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Original Research Paper

A New Note on Convergence and Data Dependence Concept for a Volterra Integral Equation by Fixed Point Iterative Algorithm

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Abstract. In this paper, we prove that a three-step iterative algorithm converges strongly to the solution of a functional Volterra integral equation of the second kind. We also show the solution presented in the convergence theorem can obtain by removing a strong restriction imposed on the control sequence. Finally, we analyse the data-dependence result by using this iterative algorithm, and to support obtained result we give an example.

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1 Introduction

Most of the problems encountered in real life can be expressed with non-linear equations. It is not always possible to find exact solutions to these non-linear equations. In this context, a lot of studies are devoted to obtaining an approximate solution of these equations (see [1], [5], [23], [29]).

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Fixed point theory, which is founded to show the existence and uniqueness of the solution of differential equations and integral equations, is a very important mathematical tool used in a wide range of science such as optimization [8], computing algorithms [2], economics [6], variational inequalities [27], complementary problems [12], and equilibrium problems [26]. Volterra integral equations are used in modelling some problems encountered in many disciplines, especially in mathematical physics, engineering, and biology. Fixed point iteration methods are used as an effective tool to reach the solution of these equations. The iterative approximation is one of the important topics in fixed point theory (see [10], [18], [30]) and references therein. Hence, fixed point iterative algorithms have been studied by many researchers to solve integral equations (see [3],[4], [9], [11], [16], [19]).

Many classical results are usually obtained using the Bielecki's norm based on the Banach Contraction Principle in demonstrating the existence and uniqueness of the solution of the second kind of Volterra functional integral equations (see [7],[17]).

In this paper, we study the following functional Volterra integral equation of the second kind which has been considered in [21] with Chebyshev's norm instead of Bielecki's norm:

$$x(t) = \lambda \int_{-t}^{t} K(t, s, x(s))\, ds + f(t),\; t \in [-I, I], \tag{1}$$

for some $I > 0$ and $\lambda \in \mathbb{R}$, where $K \in C([-I, I] \times [-I, I] \times \mathbb{R})$ and $f \in C[-I, I]$. Define the integral operator $T : C[-I, I] \to C[-I, I]$ by

$$Tx(t) = \lambda \int_{-t}^{t} K(t, s, x(s))\, ds + f(t). \tag{2}$$

In order to solve the integral equation (1) one can characterize this matter as a fixed point problem for T as under:

$$x = Tx.$$

Let $X = C[-I, I]$ equipped with Chebyshev norm

$$\|x\| = \max_{t\in[-I,I]} |x(t)|, \qquad x \in X$$

and $B_R = \{x \in C[-I, I] : \|x - f\| \leq R\}$, for some $R > 0$. Then $(X, \|.\|)$ is a Banach space and $B_R \subseteq X$ is closed in X.

The author in [21], obtained the existence and uniqueness of the solution (1) by using the contraction principle under following conditions:

Theorem 1.1 ([21])**.** *Let $K \in C([-I, I] \times [-I, I] \times \mathbb{R})$ and $f \in, C[-I, I]$. Assume that:*

i. *there exist a function $m : [-I, I] \to \mathbb{R}_+$ such that*

$$|K(t, s, u) - K(t, s, v)| \leq m(s) |u - v|,$$

for all $t, s \in [-I, I]$ and all $u, v \in [R_1 - R, R_2 + R]$ where $R_1 = \min_{t \in [-I, I]} f(t)$ and $R_2 = \max_{t \in [-I, I]} f(t)$.

ii.

$$\delta = |\lambda| \int_{-I}^{I} m(s) ds < 1.$$

iii.

$$2 |\lambda| M_K I \leq R.$$

where $M_K = \max |K(t, s, u)|$ over all $t, s \in [-I, I]$ and all $u \in [R_1 - R, R_2 + R]$.

Then

(a) *Equation (1) has a unique solution $x_* \in B_R$;*

(b) *the sequence of successive approximations*

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots$$

converges to the solution x_* for any initial point $x_0 \in B_R$.

Now, we have the following question:

Can another fixed point iterative algorithm be chosen for faster convergence to the solution of the functional integral equation (1)?

