Journal of Mathematical Extension
Vol. 17, No. 1, (2023) (10)1-21
URL: https://doi.org/10.30495/JME.2023.2412
ISSN: 1735-8299
Original Research Paper

Successive Approximations Method for Solving 2D Nonlinear Singular Fredholm Integral Equations

M. Kazemi
Ashtian Branch, Islamic Azad University

M. R. Doostdar*
Zarandieh Branch, Islamic Azad University

M. Ghorbani
Shahr-e-Qods Branch, Islamic Azad University

Abstract. In the present paper, we propose a numerical method based on the combination of the fixed point method and quadrature formula for solving two-dimensional nonlinear Fredholm integral equations of the second kind. Using uniform and partial modulus of continuity, the error estimation is given. Also, the numerical stability with respect to the choice of the first iteration is proved. Moreover, the accuracy of the method and the correctness of the theoretical results are shown by some examples.

AMS Subject Classification: 31A10; 45B05; 47H10
Keywords and Phrases: Integral equations, Iterative method, Modulus of continuity, Midpoint formula

Received: June 2022; Accepted: January 2023
*Corresponding Author

1 Introduction

Integral equations arise in many scientific and engineering fields [14, 24, 26]. Since exact methods are not efficient to solve some integral equations, high-performance numerical methods are considered for solving these equations. Various analytic and numerical methods have been proposed for approximating the solutions of integral equations. The collocation and Galerkin methods are two commonly numerical approaches for solving these equations. Other Numerical methods using block-pulse functions (BPFs) [7, 22], rationalized Haar functions [6], least-square method [10], extrapolation of Nyström method [12], discrete Galerkin method [13], Bernoulli wavelet [19], wavelets [4], Bernstein's polynomials [5], operational matrices [25], triangular functions (TFs)[15], the regularization-homotopy method [1], and so on [2, 11, 16, 21, 27] have been introduced for solving two-dimensional integral equations. Also, some numerical methods have been presented to solve integral equations using successive approximations methods based on quadrature rules [8, 9, 17, 18, 23]. Here, for two reasons we use the midpoint rule to approximate the solution of the two-dimensional nonlinear Fredholm integral equations of the second kind with singular kernel. Firstly, according to the fact that Simpson and trapezoidal rules cannot be used for approximating integrals which are not defined in the first and the endpoints of the integration. Another reason is the error bound of the midpoint rule compared to the Simpson and trapezoidal rules. In this paper, a new iterative method based on the midpoint formula is proposed for solving the following two-dimensional nonlinear Fredholm integral equations of the second kind:

$$w(s,t) = \psi(s,t) + \mu \int_{l_2}^{u_2} \int_{l_1}^{u_1} \Lambda(s,t,\tau,\nu)\ \Phi(w(\tau,\nu))\ \mathrm{d}\tau \mathrm{d}\nu, \qquad (1)$$

where $w(s,t)$ is an unknown function on $\Theta = [l_1,u_1]\times[l_2,u_2]$ while $\psi(s,t)$ and $\Lambda(s,t,\tau,\nu)$ are known functions on Θ and $\Theta \times \Theta$, respectively.
Simple application as well as the possibility of creating and applying an algorithm are the advantages of this method that encourages us to use it. The structure of this article is divided into five sections. In Sect. 2, we review preliminaries of quadrature rule for 2-D integrals and required definitions. The existence and uniqueness of the solution of the

Eq. (1) are studied using the fixed point technique in Sect. 3. Also, the convergence of the proposed method is described in this section. In Sect. 4, an iterative algorithm is presented for the implementation of the method. Finally, to show the accuracy of the method and to verify the theoretical results, some numerical examples are given in Sect. 5.

2 Preliminaries

In current section, some required concepts and some definitions are presented.

Definition 2.1. *Let $\psi : \Theta \to \mathbb{R}$ be a bounded mapping. The function $\chi_\Theta(\psi, .) : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ with the following definition*

$$\begin{aligned}\chi_\Theta(\psi,\xi) = &\sup\{|\psi(\tau,\nu) - \psi(s,t)|;\\ &\qquad \tau, s \in [l_1,u_1]; \nu, t \in [l_2,u_2]; \sqrt{(\tau-s)^2 + (\nu-t)^2} \leq \xi\},\end{aligned}$$

is called the modulus of oscillation of ψ on Θ. The function $\chi_\Theta(\psi,\xi)$ is called uniform modulus of continuity of ψ, if $\psi \in C(\Theta)$.

Theorem 2.2. *Some used properties throughout this paper are as follows:*

(a) $|\psi(\tau,\nu) - \psi(s,t)| \leq \chi_\Theta(\psi, \sqrt{(\tau-s)^2 + (\nu-t)^2})$ *for all* $\tau, s \in [l_1,u_1]$ *and* $\nu, t \in [l_2,u_2]$,

(b) $\chi_\Theta(\psi,\xi)$ *is a non-decreasing mapping in* ξ,

(c) $\chi_\Theta(\psi,0) = 0$,

(d) $\chi_\Theta(\psi,\xi_1+\xi_2) \leq \chi_\Theta(\psi,\xi_1) + \chi_\Theta(\psi,\xi_2)$ *for any* $\xi_1,\xi_2 \geq 0$,

(e) $\chi_\Theta(\psi,n\xi) \leq n\chi_\Theta(\psi,\xi)$ *for any* $\xi \geq 0$ *and* $n \in \mathbb{N}$,

(f) $\chi_\Theta(\psi,\mu\xi) \leq (\mu+1)\chi_\Theta(\psi,\xi)$ *for any* $\xi,\mu \geq 0$,

(g) $\chi_\Theta(\psi,\cdot)$ *is continuous at 0 iff* $\psi \in C(\Theta)$,

(h) If $\Theta \subseteq \Theta'$, *then* $\chi_\Theta(\psi,\xi) \leq \chi_{\Theta'}(\psi,\xi)$ *for all* $\xi \geq 0$.

