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Linear Preserves of Logarithm Majorization

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Abstract. Let $X, Y \in \mathbb{R}^n, X, Y > 0$, we say X logarithm majorized by Y, written $X \prec_{log} Y$ if $\log X \prec \log Y$. Let M_{nm}^{+} be the collection of matrices with positive entries. For $X, Y \in M_{nm}^{+}$, it is said that X is *logarithm column (row) majorized* by Y, and is denoted as $X \prec_{log}^{column} Y (X \prec_{log}^{row} Y)$, if $X_j \prec_{log} Y_j (X_i \prec_{log} Y_i)$ for all $j = 1, 2, \cdots m (i = 1, 2, \cdots n)$, where X_j and Y_j (X_i and Y_i) are the ith column (row) of X and Y respectively. In the present paper, the relations column (row) logarithm majorization on M_{nm}^{+} are studied and also all linear operators $T : M_{nm}^{+} \longrightarrow M_{nm}^{+}$ preserving column (row) logarithm majorization will be characterized.

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1 Introduction and Preliminaries

A square matrix D is said to be doubly stochastic matrix if its entries are all nonnegative and all column sums and row sums are one.

Throughout the paper, M_{nm} is the set of all $n\times m$ real matrices, M_n is the set of all $n\times n$ real matrices, M^+_{nm} is the collection of matrices with positive entries, $\mathcal{DS}(n)$ is the set of all $n\times n$ doubly stochastic matrices, $\mathcal{P}(n)$ is the set of all $n\times n$ permutation matrices, $\mathbb{R}^n = M_{n1}$ is the set of all $n\times 1$ (column) vectors, and $\mathbb{R}_n$ is the set of all $1\times n$ (row) vectors. The canonical basis of $\mathbb{R}^n$ will be denoted by $\{e_1, e_2, ...e_n\}$, $e = \sum_{j=1}^n e_j$, $J = ee^t$, and $tr(X) = e^tX$ for any $X \in \mathbb{R}^n$. The set $\{1, 2, \cdots, k\}$ denoted by $\mathbb{N}_k$.

The symbol $X = [X_1|X_2|\cdots|X_m]$ ($X = [X_1/X_2/\cdots/X_n]$) is used for the $n\times m$ matrix whose columns (rows) are $X_1, X_2, \cdots X_m \in \mathbb{R}^n$ ($X_1, X_2, \cdots X_n \in \mathbb{R}^m$).

For $X, Y \in \mathbb{R}^n, X, Y > 0$, it is said that X logarithm majorized by Y, written $X \prec_{log} Y$ if $\log X \prec \log Y$. This definition $X \prec_{log} Y$ is equivalent to

$$\begin{cases} \prod_{i=1}^k x_{[i]} \leq \prod_{i=1}^k y_{[i]}, & k = 1, 2, \cdots, n-1 \\ \prod_{i=1}^n x_{[i]} = \prod_{i=1}^n y_{[i]} \end{cases}$$

where $x_{[i]}$ denotes the ith component of the vector $X^{\downarrow}$ whose components are a decreasing rearrangement of the components of X.

Let $\mathcal{R}$ be a relation on $\mathbb{R}^n$. A linear operator $T : M_{nm} \longrightarrow M_{nm}$ is said to be a linear preserver of $\mathcal{R}$ if $X\mathcal{R}Y$ implies $T(X)\mathcal{R}T(Y)$ for all $X, Y \in M_{nm}$.

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear operator. We say T is a preserver of $\prec_{log}$ if $T(X) > 0$ whenever $X > 0$ and $T(X) \prec_{log} T(Y)$ whenever $X \prec_{log} Y$.

Theorem 1.1. *([1]) Suppose that $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear operator. Then T preserves $\prec$ if and only if T satisfies one of the following conditions (i) or (ii).*

(i) $T(X) = tr(X)a$ for some $a \in \mathbb{R}^n$.

(ii) $T(X) = rPX + str(X)e = rPx + sJX$ *for some* $r, s \in \mathbb{R}$ *and* $P \in \mathcal{P}(n)$.

