Journal of Mathematical Extension
Vol. 16, No. 12, (2022) (2)1-9
URL: https://doi.org/10.30495/JME.2022.2343
ISSN: 1735-8299
Original Research Paper

Numerical Radius Inequalities for Products of Hilbert Space Operators

M. Shah Hosseini *
Shahr-e-Qods Branch , Islamic Azad University

B. Moosavi
Safadash Branch, Islamic Azad University

Abstract. We introduce some numerical radius inequalities for products of two Hilbert space operators. Among other inequalities, it is shown that if $S, T \in \mathbb{B}(\mathbb{H})$ and $ST = TS^*$, then

$$\omega(ST) \leq \omega(S)\omega(T) + \frac{1}{2} D_S \sup_{\theta \in \mathbb{R}} D_{e^{i\theta}T + e^{-i\theta}T^*},$$

where $D_S = \inf_{\lambda \in \mathbb{C}} \|S - \lambda I\|$. Also, we show that if $S, T \in \mathbb{B}(\mathbb{H})$ and S be self-adjointable, then

$$\omega(ST) \leq \left(2\|S\| - \min_{\lambda \in \sigma(S)} |\lambda|\right) \omega(T).$$

AMS Subject Classification: Primary 47A12; secondary 47A30, 47A63.
Keywords and Phrases: Bounded linear operator, Hilbert space, norm inequality, numerical radius.

Received: March 2022; Accepted: September 2022
*Corresponding Author

1 Introduction and preliminaries

Let $B(H)$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. Let $D_S = \inf_{\lambda \in \mathbb{C}} \|S - \lambda I\|$ (the distance of S from scalar operators), and let R_S denote the radius of the smallest disk in the complex plane containing $\sigma(S)$ (the spectrum of S). It is not hard to check that the infimum in the definition of D_S is attained at some $\lambda_0 \in \mathbb{C}$, that is, $D_S = \|S - \lambda_0 I\|$. It is known (see, e.g., [10]) that $D_S = R_S$ for any normal operator S.
The numerical radius of $S \in B(H)$ is defined by

$$\omega(S) = \sup\{\ |\langle Sx, x\rangle| : \|x\| = 1\ \}.$$

It is well known that $\omega(\cdot)$ is a norm on $B(H)$ which is equivalent to the usual operator norm $\|.\|$. In fact, for all $S \in B(H)$,

$$\frac{\|S\|}{2} \leq \omega(S) \leq \|S\|. \tag{1}$$

For other results and comments on the inequalities in (1) see [3, 6, 8, 9]. In [2], Berger proved that for any $S \in B(H)$ and natural number n,

$$\omega(S^n) \leq \omega^n(S).$$

Holbrook in [4] showed that, for any $S, T \in B(H)$,

$$\omega(ST) \leq 4\omega(S)\omega(T).$$

In the case $ST = TS$, then

$$\omega(ST) \leq 2\omega(S)\omega(T).$$

If S is an isometry (or a unitary) and $ST = TS$, then

$$\omega(ST) \leq \omega(T).$$

It is shown in [5] that, for any $S, T \in B(H)$,

$$\omega(S^*T \pm TS) \leq 2\|S\|\omega(T). \tag{2}$$

If S and T are operators in $B(H)$, we write the direct sum $S \oplus T$ for the 2×2 operator matrix $\begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}$, regarded as an operator on $H \oplus H$. Thus

$$\omega(T \oplus S) = \max(\omega(T), \omega(S)).$$

Also,

$$\|T \oplus S\| = \left\| \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \right\| = \max(\|T\|, \|S\|). \tag{3}$$

The question about the best constant k such that the inequality

$$\omega(ST) \leq k\|S\|\omega(T). \tag{4}$$

holds for all operators $S, T \in B(H)$ is still open.
Concerning the inequality (4), it is shown in [1] that if $S, T \in B(H)$, then

$$\omega(ST) \leq (\|S\| + D_S)\omega(T). \tag{5}$$

Also, if $S > 0$, then

$$\omega(ST) \leq \frac{3}{2}\|S\|\omega(T). \tag{6}$$

In Section 2, we establish some numerical radius inequalities for products of two Hilbert space operators. Some applications of these inequalities are considered as well. Particularly, in some cases we obtain an improvement of inequality (5) and (6).