Proof. This theorem is proved in [20, 28] for the one-dimensional case. It can be proved for the 2D case in a similar way. □

Theorem 2.3. *Suppose $\psi : \Theta \to \mathbb{R}$ be an integrable bounded mapping. For any divisions $l_1 = \tau_0 < \tau_1 < ... < \tau_n = u_1$ and $l_2 = \nu_0 < \nu_1 < ... < \nu_n = u_2$ and any points $\gamma_i \in [\tau_{i-1}, \tau_i]$ and $\rho_j \in [\nu_{j-1}, \nu_j]$, we have*

$$\left|\int_{l_2}^{u_2}\int_{l_1}^{u_1} \psi(s,t)\ \mathrm{dsdt} - \sum_{j=1}^{n}\sum_{i=1}^{n}(\tau_i - \tau_{i-1})(\nu_j - \nu_{j-1})\psi(\gamma_i, \rho_j)\right|$$
$$\leq \sum_{j=1}^{n}\sum_{i=1}^{n}(\tau_i - \tau_{i-1})(\nu_j - \nu_{j-1})\chi_{[\tau_{i-1},\tau_i]\times[\nu_{j-1},\nu_j]}(\psi, \sqrt{(\tau_i - \tau_{i-1})^2 + (\nu_j - \nu_{j-1})^2}).$$

Proof. The proof of the theorem is the same as [17]. □

Corollary 2.4. *Assume that $\psi : \Theta \to \mathbb{R}$ be an integrable bounded mapping. Defining*

$$\chi_{\tau\nu\times st} = \chi_{[\tau,\nu]\times[s,t]}\left(\psi, \sqrt{(\nu-\tau)^2 + (t-s)^2}\right),$$

we have

$$\begin{aligned}\Bigg|\int_{l_2}^{u_2}\int_{l_1}^{u_1} \psi(s,t)\ \mathrm{dsdt} - \Big[&(\tau - l_1)(\nu - l_2)\psi(u,\alpha_1)(\tau - l_1)(u_2 - \nu)\psi(u,\alpha_2)\\ &+ (u_1 - \tau)(u_2 - \nu)\psi(l_2,\alpha_2) + (u_1 - \tau)(u_2 - \nu)\psi(v,\alpha_2)\Big]\Bigg|\\ &\leq (\tau - l_1)(\nu - l_2)\chi_{l_1\tau\times l_2\nu} + (u_1 - \tau)(\nu - l_2)\chi_{\tau u_1\times l_2\nu}\\ &+ (\tau - l_1)(u_2 - \nu)\chi_{l_1\tau\times\nu u_2} + (u_1 - \tau)(u_2 - \nu)\chi_{\tau u_1\times\nu u_2},\end{aligned}$$

$\forall\tau \in [l_1, u_1]$ $,\nu \in [l_2, u_2]$ $,u \in [l_1, \tau]$, $v \in [\tau, u_1]$, $\alpha_1 \in [l_2, \nu]$, $\alpha_2 \in [\nu, u_2]$.

Proof. Setting $n = 2$, $\tau_1 = \gamma_1 = \gamma_2 = \tau$ and $\nu_1 = \rho_1 = \rho_2 = \nu$ in Theorem 2.3, the required inequality is obtained. □

Corollary 2.5. *Let $\psi : \Theta \to \mathbb{R}$ be a 2-D integrable bounded mapping. Then, the following inequality holds:*

$$\left|\int_{l_2}^{u_2}\int_{l_1}^{u_1} \psi(s,t)\ \mathrm{dsdt}\ - (u_1 - l_1)(u_2 - l_2)\psi(\frac{l_1 + u_1}{2}, \frac{l_2 + u_2}{2})\right| \leq (u_1 - l_1)(u_2 - l_2)\ \Omega.$$

where $\Omega = \chi_{[l_1,u_1]\times[l_2,u_2]}\left(\psi, \frac{|u_1-l_1|}{2} + \frac{|u_2-l_2|}{2})\right)$.

Proof. If we take $\tau = \frac{l_1+u_1}{2}$ and $\nu = \frac{l_2+u_2}{2}$ in Corollary (2.4), the required inequality is obtained. □

3 Main Results

3.1 The sequence of successive approximations

Here, consider the Eq. (1) where $\mu > 0$ and w, ψ and Φ are continuous functions. Assume the function Λ is continuous. Therefore, it is uniformly continuous with respect to (s,t). Hence, there exists the positive constant K such that

$$K = \max_{\substack{l_1 \le s,\tau \le u_1 \\ l_2 \le t,\nu \le u_2}} \big|\Lambda(s,t,\tau,\nu)\big|.$$

Consider $\Delta = \{\psi : [l_1,u_1] \times [l_2,u_2] \to \mathbb{R}; \psi \quad \textit{is} \quad \textit{continuous}\}$ and let (Δ, d) be the space of 2-D continuous functions with the metric

$$d(\psi,\phi) = \big\|\psi - \phi\big\| = \sup_{\substack{l_1 \le s \le u_1 \\ l_2 \le t \le u_2}} \big|\psi(s,t) - \phi(s,t)\big|.$$

Now, we study the existence and uniqueness of the solution of Eq. (1) using the successive approximations method. To do this, we consider the operator $\mathcal{T} : \Delta \to \Delta$ as

$$\mathcal{T}(w)(s,t) = \psi(s,t) + \mu \int_{l_2}^{u_2} \int_{l_1}^{u_1} \Lambda(s,t,\tau,\nu)\ \Phi(w(\tau,\nu))\ \mathrm{d}\tau \mathrm{d}\nu,$$

for all $(s,t) \in [l_1,u_1] \times [l_2,u_2]$, and for all $\psi \in \Delta$.

Theorem 3.1. *Let $\Lambda(s,t,\tau,\nu)$ be continuous for $l_1 \le s,\tau \le u_1,\ \ l_2 \le t,\nu \le u_2$ and $\psi \in \Delta$. Moreover, suppose that there exists $\sigma > 0$, such that*

$$|\Phi(\psi_1) - \Phi(\psi_2)| \le \sigma|\psi_1 - \psi_2|, \qquad \forall \psi_1, \psi_2 \in \Delta. \tag{2}$$

If $M = \sigma\mu K(u_1 - l_1)(u_2 - l_2) < 1$, Eq. (1) has a unique solution $w^* \in \Delta$. Also, the following successive approximations method gives this unique solution.

$$\begin{aligned} &w_0(s,t) = \psi(s,t), \\ &w_r = \mathcal{T}(w_{r-1}), \qquad r \ge 1, \end{aligned} \tag{3}$$

where the sequence $\{w_r\}_{r\geq 1}$ *converges to* w^**. In addition, the priori and the posteriori error estimates are as follows:*

$$\|w^* - w_r\| \leq \frac{M^r}{1-M}\|w_0 - w_1\|, \tag{4}$$

$$\|w^* - w_r\| \leq \frac{M^{r+1}}{\sigma(1-M)}K_0, \tag{5}$$

where

$$K_0 = \sup_{\substack{l_1\leq s\leq u_1\\ l_2\leq t\leq u_2}} |\Phi(\psi(s,t))|.$$