We give an affirmative answer for this question by using the following iterative algorithm which is defined by Karakaya et al [14]:

$$\begin{cases} x_0 \in X, \\ x_{n+1} = Ty_n, \\ y_n = (1-\upsilon_n) z_n + \upsilon_n T z_n, \\ z_n = Tx_n, \end{cases} \tag{3}$$

in which X is a Banach space, T is a self-map of X and $\{\upsilon_n\}_{n=0}^{\infty} \in [0,1]$.

The reason why we chose the iterative algorithm (3) in this study is that it is completely independent of classical fixed point algorithms such as Picard [24], Mann [20], Ishikawa [13], Noor [22], S [25], and Picard-Mann [15] as well as being superior to these algorithms in terms of convergence speed for contraction mappings. Together with the convergence results we obtained in this study, the result of data dependency is proved for the first time for equation (1).

Lemma 1.2. *[28] Let $\{\beta_n\}_{n=0}^{\infty}$ be a sequence that satisfies the following inequality for all $n \geq n_0$:*

$$\beta_{n+1} \leq (1-\mu_n) \beta_n + \mu_n \gamma_n,$$

in which $\mu_n \in (0,1)$, for all $n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \mu_n = \infty$ and $\gamma_n \geq 0$, $\forall n \in \mathbb{N}$. Then the following inequality holds:

$$0 \leq \limsup_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \gamma_n.$$

2 Convergence Theorems and Data Dependence Analysis

In the following theorems, we show that the convergence result can be obtained for equation (1) under suitable conditions, and we also examine the data dependence result through the iterative algorithm (3).

Theorem 2.1. *Suppose that all the conditions given by Theorem 1.1 are fulfilled. Consider the control sequence $\{\upsilon_n\}_{n=0}^{\infty} \in [0,1]$ with the condition $\sum_{n=0}^{\infty} \upsilon_n = \infty$ in (3). Then*

i. *integral equation (1) has a unique solution in B_R, say x_*,*

ii. *iterative sequence (3) converges to x_*.*

Proof. Let $\{x_n\}_{n=0}^{\infty}$ be iterative sequence generated by iterative algorithm (3) for the operator $T : B_R \to B_R$ defined by (2). We will show that $x_n \to x_*$ as $n \to \infty$.

From (2), (3), and assumptions $i - iii$ given in the Theorem 1.1, we have

$$\begin{aligned} |x_{n+1}(t) - x_*(t)| &= |T(y_n(t)) - T(x_*(t))| \\ &= \left| \left[\lambda \int_{-t}^{t} K(t,s,y_n(s))\,ds + f(t) \right] \right. \\ &\quad \left. - \left[\lambda \int_{-t}^{t} K(t,s,x_*(s))\,ds + f(t) \right] \right| \\ &\leq |\lambda| \int_{-t}^{t} |K(t,s,y_n(s)) - K(t,s,x_*(s))|\,ds \\ &\leq |\lambda| \int_{-|t|}^{|t|} m(s)\,|y_n(s) - x_*(s)|\,ds \qquad (4) \\ &\leq |\lambda|\,\|y_n - x_*\| \int_{-|t|}^{|t|} m(s)ds \\ &\leq \delta\,\|y_n - x_*\|, \end{aligned}$$

and

$$\begin{aligned} |y_n(t) - x_*(t)| &\leq (1-\upsilon_n)\,|z_n(t) - x_*(t)| + \upsilon_n\,|Tz_n(t) - Tx_*(t)| \\ &\leq (1-\upsilon_n)\,|z_n(t) - x_*(t)| \\ &\quad + \upsilon_n\,|\lambda| \int_{-t}^{t} |K(t,s,z_n(s)) - K(t,s,x_*(s))|\,ds \end{aligned}$$

$$\begin{aligned}
&\leq (1-\upsilon_n)\left|z_n(t)-x_*(t)\right| \\
&\quad +\upsilon_n\left|\lambda\right|\int_{-|t|}^{|t|} m(s)\left|z_n(s)-x_*(s)\right|ds \\
&\leq (1-\upsilon_n)\left\|z_n-x_*\right\| \\
&\quad +\left|\lambda\right|\upsilon_n\left\|z_n-x_*\right\|\int_{-|t|}^{|t|} m(s)ds.
\end{aligned}$$

