(α, β)-monotone variational inequalities over arbitrary product sets

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Abstract. The purpose of this paper is to introduce some new concept and extend the usual ones are introduced for variational inequality problems over arbitrary product sets. Our result is a new version of the results obtained by Igor V. Konnov [Relatively monotone variational inequalities over product sets, Operation research letters 28(2001), 21-26].
key words : variational inequality, produce set, (α, β)-monotone, locally convex.

1. Introduction

In recent years variational inequality have been generalized and extended in various different directions in abstract see ref.[11, 12]. Moreover many authors have investigated vector variational inequalities in abstract spaces; see ref.[7, 8, 9, 16]. The development of efficient methods for proving existence of solution is one the most interesting and important in variational inequalities theory and equilibrium type problem arising in operation research, economics, mathematical, physics and other fields. It is well known that most of such problems arising game theory, transportion and network economics have a decomposable structure i.e. thay can be formulated as variational inequalities over Cartesian produce sets; see e.g. Nagurney [15] and Ferris and Pang [5]. The most existence results for such variational inequalities established under either compactness of the feasible set in the norm topology or monotonicity-type assumption regardless of the decomposable structure of the variational inequalities see [3, 10]. In fact Bianchi [2] considered the corresponding extension of P-mapping and noticed that thay are not sufficient to derive existence results with the help of Fans lemma.
In this paper we present (α, β)-monoton concept, which is suitable for

variational inequalities on arbitrary produce of locally convex spaces, and our results extend theorems of Konnov. [12].
Throughout this paper, let I be any set indexes, $\langle I\rangle$ denote the set of all nonempty finite subsete of I and let P denotes the set of all positive vector in $l^{\infty}(I)$ i.e. P= $\{(u_i)\in l^{\infty}(I): \ u_i>0\ \forall i\in I\}$, $l^{\infty}(I)=\{(u_i)_{i\in I}: \ \exists c>0\ , |u_i|<c\ \ \forall i\in I\}$.

2. Basic Definition

For each $i\in I$, let X_i be a locally convex spaces and X_i^* its dual.
Set $X=\prod_{i\in I}X_i$, so that for each $x\in X$, we have $x=(x_i)_{i\in I}$ where $x_i\in X_i$. We define the map $<,>$: $X^*\times X\to\mathbb{R}$ by $<f,x>=f(x)$ and $\ll,\gg$: $\prod_{i\in I}X_i^*\times X\to\mathbb{R}\cup\{+\infty\}$
by $\ll g,x\gg=<g,x>^+-<g,x>^-$
where $x\in X, g\in\prod_{i\in I}X_i^*$ and
$<g,x>^+=\sup_{J\in\langle I\rangle}\{\sum_{j\in J}<g_j,x_j>:<g_j,x_j>\geq 0\ \ \forall j\in J\}$, $<g,x>^-=<-g,x>^+$
we define the vector space X_w^* as follows :

$$X_w^*=\{g\in\prod_{i\in I}X_i^*\ :\ (g,x)\in D^{e'}_{\ll,\gg}\ :\ \forall x\in\prod_{i\in I}X_i\}$$

where $D^{e'}_{\ll,\gg}=\{(g,x)\in(\prod_{i\in I}X_i^*)\times X:\ll g,x\gg<\infty\}$.
It is clear that $D^{e'}_{\ll,\gg}\neq\emptyset$, $X_w^*\neq\emptyset$.
Let K_i be nonempty subset of X and let $K=\prod_{i\in I}K_i$, Next for each $i\in I$, let $G:K\to X_w^*$ be a mapping, now we define $G_i:K\to X_i^*$ by $G_i=P_ioG$, where $P_i:X_w^*\to X_i^*$ is defined to be $P_i\Big((g_j)_{j\in J}\Big)=g_i$.
we note that $G(x)=(G_i(x))_{i\in I}$ and $\ll G(x),y-x\gg=\sum_{i\in I}<G_i(x),y_i-x_i><\infty$. In this paper we study variational inequality problem as following :
a) The SyVIP(G,K) consist of finding $x^*\in K$ such that

$$<G_i(x^*),y_i-x_i^*>\geq 0\qquad\qquad \forall y_i\in K_i\ ,i\in I$$

We denote by $S_{SyVIP}(G,K)$ the solution set of the SyVIP(G,K).
b) For every given $u=(u)_{i\in I}\in P$ the $VIP(G,K,u)$ consist of finding $x^*\in K$ such that

$$\ll\big(u_iG_i(x^*)\big)_{i\in I},y-x^*\gg=\sum_{i\in I}u_i<G_i(x^*),y_i-x_i^*>\geq 0\qquad \forall y_i\in K_i\ ,i\in I$$