Proof. For any $w \in \Delta$ we have

$$\begin{aligned}
\chi_\Theta(\mathcal{T}(w),\xi) \leq & \sup_{(s_i,t_i)\in\Theta,\ i=1,2}\left\{|\psi(s_1,t_1)-\psi(s_2,t_2)| \ : \sqrt{(s_2-s_1)^2+(t_2-t_1)^2}\leq\xi\right\}\\
&+\sup_{(s_i,t_i)\in\Theta,\ i=1,2}\left\{ |\ \mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s_1,t_1,\tau,\nu)\ \Phi(w(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\right.\\
&\left.-\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s_2,t_2,\tau,\nu)\ \Phi(w(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\ |: \sqrt{(s_2-s_1)^2+(t_2-t_1)^2}\leq\xi\right\}\\
&\leq \chi_\Theta(\psi,\xi)+\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\chi_{st}(\Lambda,\xi)|\ \Phi(w(\tau,\nu))|\ \mathrm{d}\tau\mathrm{d}\nu.
\end{aligned}$$

Taking limit from the last inequality as $\xi \to 0$, using item (g) in Theorem 2.2 and Lebesgue's monotone convergence theorem, we will have $\chi_\Theta(\mathcal{T}(w),\xi)\to 0$. This implies that $\mathcal{T}$ maps Δ into itself.
Furthermore, it can be shown that the operator $\mathcal{T}$ is a contraction map. Let $w, w' \in \Delta$. Now, according to the definition of the operator $\mathcal{T}$, we can write

$$\begin{aligned}
|\mathcal{T}(w)(s,t)-\mathcal{T}(w')(s,t)| &= \mu\left|\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s,t,\tau,\nu)[\Phi(w(\tau,\nu))-\Phi(w'(\tau,\nu))]\ \mathrm{d}\tau\mathrm{d}\nu\right|\\
&\leq \mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}|\Lambda(s,t,\tau,\nu)||\Phi(w(\tau,\nu))-\Phi(w'(\tau,\nu))|\ \mathrm{d}\tau\mathrm{d}\nu\\
&\leq \sigma\mu K(u_1-l_1)(u_2-l_2)\parallel w-w'\parallel,
\end{aligned}$$

for all $(s,t)\in\Theta$. Thus,

$$\parallel \mathcal{T}(w)-\mathcal{T}(w')\parallel \leq M \parallel w-w'\parallel .$$

Since $M < 1$, it can be concluded the operator $\mathcal{T}$ is contraction on Banach space $(\Delta, \| \,.\, \|)$. Now, using the Banach's fixed point principle, the existence of a unique solution $w^* \in \Delta$ for the integral equation (1) is proved. Also, the estimates (4) and (5) are obtained using the same Banach's fixed point principle. $\square$

Note: Theorem 3.1 is a sufficient condition for the existence of an unique solution of Eq. (1).

Now, we introduce a numerical method to solve Eq. (1). We consider Eq. (1) with continuous kernel $\Lambda(s,t,\tau,\nu)$ defined on $\Theta \times \Theta$ and uniform partitions

$$D_\tau : l_1 = s_0 < s_1 < s_2 < ... < s_{n-1} < s_n = u_1,$$
$$D_\nu : l_2 = t_0 < t_1 < t_2 < ... < t_{n-1} < t_n = u_2,$$

with $s_i = l_1 + ih$, $t_j = l_2 + jh'$, where $h = \frac{u_1 - l_1}{n}$, $h' = \frac{u_2 - l_2}{n}$. Now, we present the following iterative procedure for solving Eq. (1) in point (s,t),

$$\bar{w}_0(s,t) = \psi(s,t),$$
$$\bar{w}_r(s,t) = \psi(s,t) + \mu h h' \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Lambda(s,t,s_i + \frac{h}{2}, t_j + \frac{h'}{2}) \Phi(\bar{w}_{r-1}(s_i + \frac{h}{2}, t_j + \frac{h'}{2})). \tag{6}$$

This procedure is obtained using the quadrature formula for computing the corresponding double integral.

3.2 Convergence

The current section is devoted to finding an error estimate between the exact and approximate solution of Eq. (1).

Theorem 3.2. *Under the assumptions of Theorem 3.1, the iterative process (6) converges to w^*. Furthermore, the following error estimate can be obtained:*

$$\|w^* - \bar{w}_r\|$$
$$\leq \frac{M^{r+1}}{\sigma(1-M)} K_0$$
$$+ \frac{3M}{8(1-M)} \Big(\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi, h+h') + \frac{M}{\sigma K} \eta \chi_{st}(\Lambda, h+h') + \frac{1}{\sigma K} \upsilon \chi_{\tau\nu}(\Lambda, h+h') \Big),$$

where

$$\chi_{\tau\nu}\big(\Lambda,\xi\big) = \sup\big\{\big|\Lambda(s_1,t_1,\tau,\nu)-\Lambda(s_2,t_2,\tau,\nu)\big|\ ;(s_i,t_i)\in[l_1,u_1]\times[l_2,u_2],\\ i=1,2,\ \sqrt{(s_2-s_1)^2+(t_2-t_1)^2}\leq\xi\big\},$$

$$\chi_{st}\big(\Lambda,\xi\big) = \sup\big\{\big|\Lambda(s,t,\tau_1,\nu_1)-\Lambda(s,t,\tau_2,\nu_2)\big|\ ;(\tau_i,\nu_i)\in[l_1,u_1]\times[l_2,u_2],\\ i=1,2,\ \sqrt{(\tau_2-\tau_1)^2+(\nu_2-\nu_1)^2}\leq\xi\big\},$$

and

$$\begin{aligned} K_r &= \sup_{(s,t)\in[l_1,u_1]\times[l_2,u_2]}|\Phi(\bar{w}_r(s,t))|, \quad \Gamma_r = \sup_{(s,t)\in[l_1,u_1]\times[l_2,u_2]}|\Phi(w_r(s,t))|,\\ \upsilon &= \max_{i=0,1,\ldots,r-1}\{K_i\}, \qquad\qquad \eta = \max_{i=0,1,\ldots,r-2}\{\Gamma_i\}. \end{aligned} \tag{7}$$

