An extended Legendre wavelet method for solving differential equations with non analytic solutions

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Abstract

Although spectral methods such as Galerkin, Tau and pseudospectral methods do not work well for solving ordinary differential equations in which, at least, one of the coefficient functions or solution function is not analytic [1], but it is shown that the Legendre wavelet Galerkin method is suitable for solving this kind of problems provided that the singular points have the form 2^{-k} for some positive integer k [4]. However, for the other type of singular point the Legendre wavelet basis are not an efficient method. To overcome this difficulty, in this study we use the extended Legendre wavelet basis and Tau method for solving a wide range of singular boundary value problems. The convergence properties and error analysis of the proposed method is investigated. A comparison between the standard Legendre wavelets and extended Legendre wavelets methods shows the capability of the proposed method.

Keywords: Extended Legendre wavelets, Operational matrix, Tau method, Bundary value problems, Convergency, Error analysis.

1 Introduction

In recent years, different basis functions such as orthogonal functions and wavelets have been used to approximate solutions of functional equations. Depending on the structure, the orthogonal functions may be widely classed in three families. The first includes of sets of piecewise constant basis functions (such as the Walsh functions, block pulse functions, etc.). The second consists of sets of orthogonal polynomials (such as Legendre polynomials and Chebyshev polynomials, etc.). The third is the widely used sets of sine-cosine functions in Fourier series. It is worth noting that approximating a continuous function with piecewise constant basis functions results in an approximation that is not continuous. On the other hand if a discontinuous function is approximated with continuous basis functions, the resulting approximation is continuous and can not properly model the discontinuities. Moreover, there are some functional equations that the solution vary continuously in some regions and discontinuously in others. Neither continuous basis functions nor piecewise constant basis functions taken alone can efficiently or accurately model these spatially varying properties. So, in order to properly approximate these spatially varying properties it is absolutely necessary to use approximating basis functions that can accurately model both continuous and discontinuous phenomena. For these situations, wavelet functions will be more effective. Wavelets possess several useful properties, such as orthogonality, compact support, exact representation of polynomials to a certain degree, and the ability to represent functions at different levels of resolution. It is also worth noting that the wavelets method allows the creation of very fast algorithms when compared to the algorithms ordinary used. This is due to specific attributes when they are used as basis functions.

In this paper, we consider the boundary value problem (BVP) [1, 4]:

$$y''(t) + f(t)y'(t) + g(t)y(t) = h(t), \tag{1}$$

with the boundary conditions:

$$y(a) = A, \quad y(b) = B, \tag{2}$$

where the solution $y(t)$ or coefficients $f(t), g(t)$ and $h(t)$ have some finite singular points in the interval $[0, 1]$. It can be shown that for a given ordinary boundary differential equation defined on the interval $[a, b]$, if solution and coefficients are analytic, spectral methods using orthonormal polynomials will be suitable for solving such problems and also it leads to spectral accuracy [1–3]. But when at least, one of the coefficient or solution is not analytic on $[a, b]$, these basis functions will not appropriate, so that the Gibbs phenomenon will happen around the singularity points [2, 3].

Wavelets method are very interesting to obtain approximate solutions of differential equations. It has been shown that in case that solution of problem has some singular points of the form 2^{-k}, for some positive integer k, the standard Legendre wavelets method will be a very suitable and efficient way for solving such problems with initial or boundary conditions. In [4] authors have used the Legendre wavelets Galerkin method for solving singular boundary ordinary differential equations and spectral accuracy was obtained in the cases that singular points were of the form 2^{-k} for some positive integer k. However, in cases that at least one of the coefficient functions or solution function of under consideration problem has a singular point which is not in the form 2^{-k}, the Legendre wavelets Galerkin method is not an efficient method for solution of it and also the approximate solution has not the spectral accuracy.

In this paper, an extended Legendre wavelets basis functions are introduced and their properties are described. Then, the operational matrix of derivatives for these basis functions is obtained and a general procedure for deriving this operational matrix is described. By using the extended Legendre wavelets, associated operational matrix of derivatives and Galerkin method a computational method for solving ordinary differential equations with boundary conditions is proposed. A comparison between the typical Legendre Galerkin wavelets method and the extended Legendre wavelets method for solving some numerical examples is performed. Convergence of the proposed method is investigated.

This paper is organized as follows: In section 2, the extended Legendre wavelets and their properties are presented. In section 3, shifted Legendre polynomials are introduced. In section 4, the operational matrix of derivative for the extended Legendre wavelets is obtained. In section 5, the proposed method is described. In section 6, some numerical examples are presented. Finally a conclusion is drawn in section 7.

2 Extended Legendre wavelets

In this section, the extended Legendre wavelets are presented and some their useful prpoperties are discussed.