We denote by $S_{VIP}(G,K,u)$ the solution set of the VIP(G,K,u).
c) The dual VIP(G,K,u) (abbreviated DVIP(G,K,u)) consist of finding

$x^* \in K$ such that

$$\ll \big(u_i G_i(y)\big)_{i\in I}, y - x^* \gg = \sum_{i\in I} < u_i G_i(y), y_i - x_i^* > \geq 0 \qquad \forall y_i \in K_i\,, i \in I$$

We denote by $S_{DVIP}(G,K,u)$ the solution set of the DVIP(G,K,u).

Definition 2.1. for each $u \in l^\infty(I)$, the mapping $G : K \to X_w^*$ is said to be u-hemicontinuous, if for any $x, y \in K$, the mapping $g : [0,1] \to \mathbb{R}$ by $g(\lambda) = \sum_{i\in I} u_i < G_i(x + \lambda(x-y)), y_i - x_i >$ is continuous.

We note that for each $\lambda \in [0,1]$, $g(\lambda) < \infty$.

Definition 2.2. Let $\alpha, \beta \in l^\infty(I)$, the mapping $G : K \to X_w^*$ is said to be

a) (α,β)-monotone, if for all $x, y \in K$, we have

$$\ll \beta G(x) - \alpha G(y), x - y \gg \geq 0$$

And strictly (α,β)-monotone, if the inequality is strict for all $x \neq y$.

b) (α,β)-psedumonotone, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y - x \gg \geq 0 \implies \ll \beta G(y), y - x \gg \geq 0$$

And strictly (α,β)-psedumonotone, if the second inequality is strict for all $x \neq y$.

c) (α,β)-psedumonotone-like, if for all $x, y \in K$, we have

$$\ll \alpha G(x), y - x \gg > 0 \implies \ll \beta G(y), y - x \gg \geq 0$$

And strictly (α,β)-psedumonotone-like, if the second inequality is strict for all $x \neq y$.

Lemma 2.3. Let $\alpha, \beta \in P$ and $G : K \to X_w^*$ then

a) $S_{SyVIP}(G,K) = S_{VIP}(G,K,\alpha)$

b) $S_{DVIP}(G,K,\alpha) = S_{DVIP}(G,K,\beta)$

c) $S_{VIP}(G,K,\alpha) = S_{VIP}(G,K,\beta)$

Proof: by definition 2.2 the desired result is obtained.

Lemma 2.4. Let $\alpha \in P$ and the mapping $G : K \to X_w^*$ be α-hemicontinuous, then

$$S_{DVIP}(G,K,\alpha) \subseteq S_{VIP}(G,K,\alpha)$$

Proof: let $x^* \in S_{DVIP}(G,K,\alpha)$, thus

$$\sum_{i\in I} < \alpha_i G_i(y), y_i - x_i^* > \geq 0 \qquad \forall y \in K$$

Set $y = x^* + \lambda(y - x^*)$, therefore α-hemicontinuous implies $x^* \in S_{VIP}(G,K,\alpha)$.

Lemma 2.5. Let $\alpha, \beta \in P$ and the mapping $G : K \to X_w^*$ be β-hemicontinuous, and (α,β)-psedumonotone then

$$S_{DVIP}(G,K,\beta) = S_{VIP}(G,K,\alpha).$$

The proof is parollel to the proof of lemma 2.4 and so is omited.

corollary 2.6. Let the conditions of lemma 2.5 hold, then

$$S_{DVIP}(G,K,\alpha) = S_{VIP}(G,K,\alpha) = S_{SyVIP}(G,K).$$

Definition : A set-valued $F : E \to 2^E$ is called a KKM-mapping if, for every finite subset $\{x_1, x_2, ..., x_n\}$ of E,
$\text{Co}\{x_1, x_2, ..., x_n\} \subseteq \bigcup_{i=1}^{n} F(x_i)$, where Co denotes the convexhull.