Proof. Consider Eq. (6). Using Corollary 2.5 and item (f) in Theorem 2.2, we get

$$\begin{aligned} &|w_1(s,t)-\bar{w}_1(s,t)|\\ &=\big|\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s,t,\tau,\nu)\Phi(w_0(\tau,\nu))\mathrm{d}\tau\mathrm{d}\nu\\ &-\mu hh'\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Lambda(s,t,s_i+\frac{h}{2},t_j+\frac{h'}{2})\Phi(\bar{w}_0(s_i+\frac{h}{2},t_j+\frac{h'}{2}))\big|\\ &=\big|\mu\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\Lambda(s,t,\tau,\nu)\Phi(\psi(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\\ &-\mu hh'\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\Lambda(s,t,s_i+\frac{h}{2},t_j+\frac{h'}{2})\Phi(\psi(s_i+\frac{h}{2},t_j+\frac{h'}{2}))\big|\\ &\leq\mu\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\big|\int_{s_i}^{s_{i+1}}\int_{t_j}^{t_{j+1}}\Lambda(s,t,\tau,\nu)\Phi(\psi(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\\ &-(s_{i+1}-s_i)(t_{j+1}-t_j)\Lambda(s,t,s_i+\frac{h}{2},t_j+\frac{h'}{2})\Phi(\psi(s_i+\frac{h}{2},t_j+\frac{h'}{2}))\big|, \end{aligned}$$

for all $(s,t) \in [l_1,u_1] \times [l_2,u_2]$. From the last equation, we conclude that

$$\begin{aligned}|w_1(s,t) - \bar{w}_1(s,t)| &\leq \mu h h' \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \big(\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}\Lambda\Phi(\psi), \frac{h}{2} + \frac{h'}{2}\big) \\ &\leq \frac{3\mu(u_1-l_1)(u_2-l_2)}{8}\big(\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}\Lambda\Phi(\psi), hh'\big) \\ &= \frac{3M}{8\sigma K}\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(\Lambda\Phi(\psi), hh').\end{aligned}$$

Since $\sqrt{(\alpha_1-\alpha_2)^2 + (\beta_1-\beta_2)^2} \leq h + h'$ for any $(\alpha_1,\beta_1),(\alpha_2,\beta_2) \in [s_i,s_{i+1}] \times [t_j,t_{j+1}]$, we can conclude that

$$\begin{aligned}&\big|\Lambda(s,t,\alpha_1,\beta_1)\ \Phi(\psi(\alpha_1,\beta_1)) - \Lambda(s,t,\alpha_2,\beta_2)\ \Phi(\psi(\alpha_2,\beta_2))\big| \\ &\leq \big|\Lambda(s,t,\alpha_1,\beta_1)\ \Phi(\psi(\alpha_1,\beta_1)) - \Lambda(s,t,\alpha_1,\beta_1)\ \Phi(\psi(\alpha_2,\beta_2))\big| \\ &+ \big|\Lambda(s,t,\alpha_1,\beta_1)\ \Phi(\psi(\alpha_2,\beta_2)) - \Lambda(s,t,\alpha_2,\beta_2)\ \Phi(\psi(\alpha_2,\beta_2))\big| \\ &\leq \big|\Lambda(s,t,\alpha_1,\beta_1)\big|\big|\Phi(\psi(\alpha_1,\beta_1)) - \Phi(\psi(\alpha_2,\beta_2))\big| \\ &+ \big|\Lambda(s,t,\alpha_1,\beta_1) - \Lambda(s,t,\alpha_2,\beta_2)\big|\big|\Phi(\psi(\alpha_2,\beta_2))\big| \\ &\leq K\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(\Phi(\psi), h+h') + K_0\chi_{\tau\nu}(\Lambda, h+h').\end{aligned}$$

Now, we take the supremum of the last inequality and obtain

$$\begin{aligned}&\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(\Lambda\Phi(\psi), h+h') \\ &\leq K\chi_{[s_i,s_{i+1}]\times[t_j,t_{j+1}]}(\Phi(\psi), h+h') + K_0\chi_{\tau\nu}(\Lambda, h+h'),\end{aligned}$$

and as a result we have

$$\big\|w_1 - \bar{w}_1\big\| \leq \frac{3M}{8\sigma}\chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(\psi), h+h') + \frac{3M}{8\sigma K}K_0\chi_{\tau\nu}(\Lambda, h+h'). \quad (8)$$

Now, for $r = 2$ with straightforward computing and using (2), we have

$$
\begin{aligned}
&\big|w_2(s,t) - \bar{w}_2(s,t)\big| \\
&= \big|\mu \int_{l_2}^{u_2} \int_{l_1}^{u_1} \Lambda(s,t,\tau,\nu)\ \Phi(w_1(\tau,\nu))\ \mathrm{d}\tau \mathrm{d}\nu \\
&- \mu h h' \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Lambda(s,t,s_i + \frac{h}{2}, t_j + \frac{h'}{2}) \Phi(\bar{w}_1(s_i + \frac{h}{2}, t_j + \frac{h'}{2}))\big| \\
&\leq \big|\mu \int_{l_2}^{u_2} \int_{l_1}^{u_1} \Lambda(s,t,\tau,\nu)\ \Phi(w_1(\tau,\nu))\ \mathrm{d}\tau \mathrm{d}\nu \\
&- \mu h h' \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Lambda(s,t,s_i + \frac{h}{2}, t_j + \frac{h'}{2}) \Phi(w_1(s_i + \frac{h}{2}, t_j + \frac{h'}{2}))\big| \\
&+ \big|\mu h h' \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Lambda(s,t,s_i + \frac{h}{2}, t_j + \frac{h'}{2}) \Phi(w_1(s_i + \frac{h}{2}, t_j + \frac{h'}{2})) \\
&- \mu h h' \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \Lambda(s,t,s_i + \frac{h}{2}, t_j + \frac{h'}{2}) \Phi(\bar{w}_1(s_i + \frac{h}{2}, t_j + \frac{h'}{2}))\big| \\
&\leq \frac{3M}{8\sigma} \chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_1), hh') + \mu(u_1 - l_1)(u_2 - l_2) K \alpha_1 \|w_1 - \bar{w}_1\|.
\end{aligned}
$$

So, we have the following result:

$$
\begin{aligned}
\big\|w_2 - \bar{w}_2\big\| \leq \frac{3M}{8\sigma} \chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_1), h + h') &+ \frac{3M}{8\sigma K} K_1 \chi_{\tau\nu}(\Lambda, h + h') \\
&+ M\big\|w_1 - \bar{w}_1\big\|.
\end{aligned}
$$