2.1 A brief review of constructing the extended Legendre wavelets

The extended Legendre wavelets in the framework of the recursive wavelets construction given in [6] for piecewise polynomial spaces on $[0,1]$. For this purpose, we first introduce some notations. Throughout this work, $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}_\mu = \{0, 1, \ldots, \mu - 1\}$, for a positive integer μ.
For an integer $\mu > 1$, we consider the following contractive mappings on the interval $I = [0,1]$:

$$\psi_\epsilon(t) = \frac{t+\epsilon}{\mu}, \quad t \in [0,1], \ \epsilon \in \mathbb{Z}_\mu. \tag{3}$$

It is obvious that the mappings $\{\psi_\epsilon\}$ satisfy the following properties:

$$\begin{aligned} &\psi_\epsilon(I) \subset I, \ \forall \epsilon \in \mathbb{Z}_\mu, \\ &\bigcup_{\epsilon \in \mathbb{Z}_\mu} \psi_\epsilon(I) = I. \end{aligned} \tag{4}$$

It is well known that Legendre polynomials $P_m(x)$ are orthogonal with respect to the weight function $w(x) = 1$ on the interval [-1, 1] and satisfy the following formulae [2]:

$$\begin{aligned} &P_0(x) = 1, \\ &P_1(x) = x, \\ &P_{m+1}(x) = \tfrac{2m+1}{m+1}\, xP_m(x) - mP_{m-1}(x), \quad m \in \mathbb{N} \end{aligned} \tag{5}$$

Now, let F_0 denotes the finite dimensional linear space on $[0,1]$ that is spanned by the Legendre polynomials $P_0(2x-1), P_1(2x-1), \ldots, P_{M-1}(2x-1)$, where $M \in \mathbb{N}$ and U_m is the Legendre polynomial of degree m, namely,

$$F_0 = \text{span}\{P_m(2x-1)|x \in [0,1], m \in \mathbb{Z}_\mu\}. \tag{6}$$

In order to construct an orthonormal basis for $L^2[0,1]$, for each $\epsilon \in \mathbb{Z}_\mu$ we define an isometry T_ϵ on $L^2[0,1]$:

$$(T_\epsilon f)(x) = \begin{cases} \sqrt{\mu} f(\psi_\epsilon^{-1}(x)), & x \in \psi_\epsilon(I), \\ 0, & x \notin \psi_\epsilon(I). \end{cases} \tag{7}$$

Starting from the space F_0, we define a sequence of spaces $\{F_k|k \in \mathbb{N}_0\}$ using the recurrence formula:

$$F_{k+1} = \bigoplus_{\epsilon \in \mathbb{Z}_\mu} T_\epsilon F_k, \qquad k \in \mathbb{N}_0, \tag{8}$$

where $\oplus$ denotes the direct sum, e.g., if A and B are two subspaces of $L^2[0,1]$ with $A \cap B = \{0\}$, then:

$$A \oplus B = \{f + g : f \in A, g \in B\}.$$

The sequence of spaces $\{F_k|k \in \mathbb{N}_0\}$ is nested, i.e. [9]

$$F_0 \subset F_1 \subset \ldots \subset F_k \subset F_{k+1} \subset \ldots, \tag{9}$$

and

$$\text{dim} F_k = M\mu^k, \qquad k \in \mathbb{N}_0. \tag{10}$$

Moreover, similar to theorem 2.4 in [10], it can be proved that

$$\overline{\bigcup_{k=0}^{\infty}} F_k = L^2[0,1]. \tag{11}$$

Next we construct an orthonormal basis for each of the spaces F_k. We first notice that

$$G_0 = \left\{\sqrt{2m+1} P_m(2x-1)|x \in [0,1],\ m \in \mathbb{Z}_\mu\right\},$$

is an orthonormal basis for F_0 and moreover for $f(x) \in L^2[0,1]$, with compact support and we have

$$\text{supp}\{T_\epsilon f\} \cap \text{supp}\{T_{\epsilon'} f\} = \emptyset, \qquad \epsilon \neq \epsilon',$$

where supp(f) denotes the support of the function f. It can be simplify seen that [7]:

$$G_k = \left\{T_{\epsilon 0}\, o \ldots o\, T_{\epsilon k-1}\left(\sqrt{2m+1}P_m(2x-1)\right)|m \in \mathbb{Z}_M,\, \epsilon_\ell \in \mathbb{Z}_\mu,\, \ell \in \mathbb{Z}_k\right\},$$

is an orthonormal basis for F_k, where "o" denotes composition of functions. In other words, if for $n = 1, 2, \ldots, \mu^k$, $k \in \mathbb{N}$, we set:

$$\psi_{nm}(x) = \psi(k, m, n, x) = \begin{cases} \sqrt{2m+1}\mu^{\frac{k}{2}}P_m(2\mu^k x - 2n + 1), & x \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{12}$$

then $\{\psi_{nm}(x)|n = 1, 2, \ldots, \mu^k, m \in Z_M\}$ forms an orthonormal basis for F_k with respect to the weight function $w(x) = 1$. Moreover, for any integer number $M > 1$ the functions $\{\psi_{nm}(x)|n = 1, 2, \ldots, \mu^k, m \in Z_M\}$ are called Legendre wavelets.