2 Main results

In order to derive our main results, we need the following lemmas. The first lemma is well known (see, e.g., [11]).

Lemma 2.1. *Let $S \in \mathbb{B}(\mathbb{H})$. Then*

$$\omega(S) = \sup_{\theta \in \mathbb{R}} \|Re(e^{i\theta}S)\|. \tag{7}$$

The second lemma, which can be found in [7], gives new numerical radius inequalities for products of two Hilbert space operators.

Lemma 2.2. *Let $S, T \in \mathbb{B}(\mathbb{H})$. Then*

$$w\,(ST) \leq \omega(S)\omega(T) + D_S D_T. \tag{8}$$

The following result may be stated as well.

Theorem 2.3. *Let $S, T \in \mathbb{B}(\mathbb{H})$ and $k \in \mathbb{R}$. If $S > 0$ and $D_T \leq k\|T\|$, then*

$$\omega(ST) \leq (1+k)\|S\|\omega(T).$$

Proof. By Lemma 2.2,

$$\omega(ST) \leq \omega(S)\omega(T) + D_S D_T. \tag{9}$$

Since $D_T \leq k\|T\|$, from the inequality (9) we have

$$\omega(ST) \leq \omega(T)(\omega(S) + 2kD_S).$$

If S is a projection operator, then

$$D_S = R_S = \frac{1}{2}$$

and so

$$\omega(ST) \leq (1+k)\omega(T). \tag{10}$$

First we prove that if S is a positive contraction operator, then

$$\omega(ST) \leq (1+k)\|S\|\omega(T).$$

If $R = \sqrt{S - S^2}$, then the operator $P = \begin{bmatrix} S & R \\ R & I-S \end{bmatrix}$ is a projection on $H \oplus H$ due to $S\sqrt{S-S^2} = \sqrt{S-S^2}S$. If $T_1 = \begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$, then

$$\begin{aligned} \omega\left(\begin{bmatrix} ST & RT \\ RT & (I-S)T \end{bmatrix}\right) &= \omega(PT_1) \\ &\leq (1+k)\omega(T_1) \qquad \text{(by (10))} \\ &\leq (1+k)\omega(T) \end{aligned}$$

and so

$$\omega\left(\begin{bmatrix} ST & RT \\ RT & (I-A)T \end{bmatrix}\right) \le (1+k)\omega(T). \tag{11}$$

Since $\omega(K^*SK) \le \omega(S)\|K\|^2$ for each operator $K \in \mathbb{B}(\mathbb{H})$, from the inequalities (11) we have

$$\begin{aligned} \omega(ST) &= \omega\left(\begin{bmatrix} ST & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &= \omega\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} ST & RT \\ RT & (I-ST) \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\right) \\ &\le \omega\left(\begin{bmatrix} ST & RT \\ RT & (I-ST) \end{bmatrix}\right) \\ &\le (1+k)\omega(T). \end{aligned}$$

Therefore

$$\omega(ST) \le (1+k)\omega(T).$$

Now, let S be a positive operator and $\|S\| \ge 1$. It follows from inequality

$$\omega(\frac{S}{\|S\|}T) \le (1+k)\omega(T)$$

that

$$\omega(ST) \le (1+k)\|S\|\omega(T).$$

This completes the proof. □

Theorem 2.4. *If $S, T \in \mathbb{B}(\mathbb{H})$, then*

$$\omega(ST) \le \omega(S)\omega(T) + \frac{1}{2}D_S \sup_{\theta\in\mathbb{R}} D_{e^{i\theta}T+e^{-i\theta}T^*} + \frac{1}{2}\omega(ST - TS^*).$$