By induction for $r \geq 3$, using (2), (3) and (6), we see that

$$
\begin{aligned}
\big|w_r(s,t) - \bar{w}_r(s,t)\big| \leq\ & \frac{3M}{8\sigma} \chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_{r-1}), h + h') \\
&+ \frac{3M}{8\sigma K} K_{r-1} \chi_{\tau\nu}(\Lambda, h + h') + M\|w_1 - \bar{w}_1\|.
\end{aligned} \tag{9}
$$

From (9), we obtain

$$\begin{aligned}\|w_r-\bar{w}_r\| \leq \frac{3M}{8\sigma}\Big(&\chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_{r-1}),h+h')+M\chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_{r-2}),h+h')\\&+...+M^{r-1}\chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(\psi),h+h')\Big)\\&+\frac{3M}{8\sigma K}\chi_{\tau\nu}(\Lambda,h+h')\Big(K_{r-1}+MK_{r-2}+M^2K_{r-3}+...+M^{r-1}K_0\Big).\end{aligned} \tag{10}$$

On the other hand, we have

$$\begin{aligned}&\Big|\Phi(w_r(s_1,t_1))-\Phi(w_r(s_2,t_2))\Big|\\&\leq\sigma\Big|w_r(s_1,t_1)-w_r(s_2,t_2)\Big|\\&\leq\sigma|\psi(s_1,t_1)-\psi(s_2,t_2)|\\&+\sigma\Big|\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s_1,t_1,\tau,\nu)\ \Phi(w_{r-1}(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\\&-\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Lambda(s_2,t_2,\tau,\nu)\ \Phi(w_{r-1}(\tau,\nu))\ \mathrm{d}\tau\mathrm{d}\nu\Big|\\&\leq\sigma\Big|\psi(s_1,t_1)-\psi(s_2,t_2)\Big|\\&+\sigma\mu\int_{l_2}^{u_2}\int_{l_1}^{u_1}\Big|\Lambda(s_1,t_1,\tau,\nu)-\Lambda(s_2,t_2,\tau,\nu)\Big|\Big|\Phi(w_{r-1}(\tau,\nu))\Big|\ \mathrm{d}\tau\mathrm{d}\nu\\&\leq\sigma\Big|\psi(s_1,t_1)-\psi(s_2,t_2)\Big|+\frac{M}{K}\chi_{st}(\Lambda,h+h')\Gamma_{r-1}.\end{aligned}$$

Hence, we will have

$$\chi_{[l_1,u_1]\times[l_2,u_2]}(\Phi(w_r),h+h')\leq\sigma\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi,h+h')+\frac{M}{K}\chi_{st}(\Lambda,h+h')\Gamma_{r-1}. \tag{11}$$

By inequalities (10) and (11), we see that

$$\begin{aligned}\|w_r-\bar{w}_r\|\leq\frac{3M}{8}&\Big(1+M+M^2+...+M^{r-1}\Big)\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi,h+h')\\&+\frac{3M}{8\sigma K}\chi_{st}(\Lambda,h+h')\Big(M\Gamma_{r-2}+M^2\Gamma_{r-3}+...+M^{r-1}\Gamma_0\Big)\\&+\frac{3M}{8\sigma K}\chi_{\tau\nu}(\Lambda,h+h')\Big(K_{r-1}+MK_{r-2}+M^2K_{r-3}+...+M^{r-1}K_0\Big).\end{aligned}$$

By (7), we obtain

$$\begin{aligned}\|w_r-\bar{w}_r\| &\leq \frac{3M}{8}\left(\frac{1-M^r}{1-M}\right)\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi,h+h')\\ &+\frac{3M}{8\sigma K}\chi_{st}(\Lambda,h+h')(\frac{M(1-M^{r-1})}{1-M}\eta)\\ &+\frac{3M}{8\sigma K}\chi_{\tau\nu}(\Lambda,h+h')(\frac{(1-M^r)}{1-M}\upsilon).\end{aligned}$$

Therefore, since $M<1$ we have

$$\begin{aligned}\|w_r-\bar{w}_r\| \leq \frac{3M}{8(1-M)}\bigg(&\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi,h+h')\\ &+\frac{M}{\sigma K}\eta\chi_{st}(\Lambda,h+h')+\frac{1}{\sigma K}\upsilon\chi_{\tau\nu}(\Lambda,h+h')\bigg).\end{aligned} \tag{12}$$

By inequalities (5) and (12), we deduce that

$$\begin{aligned}\|w^*-\bar{w}_r\| &\leq \|w^*-w_r\|+\|w_r-\bar{w}_r\|\\ &\leq \frac{M^{r+1}}{\sigma(1-M)}K_0+\frac{3M}{8(1-M)}\bigg(\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi,h+h')\\ &+\frac{M}{\sigma K}\eta\chi_{st}(\Lambda,h+h')+\frac{1}{\sigma K}\upsilon\chi_{\tau\nu}(\Lambda,h+h')\bigg).\end{aligned}$$

□

Remark 3.3. Since $M<1$, it can be obviously concluded that

$$\lim_{\substack{r\to\infty\\ h,h'\to 0}}\|w^*-\bar{w}_r\|=0,$$

that indicates the present method is convergent.

3.3 The stability analysis

Numerical stability is a desirable property of numerical algorithms. An algorithm is called numerically stable if small changes in the initial data create correspondingly small changes in the final results.

Definition 3.4. *Let $\bar{w}_0, \bar{z}_0 \in C(I)$ be two initial values such that $\| \bar{w}_0 - \bar{z}_0 \| < \varepsilon$, for arbitrary small $\varepsilon > 0$. A necessary and sufficient condition for the algorithm of successive approximation used to the integral equation (1) to be numerically stable with respect to the first iteration is that there exist the constants $\beta_1, \beta_2, \beta_3 > 0$ that are independent of h, h' and two continuous functions $f_1, f_2 : (0, u_1 - l_1 + u_2 - l_2] \to [0, \infty)$ with $\lim_{h\to 0} f_1(h) = 0$ and $\lim_{h\to 0} f_2(h) = 0$ such that:*

$$\| \bar{w}_r - \bar{z}_r \| < \beta_1\varepsilon + \beta_2 f_1(h) + \beta_3 f_2(h), \qquad r \in \mathbb{N} \cup \{0\},$$

where $h = h + h'$.