Then,

$$\left|y_n(t)-x_*(t)\right| \leq \left[1-\upsilon_n(1-\delta)\right]\left\|z_n-x_*\right\|, \tag{5}$$

and

$$\begin{aligned}
\left|z_n(t)-x_*(t)\right| &= \left|T(x_n(t))-T(x_*(t))\right| \\
&= \left|\left[\lambda\int_{-t}^{t}K(t,s,x_n(s))ds+f(t)\right]\right. \\
&\quad \left.-\left[\lambda\int_{-t}^{t}K(t,s,x_*(s))ds+f(t)\right]\right| \\
&\leq \left|\lambda\right|\int_{-t}^{t}\left|K(t,s,x_n(s))-K(t,s,x_*(s))\right|ds \\
&\leq \left|\lambda\right|\int_{-|t|}^{|t|} m(s)\left|x_n(s)-x_*(s)\right|ds \qquad (6) \\
&\leq \left|\lambda\right|\left\|x_n-x_*\right\|\int_{-|t|}^{|t|} m(s)ds \\
&\leq \delta\left\|x_n-x_*\right\|.
\end{aligned}$$

Substituting (6) in (5) and (5) in (4), respectively, we obtain

$$\left\|x_{n+1}-x_*\right\| \leq \delta^2\left[1-\upsilon_n(1-\delta)\right]\left\|x_n-x_*\right\|. \tag{7}$$

By repeating this process n-times, we obtain

$$\begin{aligned}
\|x_n - x_*\| &\leq \delta^2 \left[1 - \upsilon_{n-1}(1-\delta)\right] \|x_{n-1} - x_*\| \\
\|x_{n-1} - x_*\| &\leq \delta^2 \left[1 - \upsilon_{n-2}(1-\delta)\right] \|x_{n-2} - x_*\| \\
&\vdots \\
\|x_1 - x_*\| &\leq \delta^2 \left[1 - \upsilon_0(1-\delta)\right] \|x_0 - x_*\|.
\end{aligned}$$

Then

$$\|x_{n+1} - x_*\| \leq \delta^{2(n+1)} \sum_{i=0}^{n} \left[1 - \upsilon_i(1-\delta)\right] \|x_0 - x_*\|. \tag{8}$$

From the classical analysis, we have $1 - x \leq e^{-x}$. Hence by using this fact with (8), we obtain

$$\|x_{n+1} - x_*\| \leq \|x_0 - x_*\| \, \delta^{2(n+1)} e^{-(1-\delta)\sum\limits_{i=0}^{n} \upsilon_i},$$

which yields $\lim_{n\to\infty} \|x_n - x_*\| = 0$. □

The following theorem indicates that the convergence result can be obtained without the condition $\sum\limits_{n=0}^{\infty} \upsilon_n = \infty$ for the sequence of $\{\upsilon_n\}_{n=0}^{\infty} \in [0,1]$:

Theorem 2.2. *Assume that all the conditions given by Theorem 1.1 are fulfilled. Then*

- ***i.*** *integral equation (1) has a unique solution in B_R, say x_*,*
- ***ii.*** *iterative sequence (3) converges to x_*.*

Proof. Since $\{\upsilon_n\}_{n=0}^{\infty} \in [0,1]$ and $\delta < 1$, we have $[1 - \upsilon_n(1-\delta)] \leq 1$. By using this fact with the inequality (7), we obtain

$$\|x_{n+1} - x_*\| \leq \delta^2 \|x_n - x_*\|.$$

From the above inequality, we get

$$\|x_{n+1} - x_*\| \leq \delta^{2(n+1)} \|x_0 - x_*\|,$$

which implies that $\lim_{n\to\infty}\|x_n - x_*\| = 0$. □

In order to examine the data dependence result of integral equation (1), we consider the following equation:

$$S(u(t)) = \lambda \int_{-t}^{t} K_1(t,s,u(s))\,ds + g(t), \tag{9}$$

in which $K_1 \in C([-I,I]\times[-I,I]\times\mathbb{R})$ and $g \in C[-I,I]$.