Proof. *Clearly, $\|Re(ST)\| = \omega(Re(ST))$. Then*

$$\begin{aligned} \|Re(ST)\| &= \omega(\frac{ST+T^*S^*}{2}) \\ &\le \frac{1}{2}\omega(S(T+T^*)) + \frac{1}{2}\omega(ST - TS^*) \\ &\le \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} + \frac{1}{2}\omega(ST-TS^*) \end{aligned}$$

by Lemma 2.2. Hence,

$$\|Re(ST)\| \leq \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} + \frac{1}{2}\omega(ST - TS^*). \qquad (12)$$

Suppose that $\theta \in \mathbb{R}$. Replacing T by $e^{i\theta}T$ in the inequality (12) gives

$$\|Re(e^{i\theta}ST)\| \leq \|Re(e^{i\theta}T)\|\omega(S) + \frac{1}{2}D_S D_{e^{i\theta}T+e^{-i\theta}T^*} + \frac{1}{2}\omega(ST - TS^*).$$

Taking the supremum over $\theta \in \mathbb{R}$ gives

$$\omega(ST) \leq \omega(S)\omega(T) + \frac{1}{2}D_S \sup_{\theta\in\mathbb{R}} D_{e^{i\theta}T+e^{-i\theta}T^*} + \frac{1}{2}\omega(ST - TS^*),$$

by Lemma 2.1. This completes the proof. □

As a direct consequence of Theorem 2.2, we have:

Corollary 2.5. *If $S, T \in \mathbb{B}(\mathbb{H})$ and $ST = TS^*$, then*

$$\omega(ST) \leq \omega(S)\omega(T) + \frac{1}{2}D_S \sup_{\theta\in\mathbb{R}} D_{e^{i\theta}T+e^{-i\theta}T^*}.$$

Remark 2.6. Suppose that $S, T \in \mathbb{B}(\mathbb{H})$ are such as corollary 2.5 and $\theta \in \mathbb{R}$. Since $D_{e^{i\theta}T+e^{-i\theta}T^*} \leq \|e^{i\theta}T + e^{-i\theta}T^*\|$, from the Corollary 2.5 we have

$$\omega(ST) \leq \omega(S)\omega(T) + D_S\|Re(e^{i\theta}T)\|.$$

Now, taking the supremum over $\theta \in \mathbb{R}$ gives

$$\omega(ST) \leq (\omega(S) + D_S)\omega(T),$$

which is refinement of (5).

The following result may be as well.

Theorem 2.7. *Let $S, T \in \mathbb{B}(\mathbb{H})$. If there exsist $z_0 \in \mathbb{C}$ such that $\|S - z_0 I\| = D_S$ and $\|Re(ST)\| \leq \|Re(\frac{\bar{z}_0}{|z_0|}ST)\|$, then*

$$\omega(ST) \leq (2D_S + \omega(S))\omega(T)$$

Proof. By (12),

$$\|Re(ST)\| \le \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} + \frac{1}{2}\omega(ST - TS^*). \qquad (13)$$

Let $\alpha_0 = \frac{\bar{z}_0}{|z_0|}$, where $z_0 \in \mathbb{C}$ is such that $\|S - z_0 I\| = D_S$. Replacing S by $\alpha_0 S$ in the inequality (13) gives

$$\begin{aligned}
\|Re(\alpha_0 ST)\| &\le \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} + \frac{1}{2}\omega(\alpha_0 ST - \bar{\alpha}_0 TS^*) \\
&\le \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} \\
&+ \frac{1}{2}\omega(\alpha_0(S - z_0 I)T - \bar{\alpha}_0 T(S - z_0 I)^*) \\
&\le \frac{\|T+T^*\|\omega(S)}{2} + \frac{1}{2}D_S D_{T+T^*} + \|S - z_0 I\|\omega(T)
\end{aligned}$$

by (2). Therefore,

$$\|Re(\alpha_0 ST)\| \le \frac{\|T+T^*\|\omega(S)}{2} + \frac{D_S D_{T+T^*}}{2} + D_S\omega(T). \qquad (14)$$

