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On Weak Generalized Amenability of Triangular Banach Algebras

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Abstract. Let A_1, A_2 be unital Banach algebras and X be an A_1-A_2- module. Applying the concept of module maps, (inner) module generalized derivations and generalized first cohomology groups, we present several results concerning the relations between module generalized derivations from A_i into the dual space A_i^* (for $i = 1, 2$) and such derivations from the triangular Banach algebra of the form $\mathcal{T} := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$ into the associated triangular $\mathcal{T}-$ bimodule $\mathcal{T}^*$ of the form $\mathcal{T}^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$. In particular, we show that the so-called generalized first cohomology group from $\mathcal{T}$ to $\mathcal{T}^*$ is isomorphic to the directed sum of the generalized first cohomology group from A_1 to A_1^* and the generalized first cohomology group from A_2 to A_2^*. Finally, Inspiring the above concepts, we establish a one to one corresponding between weak (resp. ideal) generalized amenability of $\mathcal{T}$ and those amenability of A_i $(i = 1, 2)$.

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1. Introduction

Let A be a Banach algebra and M be a Banach $A-$ bimodule. A *module derivation* $d : A \to M$ is a linear map which satisfies $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. The linear space of all bounded derivations from A into M

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is denoted by $Z^1(A,M)$. As an example, let $x \in M$ and define $d_x : A \to M$ by $d_x(a) := xa - ax$. Then d_x is a module derivation which is called inner. Denoting the linear space of inner derivations from A into M by $N^1(A,M)$, we may consider the quotient space $H^1(A,M) := Z^1(A,M)/N^1(A,M)$, called the first cohomology group from A into M.

A linear mapping $\psi : A \to M$ is called a *module map* if $\psi(ab) = \psi(a)b$. We denote by $\mathcal{B}(A,M)$ the set of all bounded linear module maps from A into M.

Recently, a number of analysts [1, 3, 10] have studied various extended notions of derivations in the context of Banach algebras. For instance, suppose that $x, y \in M$ and define $\delta_{x,y} : A \to M$ by $\delta_{x,y}(a) := xa - ay$, then it is easily seen that $\delta(ab) = \delta(a)b + ad_y(b)$ for every $a, b \in A$. Mathieu [9] called the map of the form $\delta_{x,y}$ an inner generalized derivations. Therefore considering the relation $d(ab) = d(a)b + ad(b)$ as an special case of $\delta(ab) = \delta(a)b + ad(b)$ for all $a, b \in A$, where $d : A \to M$ is a module derivation, leads the theory of derivations to be extensively developed.

The above consideration motivated the authors in [1] to generalize the notion of derivation as follows:

Let A be a Banach algebra and M be a Banach $A-$ module. A linear mapping $\delta : A \to M$ is called *module generalized derivation* if there exists a module derivation $d : A \to M$ such that $\delta(ab) = \delta(a)b + ad(b)$ $(a, b \in A)$. For convenience, we say that a generalized derivation δ is a *$d-$ derivation*. Following as stated in [1], we call $\delta : A \to M$ an *inner generalized derivations* if there exist a $\psi \in \mathcal{B}(A,M)$ and an element $y \in M$ in such a way that $\delta(a) := \psi(a) - ay$. As an example of the above so-called inner generalized derivation suppose that x is an arbitrary element of M and define $\psi : A \to M$ by $\psi(a) := xa$, then it follows that ψ is a bounded module map and $\delta_{x,y}(a) = \psi(a) - ay$. This shows that the definition of the author in [1] covers the notion of Mathieu. From now on, we base our definition of inner generalized derivation on the interpretation of Abbaspour et al in [1]. The method has been used in [1] shows that a bounded module generalized derivation $\delta : A \to M$ is an inner generalized derivation if and only if there exists an element $y \in M$ such that δ is a d_y- derivation.