2.2 Function approximation

A function $f(x)$ defined over [0, 1) can be expanded in the terms of the extended Legendre wavelets as

$$f(x) \simeq \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{13}$$

where $c_{nm} = (f(t), \psi_{nm}(t))$ and $(.,.)$ denotes the inner product on $L^2[0, 1]$. If the infinite series in (13) is truncated, then it can be written as:

$$f(x) \simeq \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) = C^T\Psi(x), \tag{14}$$

where C and $\Psi(x)$ are $\hat{m} = \mu^k M$ column vectors as

$$C = \left[c_{10}, \ldots, c_{1(M-1)}|c_{20}, \ldots, c_{2(M-1)}|, \ldots, |c_{\mu^k 0}, \ldots, c_{\mu^k(M-1)}\right]^T, \tag{15}$$

$$\Psi(x) = \left[\psi_{10}(x), \ldots, \psi_{1(M-1)}(x)|\psi_{20}(x), \ldots, \psi_{2(M-1)}(x)|, \ldots, |\psi_{\mu^k 0}(x), \ldots, \psi_{\mu^k(M-1)}(x)\right]^T.$$

By changing indices in the vectors $\Psi(x)$ and C the series (14) can be defined as

$$f(x) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(x) = C^T\Psi(x), \tag{16}$$

where

$$C = [c_1, c_2, ..., c_{\hat{m}}], \quad \Psi(x) = [\psi_1(x), \psi_2(x), ..., \psi_{\hat{m}}(x)], \tag{17}$$

and

$$c_i = c_{nm}, \quad \psi_i(x) = \psi_{nm}(x), \quad i = (n-1)M + m + 1. \tag{18}$$

2.3 Error analysis and convergence

In the next theorems the convergence properties and error bound for the of the extended Legendre wavelets series are investigated.

Theorem 2.1. *Any function $f(x)$ defined on $[0,1)$ with bounded first and second derivatives $|f'(x)| \leq M_1$ and $|f''(x)| \leq M_2$, can be expanded as an infinite sum of the extended Legendre wavelets, and the series converges uniformly to $f(x)$, that is:*

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(x), \tag{19}$$

Proof. Let $f(x)$ be a function defined on $[0,1)$ first and second derivatives M_1 and M_2, respectively, and:

$$c_{nm} = \int_0^1 f(x)\psi_{nm}(x)dx = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)\sqrt{2m+1}\mu^{\frac{k}{2}} P_m(2\mu^k x - 2n + 1)dx. \tag{20}$$

Let $\hat{n} = 2n - 1$, then by the change of variable $t = 2\mu^k x - \hat{n}$, we have $dx = \frac{dt}{2\mu^k}$, and so:

$$\begin{aligned} c_{nm} &= \frac{(2m+1)^{\frac{1}{2}}}{2\mu^{\frac{k}{2}}} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2\mu^k}\right) P_m(t)dt \\ &= \frac{1}{2\mu^{\frac{k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f\left(\frac{\hat{n}+t}{2\mu^k}\right) d\left(P_{m+1}(t) - P_{m-1}(t)\right), \end{aligned} \tag{21}$$

where the following property of the legendre polynomials is used:

$$(2m+1)P_m(t) = P'_{m+1}(t) - P'_{m-1}(t). \tag{22}$$

Integrating by parts in (21) yields:

$$\begin{aligned} c_{nm} &= \frac{1}{2\mu^{\frac{k}{2}}(2m+1)^{\frac{1}{2}}} \left\{ \frac{1}{2\mu^k} f'\left(\frac{\hat{n}+t}{2\mu^k}\right) \left(P_{m+1}(t) - P_{m-1}(t)\right)|_{-1}^{1} \right. \\ &\left. - \frac{1}{2\mu^k} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2\mu^k}\right) \left(P_{m+1}(t) - P_{m-1}(t)\right) dt \right\}. \end{aligned} \tag{23}$$

From equation (23), we have:

$$c_{nm} = \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2\mu^k}\right) \left(P_{m+1}(t) - P_{m-1}(t)\right) dt. \tag{24}$$

Now, by considering (22), we have:

$$c_{nm} = \frac{1}{4\mu^{\frac{3k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f'\left(\frac{\hat{n}+t}{2\mu^k}\right) d\left(\frac{P_{m+2}(t)-P_m(t)}{2m+3} - \frac{P_m(t)-P_{m-2}(t)}{2m-1}\right). \quad (25)$$

Solving this equation similar to the previous step, yields:

$$c_{nm} = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2\mu^k}\right) \left(\frac{P_{m+2}(t)-P_m(t)}{2m+3} - \frac{P_m(t)-P_{m-2}(t)}{2m-1}\right) dt. \quad (26)$$

Now, let $\tau_m(t) = (2m-1)P_{m+2} - 2(2m+1)P_m(t) + (2m+3)P_{m-2}(t)$, then we have:

$$c_{nm} = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \frac{1}{(2m-1)(2m+3)} \int_{-1}^{1} f''\left(\frac{\hat{n}+t}{2\mu^k}\right) \tau_m(t) dt. \quad (27)$$

Then we have:

$$|c_{nm}| =\leq \Omega(\mu,k,m) \int_{-1}^{1} \left| f''\left(\frac{\hat{n}+t}{2\mu^k}\right)\right| |\tau_m(t)| dt, \quad (28)$$

where

$$\leq \Omega(\mu,k,m) = \frac{1}{8\mu^{\frac{5k}{2}}(2m+1)^{\frac{1}{2}}} \frac{1}{(2m-1)(2m+3)}.$$

Moreover, it is shown [11] that:

$$\int_{-1}^{1} |\tau_m(t)| dt \leq \sqrt{24} \frac{2m+3}{\sqrt{2m-3}}, \quad (29)$$