Theorem 1.2. *([3]) Suppose that* $T : M_{nm} \longrightarrow M_{nm}$ *is a linear operator. Then* T *preserves* $\prec^{column}$ *if and only if there exist* $A_1, A_2, \cdots A_m \in M_{nm}$, $b_1, b_2, \cdots b_m \in \cup_{i=1}^{m} span\{e_i\}$, $P_1, P_2, \cdots, P_m \in \mathcal{P}(n)$ *and* $S \in M_m$ *such that for every* $i \in \mathbb{N}_m, b_i = 0$ *or* $A_1 e_i = A_2 e_i = \cdots = A_m e_i = 0$ *and for all* $X = [X_1|X_2|\cdots|X_m] \in M_{nm}$,

$$T(X) = \sum_{j=1}^{m} (trX_j)A_j + [P_1 X b_1 | P_2 X b_2 | \cdots | P_m X b_m] + JXS. \tag{1}$$

In section 2, we show that every linear mapping which preserves logarithm majorization on $\mathbb{R}^n$ has the form $X \longmapsto rPX$, for all $X \in \mathbb{R}^n$, where $P \in \mathcal{P}(n)$ and some positive number r.

In section 3, we characterize linear operators $T : M_{nm}^{+} \to M_{nm}^{+}$ which preserve logarithm column (row) majorization.

For more details on multivariate majorization, we suggest the reader to [2], [4], [7], [10]. Some types of majorization such as multivariate or matrix majorization were motivated by the senses of vector majorization and were (see [8], [13]).

The study of doubly stochastic matrices in relationship to majorization, see [1], [5], [6], [9], [14], [15].

In recent years, characterize the structure of majorization preserving linear operators on certain spaces of matrices has been intensively studied (see [11], [12], [16], [17]).

2 Linear Preservers of Logarithm Majorization on $\mathbb{R}^n$

In this section we characterize linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$ which preserve $\prec_{log}$.

Lemma 2.1. *Let* $X, Y \in \mathbb{R}^n$ *and* $X, Y > 0$. *Then* $X \prec_{log} Y \prec_{log} X$ *if and only if* $X = PY$ *for some* $P \in \mathcal{P}(n)$.

Proof. Let $X, Y \in \mathbb{R}^n, X, Y > 0$. Thus $X \prec_{log} Y \prec_{log} X$ if and only if

$$\begin{cases} 0 < \prod_{i=1}^{k} x_{[i]} \leq \prod_{i=1}^{k} y_{[i]} \leq \prod_{i=1}^{k} x_{[i]}, & k = 1, 2, \cdots, n-1 \\ 0 < \prod_{i=1}^{n} x_{[i]} = \prod_{i=1}^{n} y_{[i]}. \end{cases}$$

The above condition holds, if and only if $x_{[i]} = y_{[i]}$ for every $i = 1, 2, \cdots, n$, i.e. if and only if $X = PY$ for some $P \in \mathcal{P}(n)$. □

Lemma 2.2. *Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear preserves $\prec_{log}$. Then T preserves $\prec$.*

Proof. Suppose that T is a linear preserver of $\prec_{log}$ on $\mathbb{R}^n$. We show that

$$X \prec Y \prec X \Longrightarrow T(X) \prec T(Y) \prec T(X)$$

for all $X, Y \in \mathbb{R}^n$. First, assume that $X, Y > 0$ and $X \prec Y \prec X$. It follows that $X = PY$ for some $P \in \mathcal{P}(n)$ and hence $X \prec_{log} Y \prec_{log} X$ by Lemma 2.1. Thus by hypothesis $T(X) \prec_{log} T(Y) \prec_{log} T(X)$ and hence there exist $P' \in \mathcal{P}(n)$ such that $T(X) = P'T(Y)$. It follows that $T(X) \prec T(Y) \prec T(X)$.

Next, assume $X = DY$ for some $D \in \mathcal{DS}(n)$. Then $X = \sum_{i=1}^{k} \lambda_i P_i Y$ where $\sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0$ and $P_i \in \mathcal{P}(n)$. Therfore, for every i $(1 \leq i \leq k)$ there exist $Q_i \in \mathcal{P}(n)$ such that $T(X) = \sum_{i=1}^{k} \lambda_i T(P_i Y) = \sum_{i=1}^{k} \lambda_i Q_i T(Y) = D'T(Y)$ where $D' = \sum_{i=1}^{k} \lambda_i Q_i$. It follows that $T(X) \prec T(Y)$. □

Theorem 2.3. *A linear operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear preserves $\prec_{log}$ if and only if it has the form*

$$X \longmapsto rPX, \qquad X \in \mathbb{R}^n$$

for some permutation matrix P and some positive number r.