We denote by $GZ^1(A,M)$ and $GN^1(A,M)$ the linear spaces of all bounded module generalized derivations and inner module generalized derivations from A into M, respectively. Also we call the quotient space

$$GH^1(A,M) := GZ^1(A,M)/GN^1(A,M)$$

the *generalized first cohomology group* from A to M. Applying these notations, the above characterization of inner generalized derivations immediately implies that $GH^1(A,M) = \{0\}$, whenever $H^1(A,M) = \{0\}$.

We recall that the dual space M^* of M is a Banach $A-$ module by regarding the module structure as follows

$$(a.f)(x) = f(xa), (f.a)(x) = f(ax).$$

The Banach algebra A is said to be *generalized amenable* (resp. *weakly generalized amenable*) if every generalized derivation $\delta : A \to M^*$ (resp. $\delta : A \to A^*$) is inner; i.e. $GH^1(A, M^*) = \{0\}$ (resp. $GH^1(A, A^*) = \{0\}$). The notion of an amenable Banach algebra was introduced by Johnson in [8]. Bade et al [2], later defined the concept of weak amenability for commutative Banach algebras. Let I be a closed two sided ideal of A. Then A is said to be $I-$*weakly generalized amenable* if every generalized derivation $\delta : A \to I^*$ is inner. Further, we call A *ideally generalized amenable* if $GH^1(A, I^*) = \{0\}$, for every closed two sided ideal I of A. The notion of ideal amenability was first appeared in the framework of Gorji and Yazdanpanah in [6]. The reader is referred to books [4, 11] for more information on this subject.
Let A_1, A_2 be unital Banach algebras and X be a unital $A_1 - A_2-$ module in the sense that $1_{A_1}x1_{A_2} = x$, for every $x \in X$. In this paper, we deal with the module generalized derivations from the triangular Banach algebra of the form $\mathcal{T} := \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$ into the associated triangular $\mathcal{T}-$ bimodule $\mathcal{T}^*$ of the form $\mathcal{T}^* := \begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$. Such algebras were introduced by Forrest and Marcoux in [5]. Applying several results concerning the relations between module generalized derivations from A_i into the dual space A_i^* (for $i = 1, 2$) and such derivations from $\mathcal{T}$ into $\mathcal{T}^*$, we show that the so-called generalized first cohomology group from $\mathcal{T}$ to $\mathcal{T}^*$ is isomorphic to the directed sum of the generalized first cohomology group from A_1 to A_1^* and the generalized first cohomology group from A_2 to A_2^*. Also, we establish a one to one corresponding between weak (resp. ideal) generalized amenability of $\mathcal{T}$ and those amenability of A_i $(i = 1, 2)$.

2. Module Derivations on Triangular Banach Algebras

Let $\mathcal{T} := \{\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}; a \in A_1,\ x \in X,\ b \in A_2\}$. Then $\mathcal{T}$ equipped with the usual 2×2 matrix addition and formal multiplication with the norm

$$\parallel \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \parallel := \parallel a \parallel + \parallel x \parallel + \parallel b \parallel$$

is a Banach algebra which is called the traingular Banach algebra associated to X. We define $\mathcal{T}^*$ as $\{\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}; f \in A_1^*,\ h \in X^*,\ g \in A_2^*\}$ and

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right] := f(a) + h(x) + g(b).$$

Then $\mathcal{T}^*$ is a triangular $\mathcal{T}-$ bimodule with respect to the following module structure

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} f & h \\ 0 & g \end{pmatrix} := \begin{pmatrix} af + xh & bh \\ 0 & bg \end{pmatrix},$$

$$\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} \cdot \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} := \begin{pmatrix} fa & ha \\ 0 & hx + gb \end{pmatrix}.$$

The following results show some interesting relations between module derivations from A_i to A_i^* (for $i = 1, 2$) and those from $\mathcal{T}$ to $\mathcal{T}^*$. Let $d_i : A_i \longrightarrow A_i^*$ be a bounded module derivation and $\delta_i : A_i \rightarrow A_i^*$ be a bounded module d_i- derivation, for $i = 1, 2$. Define $\triangle_1, \triangle_2 : \mathcal{T} \rightarrow \mathcal{T}^*$ by

$$\triangle_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \triangle_2(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & 0 \\ 0 & \delta_2(b) \end{pmatrix}.$$

Proposition 2.1. *$\triangle_i$ is a bounded D_i- derivation (for $i = 1, 2$), where*

$$D_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} d_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D_2(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & 0 \\ 0 & d_2(b) \end{pmatrix}.$$

Moreover $\triangle_i$ (resp. D_i) is inner if and only if so is δ_i (resp. d_i).