Therefor, since $n \leq \mu^k$, for $m > 1$ we get:

$$|c_{nm}| \leq \frac{\sqrt{6} M_2}{2n^{\frac{5}{2}}(2m-3)^2}. \quad (30)$$

Also, for $m = 1$, from (24), we have:

$$|c_{n1}| \leq \frac{M_1}{\sqrt{3} n^{\frac{3}{2}}}. \quad (31)$$

Hence, the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}$ is absolutely convergent. Moreover, it is obvious that, for $m = 0$, the sequence $\{\psi_{n0}(x)\}_{n=1}^{\infty}$ forms an orthogonal system constructed by Haar scaling function and thus $\sum_{n=1}^{\infty} c_{n0\psi_{n0}}(x)$ is convergent (for more details see appendix). Consequently, it follows that the series $\sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x)$ converges to the function $f(x)$ uniformly. □

Theorem 2.2. *Suppose $f(x)$ be a continuous function defined on $[0,1)$, with bounded first and second derivatives M_1 and M_2 respectively, and $\sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x)$ be the approximate solution using the extended Legendre wavelets. Then for the error bound we have:*

$$\sigma_{\hat{m}} < \left(\frac{M_1^2}{\mu^{2k}} + \frac{3M_2^2}{2}\sum_{n=1}^{\infty}\sum_{m=M}^{\infty} a_{nm} + \frac{3M_2^2}{2}\sum_{n=\mu^k+1}^{\infty}\sum_{m=2}^{M-1} a_{nm} + \frac{M_1^2}{3}\sum_{n=\mu^k+1}^{\infty} b_n \right)^{\frac{1}{2}}, \quad (32)$$

where

$$a_{nm} = \frac{1}{n^5(2m-3)^4}, \qquad b_n = \frac{1}{n^3},$$

and

$$\sigma_{\hat{m}} = \left(\int_0^1 \left(f(x) - \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) \right)^2 dx \right)^{\frac{1}{2}}.$$

Proof. We have:

$$\sigma_{\hat{m}}^2 = \int_0^1 \left(f(x) - \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) \right)^2 dx$$

$$= \int_0^1 \left(\sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(x) - \sum_{n=1}^{\mu^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(x) \right)^2 dx$$

$$= \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 \int_0^1 \psi_{nm}^2(x)dx + \sum_{n=\mu^k+1}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2 \int_0^1 \psi_{nm}^2(x)dx$$

$$= \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=\mu^k+1}^{\infty}\sum_{m=0}^{M-1} c_{nm}^2$$

$$= \sum_{n=1}^{\infty}\sum_{m=M}^{\infty} c_{nm}^2 + \sum_{n=\mu^k+1}^{\infty}\sum_{m=2}^{M-1} c_{nm}^2 + \sum_{n=\mu^k+1}^{\infty} \left(c_{n0}^2 + c_{n1}^2\right). \quad (33)$$

Now by considering (30), (31), (33) and remark 1 in appendix, the desired result is achieved. □

3 Operational matrix of derivative for extended Legendre wavelets

In this section, we derive a new operational matrix of derivative for the extended Legendre wavelets. At first some properties of shifted Legendre polynomials are discused. For practical use of Legendre polynomials on the interval of interest $[0,1]$ the shifted Legendre polynomials $\tilde{P}_n(x)$ on [0,1] are obtained as:

$$\tilde{P}_n(x) = P_n(2x-1).$$

The orthogonality condition for these shifted polynomials is:

$$\int_0^1 \tilde{P}_m(x)\tilde{P}_n(x)dx = \frac{1}{2m+1}\delta_{mn}.$$

In the next theorem, we derive a relation between the shifted Legendre polynomials and their derivatives that is very important for deriving the operational matrix of derivatives for the extended Legendre wavelets.

Theorem 3.1. *[5] Let $P_m(x)$ be the shifted Legendre polynomials into $[0,1]$, then we have*

$$P'_m(x) = 2\sum_{\substack{k=0 \\ k+m\ odd}}^{m-1} (2k+1)P_k(x). \tag{34}$$

Lemma 3.2. *[2] The function $f(x)$, square integrable in $[0,1]$, may be expressed in terms of shifted Legendre polynomials as:*

$$f(x) = \sum_{k=0}^{\infty} c_k \tilde{P}_k(x), \tag{35}$$

where

$$c_k = (2k+1)\int_0^1 f(x)\tilde{P}_k(x)dx.$$

Lemma 3.3. *[5] By using the shifted Legendre polynomials, any extended Legendre wavelets function $\Psi_r(x)$ of (12) can be written as:*

$$\Psi_r(x) = \sqrt{2m+1}\mu^{\frac{k}{2}}\tilde{P}_m(\mu^k x - n),$$

where $r = nM + m + 1$, $m = 0, \ldots, (M-1)$, $n = 0, \ldots, \mu^k - 1$ and $\chi_{[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}]}$ is the characteristic function defined as:

$$\chi_{[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}]}(x) = \begin{cases} 1, & x \in [\frac{n}{\mu^k}, \frac{n+1}{\mu^k}], \\ 0, & therwise. \end{cases}$$

Theorem 3.4. *Let $\Psi(x)$ be the extended Legendre wavelets vector defined in (14). The derivative of the vector $\Psi(x)$ can be defined as*

$$\frac{d\Psi(x)}{dx} = D\Psi(x), \tag{36}$$

where D is the operational matrix of derivative defined as:

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & F \end{pmatrix}, \tag{37}$$

in which F is M matrix and its (r,s)-th element is defined as:

$$F_{r,s} = \begin{cases} 2\mu^k \sqrt{(2r-1)(2s-1)} & r = 2, ..., M, \quad s = 1, ..., r-1, \ \textit{and} \ (r+s) odd, \\ \\ 0, & othewise. \end{cases} \tag{38}$$