Define the following algorithm by using the operators T given by (2) and S given by (9), respectively:

$$\begin{cases} x_{n+1} = \lambda \int_{-t}^{t} K(t,s,y_n(s))\,ds + f(t), \\ y_n = (1-\upsilon_n) z_n + \upsilon_n \left[\lambda \int_{-t}^{t} K(t,s,z_n(s))\,ds + f(t)\right], \\ z_n = \lambda \int_{-t}^{t} K(t,s,x_n(s))\,ds + f(t), \end{cases} \tag{10}$$

and

$$\begin{cases} u_{n+1} = \lambda \int_{-t}^{t} K_1(t,s,v_n(s))\,ds + g(t), \\ y_n = (1-\upsilon_n) w_n + \upsilon_n \left[\lambda \int_{-t}^{t} K_1(t,s,w_n(s))\,ds + g(t)\right], \\ w_n = \lambda \int_{-t}^{t} K_1(t,s,u_n(s))\,ds + g(t), \end{cases} \tag{11}$$

in which $\{\upsilon_n\}_{n=0}^{\infty} \in [0,1]$, $K, K_1 \in C([-I,I]\times[-I,I]\times\mathbb{R})$ and $f, g \in C[-I,I]$.

Theorem 2.3. *Let $\{x_n\}_{n=0}$ and $\{u_n\}_{n=0}^{\infty}$ be iterative sequences generated by (10) and (11). Asssume that the control sequence $\{\upsilon_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfies $\frac{1}{2} \le \upsilon_n$ for all $n \in \mathbb{N}$. Suppose*

A1. *x_* and u_* are solutions of equations (2) and (9) respectively, and all the conditions of Theorem 2.1 hold;*

A2. *there exist* $\varepsilon_1 \geq 0$ *and* $\varepsilon_2 \geq 0$ *so as to* $|K(t,s,u) - K_1(t,s,u)| \leq \varepsilon_1$ *and* $|f(t) - g(t)| \leq R_3\varepsilon_2$*, for all* $t, s \in [-I, I]$*, where* $R_3 = \frac{R}{M_K}$.

If $\{u_n\}_{n=0}^{\infty} \to u_*$ *as* $n \to \infty$*, then*

$$\|x_* - u_*\| \leq \frac{5[\varepsilon_1 + \varepsilon_2] R_3}{1-\delta}. \tag{12}$$

Proof. Using (2), (3), (9), (11) and assumptions $(i) - (iii)$ and (A2), we obtain

$$\begin{aligned} |x_{n+1}(t) - u_{n+1}(t)| &= |T(y_n)(t) - S(v_n)(t)| \\ &= \left| \left[\lambda \int_{-t}^{t} K(t,s,y_n(s))\, ds + f(t) \right] \right. \\ &\quad \left. - \left[\lambda \int_{-t}^{t} K_1(t,s,v_n(s))\, ds + g(t) \right] \right| \\ &\leq \left| \left[\lambda \int_{-t}^{t} K(t,s,y_n(s))\, ds \right] - \left[\lambda \int_{-t}^{t} K(t,s,v_n(s))\, ds \right] \right| \\ &\quad + \left| \left[\lambda \int_{-t}^{t} K(t,s,v_n(s))\, ds \right] - \left[\lambda \int_{-t}^{t} K_1(t,s,v_n(s))\, ds \right] \right| \\ &\quad + |f(t) - g(t)|. \end{aligned} \tag{13}$$

So, we get

$$\begin{aligned} |x_{n+1}(t) - u_{n+1}(t)| &\leq |\lambda| \int_{-t}^{t} |K(t,s,y_n(s)) - K(t,s,v_n(s))|\, ds \\ &\quad + |\lambda| \int_{-t}^{t} |K(t,s,v_n(s)) - K_1(t,s,v_n(s))|\, ds \\ &\quad + |f(t) - g(t)| \end{aligned}$$