Theorem 3.5. *Under the assumptions of Theorem (3.2), the procedure (6) is numerically stable with respect to the choice of the first iteration.*

Proof. Let denote $w_r = \mathcal{T}(w_{r-1}), w_0 = \psi$ and $z_r = \mathcal{T}(z_{r-1}), z_0 = g$. To study the numerical stability of this method, we reproduce the proof of Theorem 3.2 with the following non-negative constants

$$\begin{cases} K'_r = \sup\limits_{(s,t)\in[l_1,u_1]\times[l_2,u_2]} |\Phi(\bar{z}_r(s,t))|, \\ \Gamma'_r = \sup\limits_{(s,t)\in[l_1,u_1]\times[l_2,u_2]} |\Phi(z_r(s,t))|, \\ \upsilon' = \max\limits_{i=0,1,\dots,r-1} \{K'_i\}, \\ \eta' = \max\limits_{i=0,1,\dots,r-2} \{\Gamma'_i\}. \end{cases}$$

Note that

$$\begin{aligned} \| w_r - z_r \| &= \| \mathcal{T}(w_{r-1}) - \mathcal{T}(w_{r-1}) \| \\ &\le \varepsilon + M \| w_{r-1} - z_{r-1} \| \le \cdots \le (1 + M + \dots + M^k)\varepsilon \le \frac{1}{1-M}\varepsilon. \end{aligned}$$

Using (12), it follows that

$$\begin{aligned} \| \bar{w}_r - \bar{z}_r \| &\le \| \bar{w}_r - w_r \| + \| w_r - z_r \| + \| z_r - \bar{z}_r \| \\ &\le \frac{1}{1-M}\varepsilon + \frac{f_1(h)}{1-M} + \frac{f_2(h)}{1-M}. \end{aligned}$$

where

$$\beta_1 = \frac{1}{1-M}, \qquad \beta_2 = \beta_3 = \frac{8M}{8(1-M)},$$

and

$$f_1(h) = \frac{3M}{8(1-M)}\Bigg(\chi_{[l_1,u_1]\times[l_2,u_2]}(\psi, h+h') + \frac{M}{\sigma K}\eta\chi_{st}(\Lambda, h+h') + \frac{1}{\sigma K}\upsilon\chi_{\tau\nu}(\Lambda, h+h')\Bigg),$$

$$f_2(h) = \frac{3M}{8(1-M)}\Bigg(\chi_{[l_1,u_1]\times[l_2,u_2]}(g, h+h') + \frac{M}{\sigma K}\eta'\chi_{st}(\Lambda, h+h') + \frac{1}{\sigma K}\upsilon'\chi_{\tau\nu}(\Lambda, h+h')\Bigg),$$

with $h = h + h'$. $\square$

Remark 3.6. Since $M < 1$, it can be obviously concluded that

$$\lim_{f_1(h), f_2(h)\to 0, r\to+\infty} \| \bar{w}_r - \bar{z}_r \| = 0,$$

that indicates the iterative procedure (6) is numerically stable.

4 The Iterative Algorithm

In this section, the steps of the implementation of the proposed method for solving (1) are listed as follows:

Step 1: Take $\varepsilon > 0$. Set $r = 1$. Set $\bar{w}_0(s_k, t_l) = \psi(s_k, t_l)$ for $k = (0 : n)$, $l = (0 : n)$.

Step 2: Comput $\bar{w}_r(s_k, t_l)$ by (6), for $k = 0$ to n and $l = 0$ to n.

Step 3: Compute $E = | \bar{w}_r(s_k, t_l) - \bar{w}_{r-1}(s_k, t_l) |$.

Step 4: If $E < \varepsilon$. Print $\bar{w}_r(s_k, t_l)$, r, k, l, and Stop. Otherwise, set $r = r + 1$ and go to Step 2.

The notation $k = (0 : n)$ denotes the integer values of the k vary from 0 to n.

5 Numerical Examples

In the current section, the implementation of the proposed method is tested on two examples to show the accuracy of the method and the correctness of the theoretical results. All computations are performed using Maple 17 software on a laptop with the Intel Core i5-3210M CPU processor and 4GB RAM. The following notations are introduced for analyzing the error of the method:

- The error:

$$\|e_n\|_\infty := \max\{|\ w^*(s_k,t_l) - \bar{w}_r(s_k,t_l)\ |\ k,l = 1,2,...,n\}.$$

- The ratio of the successive values of $\|e_n\|_\infty$ as n is doubled [3]:

$$Ratio = \frac{\|w^* - \overline{w}_r^{(n)}\|_\infty}{\|w^* - \overline{w}_r^{(2n)}\|_\infty},$$

- The estimate of the convergence rate [3]:

$$\sigma_n = log_2\bigg(\frac{\|w^* - \overline{w}_r^{(n)}\|_\infty}{\|w^* - \overline{w}_r^{(2n)}\|_\infty}\bigg).$$

Consider w^* and $\bar{w}_r$ as the exact solution and approximate solution of the Eq. (1), respectively. Also, $NI = r$ is the number of iterations. In the following examples, the tolerance to stop the iterations is $\varepsilon = 10^{-15}$.

Example 5.1. As the first example, the following 2-D non-linear Fredholm integral equation is considered

$$w(s,t) = \psi(s,t) + \int_0^1\int_0^1 \frac{\sqrt{\nu}(t-s)}{\sqrt{1-\tau}\sqrt{1-\nu}}(w(\tau,\nu))^2\ \mathrm{d}\tau\mathrm{d}\nu, \tag{13}$$

where $(s,t) \in [0,1] \times [0,1]$ and

$$\psi(s,t) = \frac{923}{504}(s-t)\pi + s^2 + t.$$

The exact solution of (13) is $w(s,t) = s^2 + t$. We obtain the absolute errors for the grid points (s_k,t_l), for $k = l = 1,...,5$. Numerical results

Table 1: Numerical results for some values of n in Example 5.1.