Proof. By simple calculations, it can be observed that D_1 is a derivation and $\triangle_1$ is a D_1- derivation. Also

$$\| \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \| = \| \delta_1(a) \| \leqslant \| \delta_1 \| \{\| a \| + \| x \| + \| b \|\}.$$

Hence $\triangle_1$ (and similarly D_1) is bounded.
Suppose that $\triangle_1$ is inner. Then there exist a bounded linear module map $\Psi : \mathcal{T} \rightarrow \mathcal{T}^*$ and $\begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix} \in \mathcal{T}^*$ such that

$$\begin{aligned} \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} &= \triangle_1(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \\ &= \Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) - \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix} \\ &= \Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) - \begin{pmatrix} af' & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Since $\Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \in \mathcal{T}^*$, so there exists $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \in \mathcal{T}^*$ for which

$\Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}$. Define $\psi_1 : A_1 \to A_1^*$ by $\psi_1(a) := f_1$.

It follows from boundedness and linearity of Ψ that, ψ_1 is a bounded linear operator. Also, there exist $h_3 \in X^*$ and $g_3 \in A_1^*$ such that

$\Psi(\begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \psi_1(ab) & h_3 \\ 0 & g_3 \end{pmatrix}$ and

$$\begin{aligned}
\begin{pmatrix} \psi_1(ab) & h_3 \\ 0 & g_3 \end{pmatrix} &= \Psi(\begin{pmatrix} ab & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \Psi(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \psi_1(a) & h_1 \\ 0 & g_1 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \psi_1(a)b & h_1 b \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Thus ψ_1 is a module map and

$$\begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1(a) & h_1 \\ 0 & g_1 \end{pmatrix} - \begin{pmatrix} af' & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1(a) - af' & h_1 \\ 0 & g_1 \end{pmatrix}.$$

Hence $\delta_1(a) = \psi_1(a) - af'$, $g_1 = 0$ and $h_1 a = 0$ for all $a \in A_1$. Therefore δ_1 is an inner generalized derivation and $h_1 = 0 = g_1$.

Conversely, if $\delta_1(a) : A_1 \to A_1^*$ is an inner generalized derivation, then there exist $\psi_1 \in \mathcal{B}(A_1, A_1^*)$ and $f' \in A_1^*$ such that $\delta_1(a) = \psi_1(a) - af'$. Then

$$\begin{aligned}
\triangle_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) &= \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \psi_1(a) - af' & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \psi_1(a) & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} f' & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Define $\Psi : \mathcal{T} \to \mathcal{T}^*$ by $\Psi(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \psi_1(a) & 0 \\ 0 & 0 \end{pmatrix}$. It follows from boundedness and linearity of ψ_1 that, Ψ is a bounded linear module map and consequently, $\triangle_1$ is an inner generalized derivation. $\square$

Theorem 2.2. *Let $\mathcal{T}^*$ be the triangular bimodule $\begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$ associated to the triangular Banach algebra $\mathcal{T}$. Assume that $D : \mathcal{T} \to \mathcal{T}^*$ be a bounded derivation and $\triangle : \mathcal{T} \to \mathcal{T}^*$ be a bounded $D-$ derivation. Then for $i = 1, 2$, there exist a continuous derivation $d_i : A_i \to A_i^*$, a continuous d_i- derivation $\delta_i : A_i \to A_i^*$, and $h_0, h_0' \in X^*$ such that*

$$\triangle(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & h_0'x + \delta_2(b) \end{pmatrix}.$$

Proof. First we show that there exist an element $h_0 \in X^*$, a continuous derivation $d_1 : A_1 \to A_1^*$, and a continuous derivation $d_2 : A_2 \to A_2^*$ such that

$$D(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} d_1(a) - xh_0 & h_0 a - bh_0 \\ 0 & h_0 x + d_2(b) \end{pmatrix}.$$

For this aim using some ideas of [5], we can verify that

(i) There exists $h_0 \in X^*$ such that $D(\begin{pmatrix} 1_A & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}$.