Proof. We first prove the necessity of the condition. Let T be a linear preserves $\prec_{log}$. If $n = 1$, the result is trivial. So we may suppose that $n > 1$. By Lemma 2.2, T preserves $\prec$. Hence, by Theorem 1.1, T is of the form (i) or (ii).

We show that T not is of the form (i). Put

$$X = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} n \\ 1 \\ \vdots \\ 1 \\ \frac{1}{n} \end{pmatrix}.$$

Then $X \prec_{log} Y$, and hence $T(X) \prec_{log} T(Y)$. Thus $na \prec_{log} (2n + \frac{1}{n} - 2)a$, for some $a \in \mathbb{R}^n$, it follows that $n = (2n + \frac{1}{n} - 2)$ or $n = 2 - \frac{1}{n}$; a contradiction.

Now, suppose that T is of the form (ii). So,

$$\begin{aligned} T(X) = rPX + str(X)e &= P \begin{pmatrix} r+s & s & s & \cdots & s \\ s & r+s & s & \cdots & s \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s & s & s & \cdots & r+s \end{pmatrix} \\ &= P \begin{pmatrix} rx_1 + str(X) \\ rx_2 + str(X) \\ \vdots \\ rx_n + str(X) \end{pmatrix}. \end{aligned}$$

If $s < 0$, choose a positive number m such that $\left|\frac{r+s}{m}\right| < |s|$. Since $\begin{pmatrix} \frac{1}{m} \\ 1 \\ \vdots \\ 1 \end{pmatrix} > 0$, we get $T \begin{pmatrix} \frac{1}{m} \\ 1 \\ \vdots \\ 1 \end{pmatrix} > 0$ and also

$$T \begin{pmatrix} \frac{1}{m} \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{r+s}{m} + (n-1)s \\ r+ns \\ \vdots \\ r+ns \end{pmatrix} < 0,$$

which is a contradiction. Therfore s can not negative.

If $s > 0$, put

$$X := e = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} n \\ 1 \\ \vdots \\ \frac{1}{n} \end{pmatrix}.$$

Then $X \prec_{log} Y$, and hence, $T(X) \prec_{log} T(Y)$. Since

$$T(X) = P \begin{pmatrix} r + ns \\ \vdots \\ r + ns \end{pmatrix} \quad \text{and} \quad T(Y) = P \begin{pmatrix} rn + s(2n - 2 + \frac{1}{n}) \\ r + s(2n - 2 + \frac{1}{n}) \\ \vdots \\ \frac{r}{n} + s(2n - 2 + \frac{1}{n}) \end{pmatrix},$$

thus

$$\begin{aligned} (r+ns)^n &= [rn + s(2n - 2 + \frac{1}{n})][r + s(2n - 2 + \frac{1}{n})] \\ &\quad \cdots [r + s(2n - 2 + \frac{1}{n})][\frac{r}{n} + s(2n - 2 + \frac{1}{n})] \\ &> (rn + ns)(r + ns)^{n-2}(\frac{r}{n} + ns) \\ &= [r^2 + n^2 rs + n^2 s^2 + rs](r + ns)^{n-2} \\ &> [r^2 + 2nrs + n^2 s^2](r + ns)^{n-2} \\ &= (r + ns)^n, \end{aligned}$$

which is a contradiction. Therefore $s = 0$, and the form of T is

$$X \longmapsto rPX$$

where $P \in \mathcal{P}(n)$ and $r > 0$, since $T > 0$.

Clearly, the linear operator $X \longrightarrow rPX$, for $r > 0$ and $P \in \mathcal{P}(n)$, preserves logarithm majorization $\prec_{log}$. $\square$

3 Linear Preservers of Logarithm Column (Row) Majorization on M_{nm}

In this section, we characterize linear operators $T : M_{nm}^+ \to M_{nm}^+$ which preserve logarithm column (row) majorization. First we need some known facts and lemmas.

Definition 3.1. Let $X = [X_1|X_2|\cdots|X_m], Y = [Y_1|Y_2|\cdots|Y_m] \in M_{nm}^+$. The matrix X is said to be *logarithm column(row) majorized* by Y, and is denoted as $X \prec_{log}^{column} Y (X \prec_{log}^{row} Y)$, if $X_j \prec_{log} Y_j (X_i \prec_{log} Y_i)$ for all $j = 1, 2, \cdots m (i = 1, 2, \cdots n)$.

