_2, \ldots, \xi_k)$ be a partition in the structure $(V, \cdot)$ and f be state on $(V, \cdot)$. If $m_j, j = 1, 2, \ldots, k$ are the lengths of the codewords ξ_i then*

$$\lim_{\lambda \to 1} L^f_\lambda(P) = \sum_{j=1}^{k} f(\xi_j) m_j \log(S).$$

Proof. By L'Hospital's rule, we have

$$\lim_{\lambda \to 1} \frac{\phi(\lambda)}{\psi(\lambda)} = \lim_{\lambda \to 1} \frac{\phi'(\lambda)}{\psi'(\lambda)} \tag{4}$$

if $\phi(1)=\psi(1)=0$, and $\phi(\lambda)$ and $\psi(\lambda)$ are differentiable and $\lim_{\lambda\to 1}\frac{\phi'(\lambda)}{\psi'(\lambda)}$ exist. If we set $\phi(\lambda)=\lambda\left\lfloor 1-\sum_{j=1}^{k} f(\xi_j)S^{(M,(\lambda-1))/\lambda}\right\rfloor$ and $\psi(\lambda)=\lambda-1$ these functions satisfies the conditions of (4).
We find

$$\lim_{\lambda\to 1}\left[\sum_{j=1}^{k} f(\xi_j)S^{(m_j(\lambda-1))/\lambda}\log(S)m_j\lambda^{-1}\right]=\sum_{j=1}^{k} f(\xi_j)m_j\log(S).$$

□ Before proceeding to the coding theorem we need following definition of λ-norm entropy of partitions in product of MV-algebras given by Dagmar and Abolfazl [14].

Definition 3.5. *Let $P=(\xi_1,\xi_2,\dots,\xi_k)$ be any partition in the given structure $(V,\cdot)$. Then λ-norm entropy of P with respect to a state $f: V\to[0,1]$ is given by*

$$H_\lambda^f(P)=\frac{\lambda}{\lambda-1}\left[1-\left(\sum_{j=1}^{k} f(\xi_j)^\lambda\right)^{\frac{1}{\lambda}}\right], \tag{5}$$

$\lambda\in(0,1)\cup(1,\infty)$.
Concerning (5) the reader can consult [13].

The following theorem gives a relationship beween $L_\lambda^f(P)$ and $H_\lambda^f(P)$ as a coding theorem.

Theorem 3.6. *If $m_j, j=1,2,\dots,k$ are the lengths of the codewords ξ_j which satisfies equation (3) and $P=(\xi_1,\xi_2,\dots,\xi_k)$ be a partion in the structure $(V,\cdot)$ and f be state on $(V,\cdot)$. , then*

$$H_\lambda^f(P)\le L_\lambda^f(P)<S^{(1-\lambda)/\lambda}H_\lambda^f(P)+\frac{\lambda}{\lambda-1}[1-S^{(1-\lambda)/\lambda}]. \tag{6}$$

Equal sign holds iff

$$S^{-M}=f(\xi_j)^\lambda/\sum_{j=1}^{k} f(\xi_j)^\lambda,\;\; j=1,2,3,\dots,k.$$

Proof. With the help of reverse Holder's inequality, we can eastablished the lower bound of (6) as

$$\left[\sum_{j=1}^{k} f(\xi_j)^{\lambda} f(y_j)^{(\lambda-1)/\lambda}\right]^{\lambda/(\lambda-1)} \cdot \left[\sum_{j=1}^{k} f(\xi_j)^{\lambda}\right]^{1/(1-\lambda)} \leq 1, \quad \lambda \in \Re. \tag{7}$$

If $\lambda > 1$ then from (7) we have

$$\left[\sum_{j=1}^{k} f(\xi_j)^{\lambda}\right]^{1/\lambda} \geq \sum_{j=1}^{k} f(\xi_j)^{\lambda} (y_j)^{(\lambda-1)/\lambda}. \tag{8}$$

Since $\frac{\lambda}{(\lambda-1)} > 0$ for $\lambda > 1$, from (8), using definition of $H_{\lambda}^{f}(P)$ and putting $f(y_j) = S^{-m_j}, \quad j = 1, 2, \ldots k$, we have

$$H_{\lambda}^{f}(P) \leq L_{\lambda}^{f}(P), \tag{9}$$

in this equation equal sign holds iff

$$S^{-m_j} = f(\xi_j)^{\lambda} / \sum_{j=1}^{k} f(\xi_j)^{\lambda}, \;\; j = 1, 2, \ldots k. \tag{10}$$

For $0 < \lambda < 1$ we can prove inequality (9) in a same manner by reversing the inequality sign of (7), and $\frac{\lambda}{\lambda-1} < 0$ for $0 < \lambda < 1$.