Proof. By using the shifted Legendre polynomials into $[0, 1]$, the r-th element of vector $\Psi(x)$ can written as:

$$\Psi_r(x) = \psi_{nm}(x) = \mu^{\frac{k}{2}} \sqrt{2m+1}\, \tilde{P}_m(\mu^k x - n) \ \chi_{[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}]} \ , \quad r = 1, 2, ..., \hat{m}\,, \tag{39}$$

where $r = nM + m + 1$, $m = 0, 1, ..., M - 1$ and $n = 0, 1, ..., \mu^k - 1$.
By differentiation with respect to x of vector $\Psi(x)$ we have:

$$\frac{d\Psi_r(x)}{dx} = \mu^{\frac{k}{2}} \sqrt{2m+1} \mu^k \tilde{P}'_m(\mu^k x - n) \ \chi_{[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}]} \tag{40}$$

This function is zero outside the interval $\left[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}\right]$, hence its Legendre wavelets expansion only have those elements of basis extended Legendre wavelets in $\Psi(x)$ that are nonzero in the interval $\left[\frac{n}{\mu^k}, \frac{n+1}{\mu^k}\right]$ i.e. $\Psi_i(x)\,, \quad i = nM + 1\,, \, nM + 2\,, \, ..., \, nM + M.$
So its extended Legendre wavelets expansion has the following form:

$$\frac{d\Psi_r(x)}{dx} = \sum_{i=nM+1}^{(n+1)M} a_i \Psi_i(x) \tag{41}$$

This results that operational matrix D is a block matrix as defined in (37). Moreover we have:

$$\frac{d}{dx} P_0(x) = 0\ , \tag{42}$$

this implies that $\frac{d\Psi_r(x)}{dx} = 0$ for $r = 1\,,\ M+1\,,\ 2M+1\,,\ 3M+1\,,\ ...,\ (\mu^k-1)M+1$. Consequently, the first row of matrix defined in (38) is zero. Now by substituting $P'_m(\mu^k x - n)$ from Eq. (34) into (40) we have:

$$\frac{d\Psi_r(t)}{dt} = \mu^{\frac{k}{2}}\,\sqrt{2m+1}\,\mu^k \sum_{\substack{j=0\\ j+m\ odd}}^{m-1} 2(2j+1)\tilde{P}_j(\mu^k t - n)\ \chi_{[\frac{n}{\mu^k},\frac{n+1}{\mu^k}]} \tag{43}$$

Expanding this equation in the extended Legendre wavelets basis we have:

$$\frac{d\Psi_r(t)}{dt} = \mu^{\frac{k}{2}}\sqrt{2m+1}\,\mu^k \sum_{\substack{j=0\\ j+m\,odd}}^{m-1} 2\sqrt{2j+1}\sqrt{2j+1}P_j(\mu^k t - n)\,\chi_{\left[\frac{n}{\mu^k},\frac{n+1}{\mu^k}\right]}$$

$$= 2\mu^k \sum_{\substack{s=1\\ s+r\,odd}}^{r-1} \sqrt{(2r-1)\,(2s-1)}\,\Psi_{nM+s}(t)\,. \tag{44}$$

So if we choose as

$$F_{r,\,s} = \begin{cases} 2\mu^k\sqrt{(2r-1)(2s-1)} & r = 2,...,M,\quad s = 1,...,r-1,\ \ and\ (r+s)odd, \\ 0, & othewise. \end{cases}$$

then equation (36) is hold and this leads to desired result. □

The following property of the product of two Legendre wavelets vector functions $\Psi(t)$ will be used,

$$E^T\psi\psi^T = \psi^T\tilde{E}, \tag{45}$$

where E is a given vector and $\tilde{E}$ is a $\mu^k M$ matrix depended on vector E, which is called the product operation matrix of Legendre wavelets vector functions.

4 Description of the proposed method

In this section, the extended Legendre Galerkin method is applied for solving boundary ordinary differential equations with finite singular points in the interval $[0,1]$. For this purpose consider the boundary value problem:

$$y''(t) + f(t)y'(t) + g(t)y(t) = h(t), \tag{46}$$

with the boundary conditions:

$$y(0) = A, \quad y(1) = B, \tag{47}$$

in which the solution function $y(t)$ or coeficient functions $f(t), g(t)$ and $h(t)$ has finite singular point in the interval $[0,1]$ as

$$t_1 = \frac{k_1}{n_1},\ t_2 = \frac{k_2}{n_2}, \ldots, t_m = \frac{k_m}{n_m}, \qquad k_i,\ n_i \in \mathbb{N} \ \ and \ \ (k_i, n_i) = 1 \tag{48}$$

For solving the singular BVP (46) and deriving an spectral accuracy, first we choose a suitable μ according these singular points t_i. For this purpose let n_i has the prime factorization

$$n_i = p_2^{\alpha_{i_2}} \ldots p_r^{\alpha_{i_r}},\ p_j \geq 2,\ \alpha_{i_j} \geq 0. \tag{49}$$