Lemma 3.2. *Let $T : M_{nm}^+ \longrightarrow M_{nm}^+$ be a linear operator that preserve logarithm column majorization $\prec_{log}^{column}$. Then T preserves $\prec^{column}$.*

Proof. Let $X, Y \in M_{nm}^+$ and $X \prec^{column} Y \prec^{column} X$. Then $X_i \prec Y_i \prec X_i$ for all $i = 1, 2, \cdots, m$, and hence $X_i \prec_{log} Y_i \prec_{log} X_i$. Thus $X \prec_{log}^{column} Y \prec_{log}^{column} X$, and hence by hypothesis $T(X) \prec_{log}^{column} T(Y) \prec_{log}^{column} T(X)$. Therfore $(TX)_i \prec_{log} (TY)_i \prec_{log} (TX)_i$, and hence $(TX)_i \prec (TY)_i \prec (TX)_i$ for every $i = 1, 2, \cdots, m$. It follows that $T(X) \prec^{column} T(Y) \prec^{column} T(X)$.

Now, suppose that $X \prec^{column} Y$. Then there exist $D_1, D_2, \cdots, D_m \in \mathcal{DS}(n)$ such that $Y_i = D_i X_i$ for every $i = 1, 2, \cdots, m$. This implies that

$$\begin{aligned} Y &= [D_1X_1|D_2X_2|\cdots|D_mX_m] \\ &= \left[\sum_{j=1}^{k} \lambda_{1j} P_j X_1 \middle| \sum_{j=1}^{k} \lambda_{2j} P_j X_2 \middle| \cdots \middle| \sum_{j=1}^{k} \lambda_{mj} P_j X_m\right] \\ &= \sum_{i_1, i_2, \cdots, i_m = 1}^{k} (\lambda_{1i_1} \lambda_{2i_2} \cdots \lambda_{mi_m}) [P_{i_1} X_1 | P_{i_2} X_2 | \cdots | P_{i_m} X_m], \end{aligned}$$

since $\sum_{j=1}^{k} \lambda_{ij} = 1$ for $i = 1, \ldots, m$ and $P_j \in \mathcal{P}(n)$ for all $j = 1, \ldots, k$. Hence,

$$\begin{aligned} T(Y) &= \sum_{i_1, i_2, \cdots, i_m = 1}^{k} (\lambda_{1i_1} \lambda_{2i_2} \cdots \lambda_{mi_m}) T([P_{i_1} X_1 | P_{i_2} X_2 | \cdots | P_{i_m} X_m]) \\ &= \sum_{i_1, i_2, \cdots, i_m = 1}^{k} (\lambda_{1i_1} \lambda_{2i_2} \cdots \lambda_{mi_m}) [Q_{i_1} X_1' | Q_{i_2} X_2' | \cdots | Q_{i_m} X_m'] \\ &= [D_1' X_1' | D_2' X_2' | \cdots | D_m' X_m']. \end{aligned}$$

Therefore $T(X) \prec^{column} T(Y)$. □

Theorem 3.3. *Let $T : M_{nm}^+ \longrightarrow M_{nm}^+$ be a linear operator. Then T preserves $\prec_{log}^{column}$ if and only if there exist positive real numbers $r_1, r_2, \cdots, r_m$, and $P_1, P_2, \cdots, P_m \in \mathcal{P}(n)$ such that*

$$T(X) = [r_1 P_1 X e_{i_1} | r_2 P_2 X e_{i_2} | \cdots | r_m P_m X e_{i_m}]$$

where $i_1, i_2, \cdots, i_m \in \mathbb{N}_m$.