To prove upper bound for $H_{\lambda}^{f}(P)$, we use (10). Equation (10) is equivalent to

$$m_j = -\log_S f(\xi_j)^{\lambda} + \log_S [\sum_{j=1}^{k} f(\xi_j)^{\lambda}], \quad j = 1, 2, \ldots k.$$

By selecting all m_j such that

$$-\log_S f(\xi_j)^{\lambda} + \log_S \left[\sum_{j=1}^{k} f(\xi_j)^{\lambda}\right] \leq m_j$$
$$< -\log_S f(\xi_j)^{\lambda} + \log_S \left[\sum_{j=1}^{k} f(\xi_j)^{\lambda}\right] + 1.$$

it implies that

$$S^{-m_j} = f(\xi_j)^{\lambda} / \sum_{j=1}^{k} f(\xi_j)^{\lambda} S. \tag{11}$$

For $\lambda > 1$, from (11), we have

$$\sum_{j=1}^{k} f(\xi_j)^{(m_j(\lambda-1))/\lambda} > \left[\sum_{j=1}^{k} f(\xi_j)^{\lambda} S^{1-\lambda} \right]^{1/\lambda},$$

which further implies

$$L_{\lambda}^{f} < \frac{\lambda}{1-\lambda} \left[1 - \left(\sum_{j=1}^{k} f(\xi_j)^{\lambda} \right)^{1/\lambda} S^{(1-\lambda)/\lambda} \right].$$

which proves the result for $\lambda > 1$.

For $0 < \lambda < 1$, upper bound of $L_{\lambda}^{f}(P)$ can be proved in similar way.

Since $S \geq 2$ we have $\frac{\lambda}{(\lambda-1)}[1 - S^{(1-\lambda)/\lambda}] > 1$ it implies that the upper bound of $L_{\lambda}^{f}(P)$ in (6) is > 1.

For $\lambda \to 1$ then by using (2) and (5), (6) can be written as

$$\frac{H^f(P)}{\log(S)} \leq L^f(P) < \frac{H^f(P)}{\log(S)} + 1,$$

which is relatable to acclaimed result developed by Shannon(see [1]).

□

Huffman established a measure for obtaining a changeable length source code which gives results similar to results eastablishe by Shannon. The average length $L = \sum_{j=1}^{k} e_j m_j$ of Huffman code for single codeword lengths m_j, satisfies the relation $\chi(E) \leq L < \chi(E) + 1$ with Shannon's measure .This can also be eastablished with partition in a product MV-algebra $(V, \cdot)$, for this structure we have

$$L_{\lambda}^{f}(P) \geq H_{\lambda}^{f}(P).$$

Following example will illustrate this fact.

Example 3.7. Suppose the family V of measurable functions $\xi : [0,1] \to [0,1]$, and binary oeration . is defined as the natural product of fuzzy sets in V. Then the algebraic system $(V, .)$ is a product MV-algebra. Also let $f : V \to [0,1]$ be a state defined by $f(\xi) = \int_0^1 \xi(y)dy$, for every $\xi \in V$, and consider the partition $P = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$, where $\xi_1(y) = 3y^9$, $\xi_2(y) = y^3$, $\xi_3(y) = y^4$, $\xi_4(y) = y^9$, $\xi_5(y) = y^9$, $\xi_6(y) = y^{19}$, for every $y \in [0,1]$. With basic computations we have state values 0.3, 0.25, 0.2, 0.1, 0.1, 0.05 of the corresponding elements, respectively. The partition $P = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)$ has the state values (0.3 , 0.25 ,0.2 , 0.1 , 0.1 , 0.05) of the corresponding elements. A relationship between the entropy $H_\lambda^f(P)$ and average codeword length $L_\lambda^f(P)$ is obtained in Tabel 1 for Huffman codes scheme. It is clear from Table 1 that the mean codeword length $L_\lambda^s(P)$ exceeds the entropy $H_\lambda^f(P)$.

Table 1: Relation Between$H_\lambda^f(P)$ and $L_\lambda^f(P)$

m_i	Huffman code words	$f(\xi_i)$	λ	$H_\lambda^f(P)$	$L_\lambda^f(P)$	ρ	σ
2	00	0.3	2	1.0726	1.1042	0.97	0.03
2	10	0.25					
2	11	0.25					
3	011	0.1					
4	0100	0.1					
4	0101	0.05					

In Table 1 m_i is the lengths of Huffman code words, $\rho = H_\lambda^f(P) \div L_\lambda^f(P)$ is the code adequacy, and $\sigma = 1 - \rho$ is the overabundance of the code.

4 Conclusion

As a conclusion we remarked that the optimal code length is that code for which the value of mean codeword length that is $L_\lambda^f(P)$ in product MV-algebra is equal to its lower bound. From Theorem 3.2 we conclude that optimal code lengths in product MV-algebra are depends on parametric value λ in opposition to the optimal codelengths of Shannon in

MV product algebras. It is also achievable to prove coding results correspinding to λ-norm entropy in product MV-algebras such that optimal codelengths are similar to those of Shannon in product MV-algebras.The Shannon mean codelength in product MV-algebra is included in mean codelength with respect to λ-norm entropy in this structure for the limiting case that $\lambda \to 1$. This new mean codelength is generalization of classical case in product MV-algebras its further applications can be seen in fuzzy theory.

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