Now we define the μ as follow

$$\mu = p_1^{\beta_1} p_2^{\beta_2} \ldots p_r^{\beta_r} \tag{50}$$

in which $\beta_j = \min\{\alpha_{j_r} \mid\ 1 \leq r \leq m\}$. By using the extended Legendre wavelet basis for this μ, we can get an approximate solution for the BVP (46) with the spectral accuracy. For this purpose, we approximate the functions $f(t), g(t), h(t)$ and $y(t)$ as:

$$y(t) = C^T\Psi(t), \quad f(t) = F^T\Psi(t), \quad g(t) = G^T\Psi(t), \quad h(t) = H^T\Psi(t) \tag{51}$$

where $\Psi(t)$ is the extended Legendre wavelets basis vector defined in (14) and C, F, G and H are the cofficient vectors of the extended Legendre wavelet basis for the function $y(t)$, $f(t), g(t)$ and $h(t)$ respectively. Moreover, by using the operational matrix of derivatives for the extended Legendre wavelet $\Psi(t)$, we have:

$$y'(t) = C^T D\Psi(t), \quad y''(t) = C^T D^2\Psi(t). \tag{52}$$

Employing Eqs. (51) and (52), the residual for Eq. (46) can be written as

$$R(t) = \Psi(t)^T \left(D^T\right)^2 C + F^T\Psi(t)\Psi^T(t)D^T C + G^T\Psi(t)\Psi^T(t)C - \Psi(t)^T H, \tag{53}$$

By using the operational matrix of product defined in (45) we have

$$R(t) = \Psi(t)^T \left(D^T\right)^2 C + \Psi(t)^T \tilde{F} D^T C + \Psi(t)^T \tilde{G} C - \Psi(t)^T H, \tag{54}$$

As in a typical Tau method [2], we generate $\hat{m} - 2$ linear equations by applying

$$\int_0^1 \Psi_j(t)R(t)dt = 0, \qquad j = 1, ..., \hat{m} - 2. \tag{55}$$

Also, by substituting initial conditions in Eq. (47) we have

$$
\begin{aligned}
&y(0) = C^T\Psi(0) - A = 0, \\
&y(1) = C^T\Psi(1) - B = 0.
\end{aligned} \tag{56}
$$

Eqs. (55) and (56) generate a set of $\hat{m}$ linear equations. These linear equations can be solved for unknown coefficients of the vector C. The solution function $y(t)$ can be approximate by substituting vector C in (51).

5 Numerical examples

In this section, we demonstrate the efficiency of the proposed method with some illustrative examples. For all examples, spectral methods by using Legendre polynomials and typical Legendre wavelet method do not work well and can not present spectral accuracy [1,4]. It will be shown that the extended Legendre wavelet method is very efficient for solving these singular problems and spectral accuracy will be obtained. The algorithms are performed by Maple 13 with 30 digits precision.

Example 1. Consider the boundary value problem

$$
\begin{cases}
y'' + \left|t - \frac{1}{3}\right| y' + \left|t - \frac{1}{2}\right| y = h(x), \quad t \in [0,1] \\
y(0) = \left(\frac{1}{3}\right)^3, \quad y(1) = \left(\frac{2}{3}\right)^3,
\end{cases} \tag{57}
$$

where $h(t)$ is compatible with the exact solution $\left|t - \frac{1}{3}\right|^3$. In this problem the exact solution and coefficient functions have singular points at $t_1 = \frac{1}{2}$ and $t_2 = \frac{1}{3}$. According the prposed method in the section 4 by choosing $\mu = 2 \times 3 = 6$ we solve the problem for $M = 4$ and $k = 1$ and the exact solution is derived. Figure 1 shows the absolute error produced the extended Legendre wavelets method for $\mu = 6$, $M = 4$ and $k = 1$ in the interval $[0,1]$. Howover by using the typical legendre wavelets method ($\mu = 2$) we can not derive the exact solution and only an approximate solution can be obtaind. The absolute errors produced from the extended Legendre wavelets (with $\mu = 6$) and standard Legendre wavelets ($\mu = 2$) method are shown in table 1 for different values of $M = 4$ and $k = 1$. As the table shows the extended Legendre wavelets are more efficient in solving this boundary value problem.

Example 2. Consider the boundary value problem

$$
\begin{cases}
y'' + \left|t - \frac{1}{5}\right| y' + \left|t - \frac{1}{4}\right| y = h(t), \quad t \in [0,1] \\
y(0) = \left(\frac{1}{2}\right)^5, \quad y(1) = \left(\frac{1}{2}\right)^5,
\end{cases} \tag{58}
$$

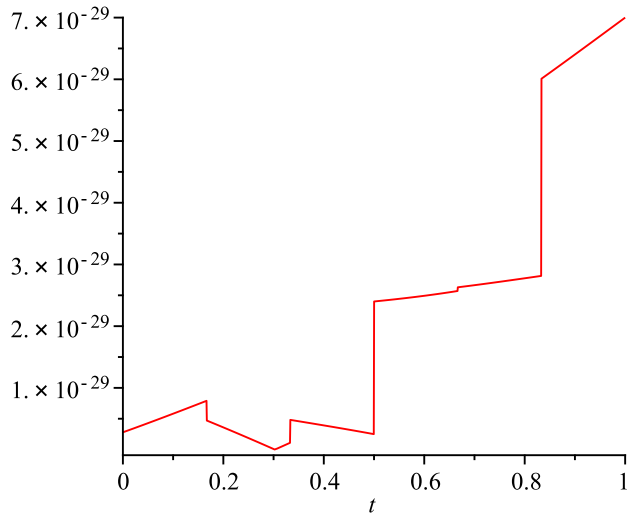

Figure 1: The absolute error produced by the extended Legendre wavelets.