(ii) There exists a bounded derivation $d_1 : A_1 \to A_1^*$ such that $D(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} d_1(a) & h_0 a \\ 0 & 0 \end{pmatrix}$.

(iii) $D(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h_0 x \end{pmatrix}$.

(iv) There exist a bounded derivation $d_2 : A_2 \to A_2^*$ such that $D(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -bh_0 \\ 0 & d_2(b) \end{pmatrix}$.

Now a similar calculation shows that

(i') There exist $f \in A_1^*$, $h_0' \in X^*$ such that $\triangle(\begin{pmatrix} 1_{A_1} & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f & h_0' \\ 0 & 0 \end{pmatrix}$.

(ii') There exists a bounded d_1-derivation δ_1: $A_1 \to A_1^*$ such that $\triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} \delta_1(a) & h_0'a \\ 0 & 0 \end{pmatrix}$.

(iii') $\triangle(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} -xh_0 & 0 \\ 0 & h_0'x \end{pmatrix}$.

(iv') There exist a bounded d_2- derivation $\delta_2 : A_2 \to A_2^*$ such that $\triangle(\begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}) = \begin{pmatrix} 0 & -bh_0 \\ 0 & \delta_2(b) \end{pmatrix}$.

and finally $\triangle(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & h_0'x + \delta_2(b) \end{pmatrix}$.

For this aim following the parts (i) and (ii), we only prove the parts (i') and (ii').The other parts are similar.

(i') There exist $f \in A_1^*$, $h \in X^*$, and $g \in A_2^*$ such that $\triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}$.

On the other hand

$$\begin{aligned}
\begin{pmatrix} f & h \\ 0 & g \end{pmatrix} &= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & h \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} f & h \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

It follows that $g = 0$. Taking $h_0' := h$, completes the proof.

(ii') There exist $f_1 \in A_1^*$, $h_1 \in X^*$, and $g_1 \in A_2^*$ such that $\triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) = \begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}$. On the other hand

$$\begin{aligned}
\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} &= \triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \triangle(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix})\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} D(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \begin{pmatrix} f & h_0' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1(a) & h_0a \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} fa & h_0'a \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} d_1(a) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} fa + d_1(a) & h_0'a \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

It follows that $g_1 = 0$, $h_1 = h'_0 a$, and $f_1 = fa + d_1(a)$.
Take $\delta_1(a) := f_1$. We show that δ_1 is a d_1- derivation. Trivially δ_1 is linear. Moreover

$$\begin{aligned}
\begin{pmatrix} \delta_1(a_1a_2) & h'_0a_1a_2 \\ 0 & 0 \end{pmatrix} &= \triangle(\begin{pmatrix} a_1a_2 & 0 \\ 0 & 0 \end{pmatrix}) \\
&= \triangle\left[\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}\right] \\
&= \begin{pmatrix} \delta_1(a_1) & h'_0a_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & 0 \end{pmatrix}+\begin{pmatrix} a_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} d_1(a_2) & h_0a_2 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta_1(a_1)a_2 & h'_0a_1a_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a_1d_1(a_2) & 0 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \delta_1(a_1)a_2 + a_1d_1(a_2) & h'_0a_1a_2 \\ 0 & 0 \end{pmatrix}.
\end{aligned}$$

Therefore δ_1 is a d_1- derivation. Further since $\triangle$ is bounded, so

$$\begin{aligned}
\| \delta_1(a) \| &\leqslant \| \delta_1(a) \| + \| h'_0a \| \\
&= \| \begin{pmatrix} \delta_1(a) & h'_0a \\ 0 & 0 \end{pmatrix} \| \\
&= \| \triangle(\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}) \| \\
&\leqslant \| \triangle \| \| a \| .
\end{aligned}$$