Lemma 2.7. [Fan-4] Let E be a Hausdorff topological vector space and $F : E \to 2^E$ be a KKM-mapping such that for any $x \in E, F(x)$ is closed and $F(x_0)$ contained in a compact set $D \subseteq E$ for some $x_0 \in E$. Then $\bigcap_{x\in E} F(x) \neq \emptyset$.

3. Main results

Theorem 3.1. suppose that $\alpha, \beta \in P, X$ locally convex space,
$K \subseteq X$ is nonempty weakly compact and let the mapping
$G : K \to X_w^*$ be β-hemicontinuous, and (α, β)-psedumonotone then $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Proof : Define set-valued mapping $H, T : K \to 2^K$ by

$$T(y) = \{x \in K \ : \ \sum_{i\in I} < \alpha_i G_i(x), y_i - x_i > \geq 0\}$$

$$T(y) = \{x \in K \ : \ \sum_{i\in I} < \beta_i G_i(y), y_i - x_i > \geq 0\}.$$

We denote T is KKM-mapping. Let $\{y^1, y^2, ..., y^n\}$ be any finite subset of K and $z \in Co\{y^1, y^2, ..., y^n\}$ then $z = \sum_{j=1}^{n} \lambda_j y^j$, for some $\lambda_j \geq 0, j = 1, 2, ..., n$. If $z \notin \bigcup_{j=1}^{n} T(y^j)$, then

$$\sum_{i\in I} \alpha_i < G_i(z), y_i^j - z_i > < 0 \qquad \forall j = 1, 2, ..., n.$$

Therefore, $0 = \sum_{i\in I} \alpha_i < G_i(z), z_i - z_i > < 0$, is a contradiction, hence T is a KKM-mapping. Since $\overline{T(y)}^w \subseteq K$, by lemma 2.7 $\bigcap_{y\in K} \overline{T(y)}^w \neq \emptyset$. Since G is (α, β)-psedumonotone we have $T(y) \subseteq H(y)$, that is clear $H(y)$ is weakly closed, therefore $\bigcap_{y\in K} H(y) \neq \emptyset$, that is

$$S_{DVIP}(G,K,\alpha) \neq \emptyset$$

But Lemma 2.5. implies that $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Corollary 3.2. Suppose that $\alpha, \beta \in P, X$ locally convex space,
$K \subseteq X$ is nonempty weakly compact and let the mapping
$G : K \to X_w^*$ be β-hemicontinuous, and stictly (α, β)-psedumonotone then $VIP(G,K,\alpha)$ has a uniqe solution.

Proof : Theorem 3.1 implies that $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Assume P for contradiction, that $x^1, x^2 \in S_{VIP}(G,K,\alpha)$, and $x^1 \neq x^2$ for any $y \in K$, we have $\sum_{i\in I} \alpha_i < G_i(x^1), x_i^2 - x_i^1 > \geq 0$, since strictly (α,β)-psedumonotone, implies

$$\sum_{i\in I} \beta_i < G_i(x^2), x_i^1 - x_i^2 > < 0 \Longrightarrow x^2 \notin S_{VIP}(G,K,\beta) = S_{VIP}(G,K,\alpha).$$

Corollary 3.3. Suppose that $\alpha, \beta \in P, X$ locally convex space and the mapping $G : K \to X_w^*$ be β-hemicontinuous, and stictly (α,β)-psedumonotone and let there exist a weakly compact subset E of K, and a piont $e \in E \cap K$ such that
$\sum_{i\in I} \alpha_i < G_i(x), e_i - x_i > < 0 \quad \forall x \in K \setminus E$ then
$S_{VIP}(G,K,\alpha) \neq \emptyset$.