Table 1: The absolute error produced by the extended Legendre wavelets (M=4, k=1).

μ	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\mu = 6$	8.18×10^{-30}	1.22×10^{-30}	2.50×10^{-29}	2.66×10^{-29}	6.40×10^{-29}
$\mu = 2$	6.16×10^{-4}	6.78×10^{-4}	3.30×10^{-3}	2.17×10^{-7}	7.04×10^{-8}

where $h(t)$ is compatible with the exact solution $\left|t - \frac{1}{2}\right|^5$. In this problem the exact solution and coefficient functions have singular points at $t_1 = \frac{1}{5}, t_2 = \frac{1}{4}$ and $t_3 = \frac{1}{2}$. According the prposed method in the section 4 by choosing $\mu = 2 \times 5 = 10$ the exact solution of the above boundary value problem can be derived. Here we solve the problem with $\mu = 10$, $M = 6$ and $k = 2$ and we get the exact solution. Howover by using the standard legendre wavelets method ($\mu = 2$) the exact solution can not be derived. The absolute errors produced from the extended Legendre wavelets (with $\mu = 10$) method and standard Legendre wavelets ($\mu = 2$) method are shown in table 2 for different values of $M = 6$ and $k = 2$.

Table 2: The maximum absolute error produced by the extended Legendre wavelets.

μ	$t = 0.1$	$t = 0.3$	$t = 0.5$	$t = 0.7$	$t = 0.9$
$\mu = 10$	2.99×10^{-30}	4.91×10^{-30}	2.25×10^{-29}	4.70×10^{-30}	2.71×10^{-29}
$\mu = 2$	1.60×10^{-7}	1.36×10^{-8}	1.90×10^{-8}	5.49×10^{-9}	1.73×10^{-9}

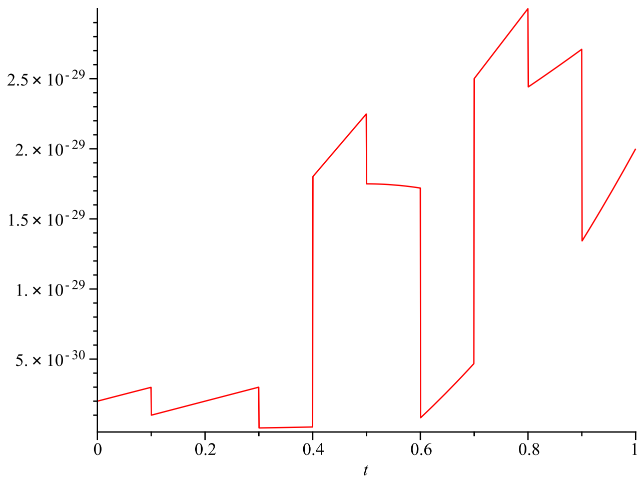

Figure 2: The absolute error produced by the extended Legendre wavelets.

Example 3. In this example we consider a boundary value problem in which coefficient functions are non analytical but its solution function is infinite differentiable on [0, 1]. So, consider the boundary value problem

$$\begin{cases} y'' + g(t)y' + \left|t - \frac{1}{2}\right| y = h(t), \quad t \in [0,1] \\ \\ y(0) = y(1) = 0, \end{cases} \tag{59}$$

in which

$$g(t) = \begin{cases} -4t+\frac{4}{3} & 0 \le t < \frac{1}{3} \\ t-\frac{1}{3} & \frac{1}{3} \le t < 1 \end{cases} \tag{60}$$

and $h(t)$ is compatible with the exact solution $y(t) = \frac{e^t \sin(\pi t)}{1+t^2}$. Here the exact solution is infinitely differentiable but the coefficient functions have singular points at $t_1 = \frac{1}{3}$ and $t_2 = \frac{1}{2}$. As the Taylor series of the exact solution has infinite terms, the exact solution can not be obtained by either standard Legendre wavelet method or extended Legendre wavelet method. Howover a comparision between these methods shows the efficiency of the extended Legendre wavelet method. Table 3 shows the absolute error of these methods for different values of M and k. As this table shows the extended Legendre wavelet method with $\mu = 6$ is more efficient for approximate the solution of this problems and by increasing the M and k give a good approximation of the exact solution.

Table 3: The maximum absolute error for the standard Legendre wavelet ($\mu = 2$) and extended Legendre wavelets (with $\mu = 6$).

M, k	$M = 6, k = 1$	$M = 6, k = 2$	$M = 8, k = 1$	$M = 8, k = 2$	$M = 10, k = 1$	$M = 10, k = 2$
$\mu = 2$	5.50×10^{-1}	3.43×10^{-2}	3.12×10^{-1}	4.51×10^{-4}	1.32×10^{-3}	4.50×10^{-5}
$\mu = 6$	1.37×10^{-2}	1.24×10^{-4}	2.50×10^{-4}	3.59×10^{-7}	4.73×10^{-7}	7.15×10^{-10}

6 Conclusion

Wavelets method have been successful for solving singular boundary value problems with non analytic solution. However, difficulties arise in dealing with ordinary differential equations in which singular points are not of the form 2^{-k} for some positive integer k [4]. To overcome these difficulties, in this paper an extended Legendre wavelet method is proposed and is applied for solving ordinary differential equations in which, at least, one of the coefficient functions or solution function is not analytic. The comparison shows the high efficiency of the proposed method.