It follows that δ_1 is bounded and $\| \delta_1 \| \leqslant \| \triangle \|$. □

Theorem 2.3. *Let A_1, A_2 be unital Banach algebras, X be an A_1-A_2-module and let $\mathcal{T}^*$ be the triangular bimodule $\begin{pmatrix} A_1^* & X^* \\ 0 & A_2^* \end{pmatrix}$ associated to the triangular Banach algebra $\mathcal{T}$. Then*

$$GH^1(\mathcal{T}, \mathcal{T}^*) \cong GH^1(A_1, A_1^*) \oplus GH^1(A_2, A_2^*).$$

Proof. Define $\pi : GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*) \to GH^1(\mathcal{T}, \mathcal{T}^*)$ by $\pi(\delta_1, \delta_2) :=$

$[\triangle^{\delta_1}_{\delta_2}]$, where
$\triangle^{\delta_1}_{\delta_2} := \triangle_1 + \triangle_2$ (as we defined in Proposition 2.1) and $[\triangle^{\delta_1}_{\delta_2}]$ represents the equivalent class of $\triangle^{\delta_1}_{\delta_2}$ in $GH^1(\mathcal{T}, \mathcal{T}^*)$. Clearly π is linear. We are going to show that π is surjective. For, let $\triangle$ be a bounded $D-$ derivation from $\mathcal{T}$ to $\mathcal{T}^*$. Let δ_1, δ_2, h_0 and h'_0 be as in Theorem 2.2. Then trivially

$$(\triangle - \triangle_{\delta_2}^{\delta_1})(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & \delta_2(b) + h_0'x \end{pmatrix} - \begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix}$$
$$= \begin{pmatrix} -xh_0 & h_0'a - bh_0 \\ 0 & h_0'x \end{pmatrix}$$
$$= \begin{pmatrix} 0 & h_0' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} - \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & h_0 \\ 0 & 0 \end{pmatrix}.$$

Taking $\Psi(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & h_0' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$, we conclude that

$$\triangle - \triangle_{\delta_2}^{\delta_1} \in GN^1(\mathcal{T}, \mathcal{T}^*).$$

This implies that $[\triangle] = [\triangle_{\delta_2}^{\delta_1}]$ and π is surjective. Therefore

$$GH^1(\mathcal{T}, \mathcal{T}^*) \cong \ GZ^1(A_1, A_1^*) \oplus GZ^1(A_2, A_2^*)/\ker\pi.$$

It is enough to show that $\ker\pi = GN^1(A_1, A_1^*)\oplus GN^1(A_2, A_2^*)$. For this aim, note that if
$(\delta_1, \delta_2) \in \ker\pi$, then $\triangle_{\delta_2}^{\delta_1}$ is an inner generalized derivation. So there exist $\Psi \in \mathcal{B}(\mathcal{T}, \mathcal{T}^*)$ and $\begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix} \in \mathcal{T}^*$ such that

$$\begin{pmatrix} \delta_1(a) & 0 \\ 0 & \delta_2(b) \end{pmatrix} = \triangle_{\delta_2}^{\delta_1}(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})$$
$$= \Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) - \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} f' & h' \\ 0 & g' \end{pmatrix}$$
$$= \Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) - \begin{pmatrix} af' & bh' \\ 0 & bg' \end{pmatrix}.$$

Since $\Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}) \in \mathcal{T}^*$, so there exists $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix} \in \mathcal{T}^*$ for which $\Psi(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})=$ $\begin{pmatrix} f_1 & h_1 \\ 0 & g_1 \end{pmatrix}$. Define $\psi_i : A_i \to A_i^*$ $(i = 1, 2)$, by $\psi_1(a) := f_1$ and $\psi_2(b) := g_1$. Following the arguments as stated in the proof of Proposition 2.1, it can be obtained that for $i = 1, 2$, $\psi_i : A_i \to A_i^*$ is a bounded linear module map and Hence $\delta_1(a) = \psi_1(a) - af'$ and $\delta_2(b) = \psi_2(b) - bg'$ for all $a \in A_1$, $b \in A_2$. So δ_1 and δ_2 are the inner generalized derivations. Hence

$$(\delta_1, \delta_2) \in GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*).$$