Proof : Since proof of theorem 3.1 and under the above assumption we have $T(e) \subseteq E$, therefore $\overline{T(y)}^w$ is weakly compact, hence by lemma 2.7 we have $S_{VIP}(G,K,\alpha) \neq \emptyset$.

Next theorem shows that this paper generalized theorems of V.Konnov [3].

Theorem 3.4. Suppose that $|I| = n < \infty$ and $\{X_i\}_{i\in I}$ be finite family of locally convex spaces. Then

$$\prod_{i\in I} X_i^* = X_w^*$$

Proof : For each $f \in X^*$, we define $< f, \overline{x_i} >=< f_i, x_i >$
where $\overline{x_i} = (0, ..., x_i, 0, ...), f_i \in X_i{}^*$. Now we define $\Gamma : X^* \to X_w^*$ by $\Gamma(f) = (f_i)_{i\in I}$.
It is easy to see that Γ is homeomorphism , that complate proof .

References

[1] C. Baiocchi and A. Capelo, Variational and Quasi-Variational Inequalities, Application to Free boundary Problems, J. Wiley, New York 1984.

[2] M. Bianchi, Pseudo P-monotone operators and variational inequalities, Research Report No. 6, Institute di Econometrica e Matematica per le Decision Economiche, Universita Cattolica del Sa Cuox, Milan, 1993.

[3] G.Y. Chen, Existence of solution for a vector variational inequality: An extension of Hartman-Stampcchia thoery, J. Optim. Theory Appl. 74(1992), 445-456

[4] K. Fan, A generalization of Tychonoff fixed point theorem, Math. Ann. 142(1961), 305-310

[5] M. Ferris and J.S. Pang, Engineering and economics applications of complementarity problems, SIAM Rev. 39(1997), 669-713

[6] M. Fiedler and V. Ptak, On matrices with non positive off diagonal elements and principal minors, Czechoslovak Math. J. 12(1962), 382-400

[7] F. Giannessi, Vector Variational Inequalities and Vector Vatiational Equilibrium, Kluwer Academic Publisher, Dordrecht, 2000.

[8] F. Giannessi, Theorem of alternative programs and complementarity problems, In Variational Inequalities and Complmentarity Problems, (Edited by R.W. Cottle, F. Giannessi and J.L. Lions), Wiley, New York, 1980.

[9] F. Giannessi, and A. Maugeri, Variational Inequalities and Network Equilibrium Plenum, New York, 1995.

[10] N. Hadjisavvas and S. Schahble, Quasi monotonicity and pseudo monotonicity in variational inequqalities and equlibrium: in J.P. Crouzeix, J.E. Martinez-Legaz, M. Volle (Eds), Generalized Convexity, Generalized Monotonicity, Kluwer Academic Publishers, Dordecht, 1998, pp.257-275.

[11] D. Kinderlehrer and G. Stampacchia, An Introduction to Variational Inequalities and Their Applications, Academic Press, New York, 1980.

[12] V. konnov, Relatively monotone variational inequalities over product sets, Oper. Res. Letts. 28(2001), 21-26

[13] V. Konnov, Combined Relaxation methods for variational inequality problems over the product sets, Labachevskii J. Math. 2(1999), 711-728.

[14] V. Konnov, A combined Relaxation methods for decomposable variational inequalities, Optimiz. Methods and Software 10(1999), 711-728.

[15] A. Nagurney, Network Economics: A Variational Inequality Approach, Kluwer Academic Publishers Group, Dordrecht, 1993.

[16] W. Oettli, A remark on vector valued equibria and generalized monotonicity, Acta Math. Vietnam 22(1997), 213-221

[17] J. Parida, M. Sahoo and A. Kumar, A variational-like inequality Problem, Bull. Austral. Math. Soc. 39(1989), 225-231

[18] M. Patriksson, Nonlinear Programming and Variational Inequality Problems: A Unified Approach, Kluwer Academic Publishers, Dordrecht, 1999.

[19] W.C. Rheinboldt, On M-functions and their applications to nonlinear Gauss-Seidel iteration to network flows, J. Math. Anal. Appl. 32(1970), 274-307

[20] L. Tartar, The novelle characterization des M-matrices, RAIRO 5(1971), 127-128