Appendix

In this appendix we will obtain an upper bound for c_{nm} in the extended Legendre wavelets expansions in case $m = 0$. For $m = 0$, the extended Legendre wavelets form an orthonormal system on $[0, 1)$ as:

$$\psi_{n0}(x) = \begin{cases} \mu^{\frac{k}{2}}, & x \in [\frac{n-1}{\mu^k}, \frac{n}{\mu^k}), \\ 0, & otherwise, \end{cases} \tag{61}$$

for $n = 1, 2, \ldots, \mu^k$.

By expanding any square integrable function $f(x)$ in terms of these basis functions on $[0, 1)$ we have:

$$f(x) = \sum_{n=1}^{\infty} c_{n0} \psi_{n0}(x), \tag{62}$$

where

$$c_{n0} = \mu^{\frac{k}{2}} \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x) dx. \tag{63}$$

If the infinite series in (62) is truncated, then it can be written as:

$$f_{\mu^k}(x) \simeq \sum_{n=1}^{\mu^k} c_{n0}\psi_{n0}(x). \tag{64}$$

Theorem 6.1. *Suppose $f_{\mu^k}(x)$ be the truncated expansion of $f(x)$ in the above basis functions and $e_{\mu^k}(x) = f_{\mu^k}(x) - f(x)$ be the corresponding error, then the expansion will converge in the sense that $e_{\mu^k}(x)$ approaches zero as μ^k tends to infinity. Moreover the convergence order is one, that is:*

$$\|e_{\mu^k}(x)\| = \mathcal{O}\left(\frac{1}{\mu^k}\right). \tag{65}$$

Proof. By defining the error between $f(x)$ and its expansion over any subinterval as:

$$e_n(x) = c_{n0}\psi_{n0}(x) - f(x), \qquad x \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right), \; n = 1, 2, \ldots, \mu^k, \tag{66}$$

we obtain

$$\begin{aligned} \|e_n(x)\|^2 &= \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} e_n(x)^2 dx = \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} \left(c_{n0}\psi_{n0}(x) - f(x)\right)^2 dx \\ &= \left(c_{n0}\mu^{\frac{k}{2}} - f(\eta_n)\right)^2 \frac{1}{\mu^k}, \qquad \eta_n \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right), \end{aligned} \tag{67}$$

where we have used the weighted mean value theorem for integrals.
From (63) and the weighted mean value theorem, we also have:

$$c_{n0} = \mu^{\frac{k}{2}} \int_{\frac{n-1}{\mu^k}}^{\frac{n}{\mu^k}} f(x)dx = \mu^{\frac{k}{2}} \frac{1}{\mu^k} f(\zeta_n) = \frac{1}{\mu^{\frac{k}{2}}} f(\zeta_n), \qquad \zeta_n \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right). \tag{68}$$

By substituting (68) into (67), we obtain:

$$\|e_n(x)\|^2 = \left(f(\zeta_n) - f(\eta_n)\right)^2 \frac{1}{\mu^k}. \tag{69}$$

Now, since $|f'(x)| < M_1$, then $f(x)$ satisfies a Lipschitz condition on each subinterval, i.e.:

$$|f(\zeta_n) - f(\eta_n)| \leq M_1|\zeta_n - \eta_n|, \qquad \forall \zeta_n, \eta_n \in \left[\frac{n-1}{\mu^k}, \frac{n}{\mu^k}\right) \tag{70}$$

Then, from (69) and (70), we have:

$$\|e_n(x)\|^2 \leq \frac{M_1^2}{\mu^{3k}}, \tag{71}$$

which leads to:

$$\|e_{\mu^k}(x)\|^2 = \int_0^1 e_{\mu^k}(x)^2 dx = \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n(x)\right)^2 dx$$

$$= \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n(x)^2\right) dx + 2\sum_{n\leq n'} \int_0^1 e_n(x) e_{n'}(x) dx. \tag{72}$$

Now, due to disjointness of the supports of these basis functions we have:

$$\|e_{\mu^k}(x)\|^2 = \int_0^1 \left(\sum_{n=1}^{\mu^k} e_n(x)^2\right) dx = \sum_{n=1}^{\mu^k} \|e_n(x)\|^2. \tag{73}$$

Substituting (71) into (73), we obtain:

$$\|e_{\mu^k}(x)\|^2 \leq \frac{M_1^2}{\mu^{2k}}, \tag{74}$$

or, in other words, $\|e_{\mu^k}(x)\| = \mathcal{O}\left(\frac{1}{\mu^k}\right)$.
This completes the proof. □

Corollary 6.2. Let $f_{\mu^k}(x)$ be the expansion of $f(x)$ by the above basis functions and $e_{\mu^k}(x)$ be the corresponding error, then we have:

$$\|e_{\mu^k}(x)\| \leq \frac{M_1}{\mu^k}. \tag{75}$$

Proof. This is an immediate consequence of the theorem 6.1. □

Remark 1. We notice that according to the above information we have:

$$\sum_{n=\mu^k+1}^{\infty} c_{n0}^2 = \|f_{\mu^k}(x) - \sum_{n=1}^{\infty} c_{n0}\psi_{n0}(x)\|^2 = \|e_{\mu^k}(x)\|^2 \leq \frac{M_1^2}{\mu^{2k}}. \tag{76}$$

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