Conversely, if $(\delta_1, \delta_2) \in GN^1(A_1, A_1^*) \oplus GN^1(A_2, A_2^*)$, then δ_1 and δ_2 are inner. By Proposition 2.1, $\triangle_i$ is an inner D_i- derivation, for $i = 1, 2$. Hence $\triangle_1 + \triangle_2 = \triangle_{\delta_2}^{\delta_1}$ is an inner generalized derivation. Therefore

$$\ker\pi = HG_{d_1}N^1(A_1, A_1^*) \oplus HG_{d_2}N^1(A_2, A_2^*).$$

Now we have

$$\begin{aligned} GH^1(\mathcal{T}, \mathcal{T}^*) &= \tfrac{GZ^1(A_1,A_1^*)\oplus GZ^1(A_2,A_2^*)}{GN^1(A_1,A_1^*)\oplus GN^1(A_2,A_2^*)} \\ &\cong \tfrac{GZ^1(A_1,A_1^*)}{GN^1(A_1,A_1^*)} \oplus \tfrac{GZ^1(A_2,A_2^*)}{GN^1(A_2,A_2^*} \\ &= GH^1(A_1, A_1^*) \oplus GH^1(A_2, A_2^*). \quad \square \end{aligned}$$

In the rest of this paper, we investigate ideal generalized amenability of triangular Banach algebras. For this aim, first we characterize the form of each closed two sided ideal of triangular Banach algebras as following:

Proposition 2.4. *Let A_1, A_2 be unital Banach algebras, X be a unital A_1-A_2- module and let $\mathcal{T}$ be the unital triangular Banach algebra associated to X. Then, $\mathcal{I}$ is a closed two sided ideal of $\mathcal{T}$ if and only if there exist a closed two sided ideal I_1 of A_1, a closed two sided ideal I_2 of A_2 and a closed A_1-A_2-submodule Y of X in such a way that $\mathcal{I} = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}$ and $I_1X \cup XI_2 \subseteq Y$.*

Proof. Suppose that $\mathcal{I}$ is a closed two sided ideal in $\mathcal{T}$. Define I_1, I_2 and Y as follows:

$$I_1 := \{a \in A_1; \text{there exist } b \in A_2 \text{ and } x \in X \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \mathcal{I}\},$$

$$I_2 := \{b \in A_2; \text{there exist } a \in A_1 \text{ and } x \in X \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \mathcal{I}\} \text{ and}$$

$$Y := \{x \in X; \text{there exist } a \in A_1 \text{ and } b \in A_2 \text{ with } \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \mathcal{I}\}.$$

It is easy to check that I_1 is a closed two sided ideal in A_1, I_2 is a closed two sided ideal in A_2 and Y is a closed $A_1 - A_2-$submodule of X. Also if $a \in I_1$ and $x_1 \in X$, then there exist $b \in A_2$ and $x \in X$ with $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} \in \mathcal{I}$ and consequently, $\begin{pmatrix} 0 & ax_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & x_1 \\ 0 & 0 \end{pmatrix} \in \mathcal{I}$ which means $ax_1 \in Y$. Similarly, one can show that $XI_2 \subseteq Y$. To prove the closedness of Y, let $\{x_n\}$ be a sequence of Y fulfilling $x_n \to x$, for some $x \in X$. Hence, for each $n \in \mathbb{N}$ there exist $a_n \in A_1$ and $b_n \in A_2$ such that $\begin{pmatrix} a_n & x_n \\ 0 & b_n \end{pmatrix} \in \mathcal{I}$.