$s=t$	Exact	$e_{k,l}, n=10$	$e_{k,l}, n=20$	$e_{k,l}, n=40$	$e_{k,l}, n=80$
0.1	0.11	7.0909×10^{-7}	1.7928×10^{-7}	4.4948×10^{-8}	1.1244×10^{-8}
0.3	0.39	6.1274×10^{-6}	1.5492×10^{-6}	3.8840×10^{-7}	9.7169×10^{-8}
0.5	0.75	1.5635×10^{-5}	3.9532×10^{-6}	9.9109×10^{-7}	2.4794×10^{-7}
0.7	1.19	2.6708×10^{-5}	6.7530×10^{-6}	1.6929×10^{-6}	4.2354×10^{-7}
0.9	1.71	3.5882×10^{-5}	9.0726×10^{-6}	2.2745×10^{-6}	5.6902×10^{-7}
$\|e_n\|_\infty$	–	3.8504×10^{-5}	9.7357×10^{-6}	2.4407×10^{-6}	6.1061×10^{-7}
Ratio	–	-	3.9549	3.9888	3.9971
σ_n	–	-	1.9836	1.9959	1.9989
NI	–	7	7	7	7

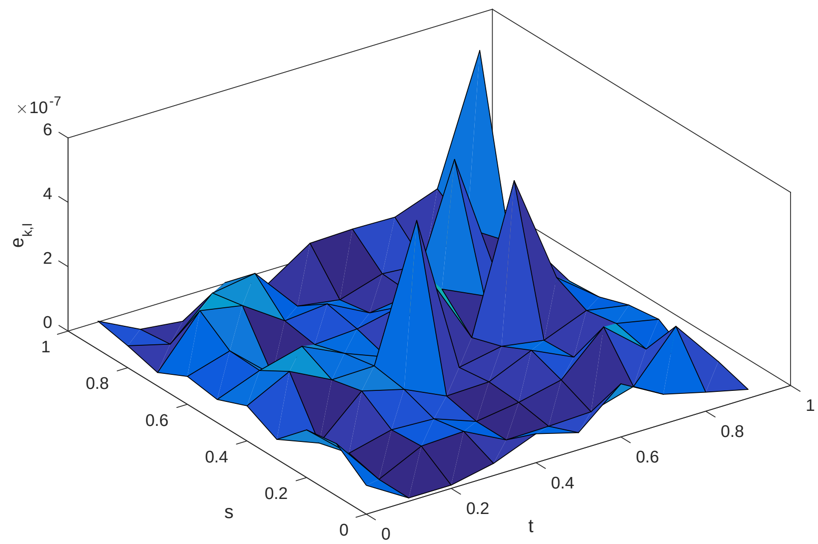

Figure 1: The absolute error graph of $e_{k,l}(s,t)$ for $n=80$ for Example 5.1.

(error between exact and approximate value of $\bar{w}(s,t)$) for $n=10,20,40$ and 80 are given in Table 1. Also, Fig. 1 displays the absolute error graph of $e_{k,l}(s,t)$ for $n=80$. The obtained results show that the algorithm is convergent. With a simple check, we can see that by doubling the values of n, the errors $\|e_n\|_\infty$ are decreasing by factor of approximately 4.

Table 2: Numerical results for some values of n in Example 5.2.

$s=t$	Exact	$e_{k,l}, n=10$	$e_{k,l}, n=20$	$e_{k,l}, n=40$	$e_{k,l}, n=80$
0.1	1.02	5.2098×10^{-8}	1.3923×10^{-8}	3.5396×10^{-9}	8.8863×10^{-10}
0.3	1.18	1.6767×10^{-6}	4.4680×10^{-7}	1.1350×10^{-7}	2.8489×10^{-8}
0.5	1.50	9.0128×10^{-6}	2.3966×10^{-6}	6.0849×10^{-7}	1.5271×10^{-7}
0.7	1.98	2.8161×10^{-5}	7.4765×10^{-6}	1.8975×10^{-6}	4.7617×10^{-7}
0.9	2.62	6.7145×10^{-5}	1.7803×10^{-5}	4.5171×10^{-6}	1.1334×10^{-6}
$\|e_n\|_\infty$	–	9.7107×10^{-5}	2.5734×10^{-5}	6.5283×10^{-6}	1.6380×10^{-6}
Ratio	–	-	3.7734	3.9419	3.9855
σ_n	–	-	1.9158	1.9788	1.9947
NI	–	5	5	5	5

Example 5.2. The following 2-D non-linear Fredholm integral equation is considered as the next example

$$w(s,t) = \psi(s,t) + \int_0^1 \int_0^1 \frac{\sqrt{\nu}(t^2+s+1)}{5\tau^{\frac{3}{2}}\sqrt{1-\nu}} (w(\tau,\nu))^3 \, \mathrm{d}\tau \mathrm{d}\nu, \tag{14}$$

where $(s,t) \in [0,1] \times [0,1]$ and

$$\psi(s,t) = 2st + 1 - \frac{11}{8}\pi(s-t)(s^2+t+1),$$

with the exact solution $w(s,t) = 2st+1$. Table 2 illustrates the numerical results for this example. The absolute error graph of $e_{k,l}(s,t)$ for $n = 80$ is displayed in Fig. 2.

6 Conclusions

In this work, an attempt was made to present an iterative numerical technique based on the combination of the successive approximations method and the midpoint formula for the numerical solution of two-dimensional nonlinear Fredholm integral equations of the second kind. Using uniform and partial modulus of continuity, the convergence was proved and the error was estimated in Theorem 3.2. Also, the numerical stability with respect to the choice of the first iteration was studied in Theorem 3.5. Moreover, the accuracy of the method and the correctness of the theoretical results were shown by some examples.

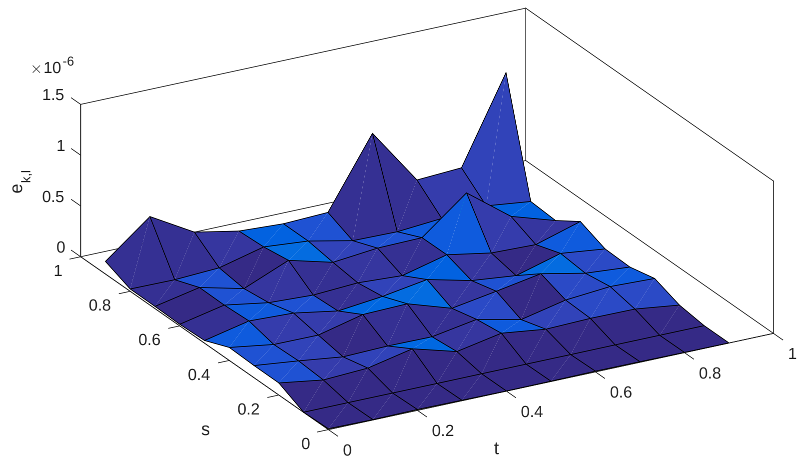

Figure 2: The absolute error graph of $e_{k,l}(s,t)$ for $n = 80$ for Example 5.2.

References

[1] A. Altürk, The regularization-homotopy method for the two-dimensional Fredholm integral equations of the first kind, *Math. Comput. Appl.*, **21**(2) (2016).