$$\begin{aligned}
&\leq |\lambda| \int_{-|t|}^{|t|} m(s) \left|y_n(s) - \upsilon_n(s)\right| ds \\
&\quad + |\lambda| \int_{-|t|}^{|t|} \varepsilon_1 ds + |f(t) - g(t)| \\
&\leq |\lambda| \left\|y_n - \upsilon_n\right\| \int_{-|t|}^{|t|} m(s)ds + 2|\lambda| |t| \varepsilon_1 + |f(t) - g(t)| \\
&\leq \delta \left\|y_n - \upsilon_n\right\| + 2|\lambda| |T| \varepsilon_1 + |f(t) - g(t)| \qquad (|t| \leq T) \\
&\leq \delta \left\|y_n - \upsilon_n\right\| + R_3\varepsilon_1 + R_3\varepsilon_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
|y_n(t) - \upsilon_n(t)| &\leq (1-\upsilon_n)|z_n(t) - w_n(t)| + \upsilon_n |Tz_n(t) - Sw_n(t)| \\
&\leq (1-\upsilon_n)|z_n(t) - w_n(t)| \\
&\quad + \upsilon_n \left| \left[\lambda \int_{-t}^{t} K(t,s,z_n(s))\, ds\right] - \left[\lambda \int_{-t}^{t} K(t,s,w_n(s))\, ds\right] \right| \\
&\quad + \upsilon_n \left| \left[\lambda \int_{-t}^{t} K(t,s,w_n(s))\, ds\right] - \left[\lambda \int_{-t}^{t} K_1(t,s,w_n(s))\, ds\right] \right| \\
&\quad + \upsilon_n |f(t) - g(t)| \\
&\leq (1-\upsilon_n)|z_n(t) - w_n(t)| \\
&\quad + \upsilon_n |\lambda| \int_{-t}^{t} |K(t,s,z_n(s)) - K(t,s,w_n(s))|\, ds \\
&\quad + \upsilon_n |\lambda| \int_{-t}^{t} |K(t,s,w_n(s)) - K_1(t,s,w_n(s))|\, ds \\
&\quad + \upsilon_n |f(t) - g(t)|.
\end{aligned}$$

So, we get

$$|y_n(t) - \upsilon_n(t)| \leq (1-\upsilon_n)|z_n(t) - w_n(t)|$$

$$
\begin{aligned}
&+\upsilon_n |\lambda| \int_{-|t|}^{|t|} m(s) \left| z_n(s) - w_n(s) \right| ds \\
&+\upsilon_n |\lambda| \int_{-|t|}^{|t|} \varepsilon_1 ds + \upsilon_n \left| f(t) - g(t) \right| \\
\leq\ & [1 - \upsilon_n(1-\delta)] \left\| z_n - w_n \right\| \\
&+2\upsilon_n |\lambda| |T| \varepsilon_1 + \upsilon_n \left| f(t) - g(t) \right| \\
\leq\ & [1 - \upsilon_n(1-\delta)] \left\| z_n - w_n \right\| \\
&+\upsilon_n R_3 \varepsilon_1 + \upsilon_n R_3 \varepsilon_2. \qquad (14)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\left| z_n(t) - w_n(t) \right| &= \left| T(x_n)(t) - S(u_n)(t) \right| \\
&= \left| \left[\lambda \int_{-t}^{t} K(t, s, x_n(s))\, ds + f(t) \right] \right. \\
&\quad \left. - \left[\lambda \int_{-t}^{t} K_1(t, s, u_n(s))\, ds + g(t) \right] \right| \\
&\leq \left| \left[\lambda \int_{-t}^{t} K(t, s, x_n(s))\, ds \right] - \left[\lambda \int_{-t}^{t} K(t, s, u_n(s))\, ds \right] \right| \\
&\quad + \left| \left[\lambda \int_{-t}^{t} K(t, s, u_n(s))\, ds \right] - \left[\lambda \int_{-t}^{t} K_1(t, s, u_n(s))\, ds \right] \right| \\
&\quad + \left| f(t) - g(t) \right| \\
&\leq |\lambda| \int_{-t}^{t} \left| K(t, s, x_n(s)) - K(t, s, u_n(s)) \right| ds \\
&\quad + |\lambda| \int_{-t}^{t} \left| K(t, s, u_n(s)) - K_1(t, s, u_n(s)) \right| ds \\
&\quad + \left| f(t) - g(t) \right|
\end{aligned}
$$

$$\leq \quad |\lambda| \int_{-|t|}^{|t|} m(s) \left| x_n(s) - u_n(s) \right| ds + |\lambda| \int_{-|t|}^{|t|} \varepsilon_1 ds + \left| f(t) - g(t) \right|.$$

So, we get

$$\begin{aligned} \left| z_n(t) - w_n(t) \right| &\leq |\lambda| \left\| x_n - u_n \right\| \int_{-|t|}^{|t|} m(s) ds + 2 |\lambda| |t| \varepsilon_1 + \left| f(t) - g(t) \right| \\ &\leq \delta \left\| x_n - u_n \right\| + 2 |\lambda| |T| \varepsilon_1 + \left| f(t) - g(t) \right| \qquad (|t| \leq T) \\ &\leq \delta \left\| x_n - u_n \right\| + R_3 \varepsilon_1 + R_3 \varepsilon_2. \end{aligned} \tag{15}$$