On the other hand, for each $\epsilon > 0$ there exist $N \in \mathbb{N}$ such that for each $n \geqslant N$, $\|x_n - x\| < \epsilon$. It follows that

$$\|\begin{pmatrix} a_n & x_n \\ 0 & b_n \end{pmatrix} - \begin{pmatrix} a_n & x \\ 0 & b_n \end{pmatrix}\| = \|x_n - x\| < \epsilon.$$

Applying the closedness of $\mathcal{I}$, we conclude that $x \in Y$ which completes the proof of the closedness of Y. The converse is similar. □

Remark 2.5. *Let A_1, A_2 be unital Banach algebras, X be a unital $A_1 - A_2-$ module and let*

$\mathcal{I} = \begin{pmatrix} I_1 & Y \\ 0 & I_2 \end{pmatrix}$ *be a closed ideal of the unital triangular Banach algebra $\mathcal{T}$ associated to X. Define $\mathcal{I}^*$ as*

$$\{\begin{pmatrix} f & h \\ 0 & g \end{pmatrix}; f \in I_1^*,\ h \in Y^*,\ g \in I_2^*\} \quad \text{and} \quad \begin{pmatrix} f & h \\ 0 & g \end{pmatrix}\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}\right] := f(a)+h(x)+g(b).$$

It is trivial that $\mathcal{I}^*$ is a triangular $\mathcal{I}-$ bimodule with respect to the module structure as stated at the first part of Section 2. Let $d_i : A_i \to I_i^*$ be a bounded derivation and $\delta_i : A_i \to I_i^*$ be a bounded d_i- derivation, for $i = 1, 2$. Define $\triangle_1, \triangle_2 : \mathcal{T} \to \mathcal{I}^*$ by

$$\triangle_1 \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} := \begin{pmatrix} \delta_1(a) & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \triangle_2 \begin{pmatrix} a & x \\ 0 & b \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ 0 & \delta_2(b) \end{pmatrix}.$$

(i) Following exactly the method has been used in Proposition 2.1, shows that $\triangle_i$ is a bounded D_i- derivation (for $i = 1, 2$), where

$$D_1(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} d_1(a) & 0 \\ 0 & 0 \end{pmatrix} \text{ and } D_2(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) := \begin{pmatrix} 0 & 0 \\ 0 & d_2(b) \end{pmatrix}.$$

Moreover $\triangle_i$ (resp. D_i) is inner if and only if so is δ_i (resp. d_i).
(ii) Assume that $D : \mathcal{T} \to \mathcal{I}^*$ be a bounded derivation and $\triangle : \mathcal{T} \to \mathcal{I}^*$ be a bounded D_i-derivation. Then similar to the proof of Theorem 2.2, it can be shown that for $i = 1, 2$ there exist a bounded derivation $d_i : A_i \to I_i^*$, a bounded d_i- derivation $\delta_i : A_i \to I_i^*$, and $h_0, h_0' \in X^*$ such that

$$\triangle(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}) = \begin{pmatrix} \delta_1(a) - xh_0 & h_0'a - bh_0 \\ 0 & h_0'x + \delta_2(b) \end{pmatrix}.$$

(iii) As an immediate consequence of the parts (i) and (ii), it is easily seen that

$$GH^1(\mathcal{T},\mathcal{I}^*) \cong GH^1(A_1,I_1^*) \oplus GH^1(A_2,I_2^*).$$

Corollary 2.6. *$\mathcal{T}$ is weakly (resp. ideally) generalized amenable if and only if A_i is weakly (resp. ideally) generalized amenable, for $i = 1, 2$.*
In [7] (resp. [6]), it has been proved that every C^*-algebra is weakly (resp. ideally) amenable. Therefore $H^1(A,A^*) = \{0\}$ (resp. $H^1(A,I^*) = \{0\}$, for every closed two sided ideal I of A) and it follows from the characterization of generalized inner derivation that $GH^1(A,A^*) = \{0\}$ (resp. $GH^1(A,I^*) = \{0\}$, for every closed two sided ideal I of A) which means every C^*-algebra is weakly (resp. ideally) generalized amenable.

Corollary 2.7. *For each C^*-algebra A, the triangular Banach algebras $\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$ and $\begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$ are weakly (resp. ideally) generalized amenable.*

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