[2] A. Amara, S. Etemad, S. Rezapour, Approximate solutions for a fractional hybrid initial value problem via the Caputo conformable derivative, *Adv. Differ. Equ.* **2020** (2020) 608.

[3] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 2011.

[4] I. Aziz, F. Khan, et al, A new method based on Haar wavelet for the numerical solution of two-dimensional non-linear integral equations, *J. Comput. Appl. Math.*, **272** (2014) 70–80.

[5] A. Babaaghaie, K. Maleknejad, A new approach for numerical solution of two-dimensional non-linear Fredholm integral equations in the most general kind of kernel, based on Bernstein polynomials and its convergence analysis, *J. Comput. Appl. Math.*, **344** (2018) 482–494.

[6] E. Babolian, S. Bazm, and P. Lima, Numerical solution of nonlinear two-dimensional integral equations using rationalized Haar functions, *Commun. Nonlinear Sci. Numer. Simul.*, **16**(3) (2011) 1164–1175 .

[7] E. Babolian, A. J. Shaerlar, Two-dimensional block pulse functions and application to solve Volterra-Fredholm integral equations with Galerkin method, *Int. J. Contemp. Math. Sci*, **6**(16) (2011) 763–770.

[8] A. M. Bica, The numerical method of successive interpolations for Fredholm functional integral equations, *Numer Algorithms*, **58**(3) (2011) 351–377.

[9] A. M. Bica , M. Curila, S. Curila, About a numerical method of successive interpolations for functional Hammerstein integral equations, *J. Comput. Appl. Math.*, **236** (2012) 2005–2024.

[10] Z. El Majouti, R. El Jid and A. Hajjaj, Numerical solution of two-dimensional Fredholm–Hammerstein integral equations on 2D irregulardomains by using modified moving least-square method, *Int. J. Comput. Math.*, (2020).

[11] S. Etemad, B. Tellab, J. Alzabut, S. Rezapour, M. I. Abbas, Approximate solutions and Hyers–Ulam stability for a system of the coupled fractional thermostat control model via the generalized differential transform, *Adv. Differ. Equ.* **2021** (2021) 428. https://doi.org/10.1186/s13662-021-03563-x

[12] H. Guoqiang, W. Jiong, Extrapolation of Nyström solution for two-dimensional non-linear Fredholm integral equations, *J. Comput. Appl. Math.*, **134**(1) (2001) 259–268.

[13] G. Han, R. Wang, Richardson extrapolation of iterated discrete Galerkin solution for two-dimensional Fredholm integral equations, *J. Comput. Appl. Math.*, **139**(1) (2002) 49–63.

[14] R. J. Hanson, J. L. Phillips, Numerical solution of two-dimensional integral equations using linear elements, *SIAM J. Numer. Anal.*, **15**(1) (1978) 113–121.

[15] M. Kazemi, Triangular functions for numerical solution of the nonlinear Volterra integral equations, *J. Appl. Math. Comput.*, (2021) 1–24.

[16] M. Kazemi, M. R. Doostdar, On the numerical scheme for solving non-linear two-dimensional Hammerstein integral equations, *Comput. Methods Differ. Equ.*, **10**(4) (2022) 1059–1074.

[17] M. Kazemi, R. Ezzati, Numerical solution of two-dimensional nonlinear integral equations via quadrature rules and iterative method, *Adv. Differ. Equat. Contr. Process.*, **17** (2016) 27–56.

[18] M. Kazemi, V. Torkashvand, R. Ezzati, On a method based on Bernstein operators for 2D non-linear Fredholm-Hammerstein integral equations, *U.P.B. Sci. Bull., Series A*, **83** (2021) 178–198.

[19] A. A. Khajehnasiri, R. Ezzati, M. A. Kermani, Solving fractional two-dimensional nonlinear partial Volterra integral equation by using Bernoulli wavelet, *Iran. J. Sci. Technol. Trans. A Sci.*, (2021) 1-13.

[20] I. Kolodii, F. Khildebrand, Some properties of the modulus of continuity, *Mathematical notes of the academy of sciences of the USSR*, **9**(5) (1971) 285–288.

[21] M. I. Liaqat, A. Khan, M. Alam, M. K. Pandit, S. Etemad, S. Rezapour, Approximate and closed-form solutions of Newell-Whitehead-Segel equations via modified conformable Shehu transform decomposition method, *Mathematical Problems in Engineering*, (2022). https://doi.org/10.1155/2022/6752455

[22] K. Maleknejad, S. Sohrabi, B. Baranji, Application of 2D-BPFs to non-linear integral equations, *Commun. Nonlinear Sci. Numer. Simul.*, **15**(3) (2010) 527–535.

[23] K. Maleknejad, P. Torabi, Application of fixed point method for solving non-linear Volterra-Hammerstein integral equation, *U.P.B. Sci. Bull., Series A*, **74** (2012) 45–56.

[24] S. McKee, T. Tang, T. Diogo, An Euler-type method for two-dimensional Volterra integral equations of the first kind, *IMA J. Numer. Anal.*, **20**(3) (2000) 423–440.

[25] K. Parand, H. Yari, M. Delkhosh, Solving two-dimensional integral equations of the second kind on non-rectangular domains with error estimate, *Eng. Comput.*, (2019).

[26] Qi. Tang, D. Waxman, An integral equation describing an asexual population in a changing environment, *Nonlinear Anal.*, **53**(5) (2003) 683–699.

[27] S. Rezapour, S. Etemad, B. Tellab, P. Agarwal, J. L. Garcia Guirao, Numerical solutions caused by DGJIM and ADM methods for multi-term fractional BVP involving the generalized Ψ-RL-operators, *Symmetry*, **13**(4) (2021) 532.

[28] T. J. Rivlin, *An Introduction to the Approximation of Functions*, Courier Corporation, 2003.

Manochehr Kazemi
Department of Mathematics
Assistant Professor of Mathematics
Ashtian Branch, Islamic Azad University, Ashtian, Iran
E-mail: univer_ka@yahoo.com

Mohammad Reza Doostdar
Department of Mathematics
Assistant Professor of Mathematics
Zarandieh Branch, Islamic Azad University, Zarandieh, Iran
E-mail: mrdoostdar@yahoo.com

Morteza Ghorbani
Department of Mathematics, Faculty of Science
Assistant Professor of Mathematics
Shahr-e-Qods Branch, Islamic Azad University, Tehran, Iran
E-mail: morteza.ghorbani77@qodsiau.ac.ir