The hypothesis, $\|Re(ST)\| \le \|Re(\frac{\bar{z}_0}{|z_0|}ST)\|$, gives

$$\begin{aligned}
\|Re(ST)\| &\le \frac{\|T+T^*\|\omega(S)}{2} + \frac{D_S D_{T+T^*}}{2} + D_S\omega(T) \qquad \text{(by (14))} \\
&\le \|Re(T)\|\omega(S) + D_S\frac{\|T+T^*\|}{2} + D_S\omega(T) \\
&\le \|Re(T)\|(\omega(S) + D_S) + D_S\omega(T).
\end{aligned}$$

Suppose that $\theta \in \mathbb{R}$. Replacing T by $e^{i\theta}T$ gives

$$\|Re(e^{i\theta}ST)\| \le \|Re(e^{i\theta}T)\|(\omega(S) + D_S) + D_S\omega(T).$$

Taking the supremum over $\theta \in \mathbb{R}$ gives

$$\omega(ST) \le \omega(T)(\omega(S) + D_S) + D_S\omega(T),$$

which is exactly the desired result. $\square$

Remark 2.8. The conditions stated in the Theorem 2.7 apply to many cases. For example, it can be described as when there exist $z_0 \in \mathbb{R}$ such that $\|S - z_0 I\| = D_S$. Also, for the case where the operator S is self-adjointable, then by considering $z_0 = \frac{\max_{\lambda\in\sigma(S)}(\lambda)+\min_{\lambda\in\sigma(S)}(\lambda)}{2}$, the conditions of Theorem 2.7 are established.

Corollary 2.9. *If $S, T \in \mathbb{B}(\mathbb{H})$ and S be self-adjointable, then*

$$\omega(ST) \leq \left(2\|S\| - \min_{\lambda\in\sigma(S)}(|\lambda|)\right)\omega(T).$$

Proof. *Since S is self-adjointable operator, then*

$$D_S = R_S = \frac{\|S\| - \min_{\lambda\in\sigma(S)}(|\lambda|)}{2}.$$

The result follows from Theorem 2.7. □

Acknowledgements
The authors thank the Editorial Board and the referees for their valuable comments that helped to improve the article.

References

[1] A. Abu-Omar and F. Kittaneh, Numerical radius inequalities for products of Hilbert space operators, *J. Operator Theory.* 72(2) (2014), 521-527.

[2] C. Berger, A strange dilatation theorem, *Notices Amer. Math. Soc*, 12 (1965), 590.

[3] S. S. Dragomir, Some inequalities for the norm and the numerical radius of linear operators in Hilbert spaces, *Tamkang J. Math*, 39(1) (2008), 1-7.

[4] J.A. R. Holbrook, Multiplicative properties of the numerical radius in operator theory, *J. Reine Angew. Math*, 237 (1969), 166-174.

[5] C.K. Fong, J.A. Holbrook, Unitarily invariant operator norms, *Canad. J. Math*, 35 (1983), 274-299.

[6] F. Kittaneh, Numerical radius inequalities for Hilbert space operators, *Studia Math*, 168(1) (2005), 73-80.

[7] M. Shah Hosseini, B. Moosavi, Some numerical radius inequalities for products of Hilbert space operators, *Filomat*, 33(7) (2019), 2089-2093.

[8] M. Shah Hosseini, M. E. Omidvar, Some reverse and numerical radius inequalities, *Math. Slovaca*, 68(5) (2018), 1121-1128.

[9] M. Shah Hosseini, B. Moosavi, H. R. Moradi, An alternative estimate for the numerical radius of Hilbert space operators, *Math. Slovaca*, 70(1) (2020), 233-237.

[10] J.G. Stampfli, The norm of a derivation, *Pacific J. Math*, 33 (1970), 737-747.

[11] T. Yamazaki, On upper and lower bounds of the numerical radius and an equality condition, *Studia Math*, 178 (2007), 83-89.

Mohsen Shah Hosseini
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran.
E-mail: mohsen_shahhosseini@yahoo.com

Baharak Moosavi
Department of Mathematics
Assistant Professor of Mathematics
Department of Mathematics, Safadasht Branch, Islamic Azad University, Tehran, Iran.
E-mail: baharak_moosavie@yahoo.com