Proof. The case $n = 1$ being clear, we let $n \geq 2$. Assume the linear operator $T : M_{nm}^+ \longrightarrow M_{nm}^+$ preserves $\prec_{log}^{column}$. Then, by Lemma 3.2, T preserves $\prec^{column}$. Thus, by Theorem 1.2, T is of the form (1). So it is enough to show that $A_1 = A_2 = \cdots = A_m = 0$, and $S = 0$.
First, we prove that $A_j = 0$ for every $j \in \mathbb{N}_m$. Assume that $A_j \neq 0$ for some $j \in \mathbb{N}_m$. Without loss of generality suppose that $A_j e_1 \neq 0$, then $b_1 = 0$. Put

$$X := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{n\times m}, \quad Y := \begin{pmatrix} n & n & \cdots & n \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \\ \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}_{n\times m}.$$

It is clear that $X \prec_{log}^{column} Y$, so $T(X) \prec_{log}^{column} T(Y)$. It easy to see that

$$(TX)_1 = \sum_{j=1}^{m} (trX_j) A_j e_1 + JXSe_1 = n \left(\sum_{j=1}^{m} A_j e_1 + JSe_1 \right)$$

and,

$$(TY)_1 = \sum_{j=1}^{m} (trY_j) A_j e_1 + JYSe_1 = (2n + \frac{1}{n} - 2) \left(\sum_{j=1}^{m} A_j e_1 + JSe_1 \right),$$

where $(TX)_1$ is the first column of $T(X)$ and $(TY)_1$ is the first column of $T(Y)$. Since $(TX)_1 \prec_{log} (TY)_1$, it follows that $n = 2n + \frac{1}{n} - 2$, which is a contradiction. Therefore $A_j = 0$, for every $j \in \mathbb{N}_m$.

Now, we show that $S = 0$. By Theorem 1.2, for every $i, j \in \mathbb{N}_m$, there exist $r_j \in \mathbb{R}$ such that $b_j = r_j e_{i_j}$.

For every $i, j \in \mathbb{N}_m$, consider the embedding $\psi, \psi_j : \mathbb{R}^n \longrightarrow M_{nm}$ by $\psi(X) := [X|X|\cdots|X] = \sum_{i=1}^{m} Xe_i^t$, $\psi_j(X) := [X|\cdots|X|2X|X|\cdots|X] = \psi(X) + Xe_j^t$ and projection $E_i : M_{nm} \longrightarrow \mathbb{R}^n$ by $E_i(X) := Xe_i$. Put $S = [S_1|S_2|\cdots|S_m]$. It is easy to show that for every $j \in \mathbb{N}_m$, $E_jT\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $E_jT\psi(X) = r_jP_jX + tr(S_j)tr(X)e$ preserves $\prec_{log}$. Then by Theorem 2.3, the form of $E_jT\psi$ is $X \longmapsto rPX$, where $r > 0$. Therefore

$$T(X) = [r_1P_1Xe_{i_1}|r_2P_2Xe_{i_2}|\cdots|r_mP_mXe_{i_m}] + JXS,$$

where r_j is positive real number, for every $j \in \mathbb{N}_m$.

Fix $j \in \mathbb{N}_m$. For every $i \in \mathbb{N}_m$, $E_jT\psi_i : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by $E_jT\psi_i(X) = t_iP_iX + \sum_{k=1}^{m} S_{kj}tr(X)e + S_{ij}tr(X)e$ preserves $\prec_{log}$. So, the form of $E_jT\psi_i$ is $X \longmapsto rPX$, where $r > 0$. It follows that for every $i \in \mathbb{N}_m$, we obtain

$$\sum_{k=1}^{m} S_{kj} + S_{ij} = 0 \tag{2}$$

and $t_i = r_i$ or $t_i = 2r_i$. So, $S_{1j} = S_{2j} = \cdots = S_{mj} = 0$ is a solution of the system of m linear equations in m variables (2). Since $j \in \mathbb{N}_m$ is arbitrary, therefore $S = 0$ □

Theorem 3.4. *Let $T : M_{nm}^+ \longrightarrow M_{nm}^+$ be a linear operator. Then T preserves $\prec_{log}^{row}$ if and only if there exist positive real numbers $r_1, r_2, \cdots, r_n$, and $P_1, P_2, \cdots, P_n \in \mathcal{P}(m)$ such that*

$$T(X) = [r_1e_{i_1}XP_1/r_2e_{i_2}XP_2/\cdots/r_me_{i_m}XP_n]$$

where $i_1, i_2, \cdots, i_n \in \mathbb{N}_n$.

Proof. Define $T' : M_{nm}^+ \longrightarrow M_{nm}^+$ by $T'(X) = [T(X^t)]^t$ for all $X \in M_{nm}^+$. It is easy to see that T is a linear preserve of $\prec_{log}^{row}$ if and only if T' is a linear preserver of $\prec_{log}^{column}$. By Theorem 3.3, the desired conclusion is obtained. □

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