Substituting (15) in (14) and (14) in (13), respectively, and using $\delta < 1$, we obtain

$$\begin{aligned} \left\| x_{n+1} - u_{n+1} \right\| \leq\ & \left[1 - \upsilon_n (1 - \delta) \right] \left\| x_n - u_n \right\| \\ & + (2R_3 + \upsilon_n R_3) \varepsilon_1 \\ & + (2R_3 + \upsilon_n R_3) \varepsilon_2. \end{aligned}$$

By using $\frac{1}{2} \leq \upsilon_n$ in the above inequality, we get

$$\begin{aligned} \left\| x_{n+1} - u_{n+1} \right\| &\leq \left[1 - \upsilon_n (1 - \delta) \right] \left\| x_n - u_n \right\| \\ &\quad + 5 \upsilon_n R_3 \varepsilon_1 + 5 \upsilon_n R_3 \varepsilon_2 \\ &= \left[1 - \upsilon_n (1 - \delta) \right] \left\| x_n - u_n \right\| \\ &\quad + \upsilon_n (1 - \delta) \frac{5(\varepsilon_1 + \varepsilon_2) R_3}{1 - \delta}. \end{aligned} \tag{16}$$

Choose

$$\begin{aligned} \beta_n &= \left\| x_n - u_n \right\|, \\ \mu_n &= \upsilon_n (1 - \delta) \in (0, 1), \\ \gamma_n &= \frac{5(\varepsilon_1 + \varepsilon_2) R_3}{1 - \delta} \geq 0. \end{aligned}$$

Since $\frac{1}{2} \leq \upsilon_n$ for all $n \in \mathbb{N}$ we have $\sum\limits_{n=0}^{\infty} \upsilon_n = \infty$. So, all the conditions of Lemma 1.2 are hold. Hence

$$0 \leq \limsup_{n\to\infty} \|x_n - u_n\| \leq \limsup_{n\to\infty} \frac{5(\varepsilon_1 + \varepsilon_2)R_3}{1-\delta}.$$

From Theorem 2.1, we have $\lim_{n\to\infty} x_n = x_*$. By using this and the assumption $u_n \to u_*$ as $n \to \infty$, we get

$$\|x_* - u_*\| \leq \frac{5(\varepsilon_1 + \varepsilon_2)R_3}{1-\delta}.$$

□

3 Example of Data Dependence

Example 3.1. Let's consider the following equation

$$x(t) = \frac{1}{180}\int\limits_{-t}^{t} \frac{5s - 2x(s)}{10} ds + \left(-2t^2 + 3t + 2\right), \quad t \in [-1, 1],$$

where $\lambda = \frac{1}{180}$, $K(t, s, u) = \frac{5s-2u}{10}$ and $f(t) = -2t^2 + 3t + 2$. Let $R = 1$. Then

$$\begin{aligned} R_1 &= f(-1) = -3, \\ R_2 &= f(1) = 3. \end{aligned}$$

We have $M_K = (R_2 + R)^2 = 16$ and hence $2|\lambda| M_K T = \frac{16}{30} < 1 = R$ and $R_3 = \frac{1}{16}$. Also we have

$$|K(t, s, u) - K(t, s, v)| = \frac{1}{5}|u - v|.$$

By choosing $m(s) = \frac{1}{5}$, we obtain

$$\delta = \frac{1}{180}\int\limits_{-1}^{1} \frac{1}{5} ds = \frac{1}{450} < 1.$$

Let $T : C[-4,4] \to C[-4,4]$ be as follows:

$$T(x(t)) = \frac{1}{180}\int_{-t}^{t} \frac{5s - 2x(s)}{10} ds + \left(-2t^2 + 3t + 2\right). \tag{17}$$

It can be seen from the following operations that T is a contraction mapping:

$$\begin{aligned} & |Tx(t) - Ty(t)| \\ = & \left| \frac{1}{180}\int_{-1}^{1} \left[\frac{5s - 2x(s)}{10} ds - \frac{5s - 2y(s)}{10}\right] ds \right| \\ \leq & \frac{2}{1800}\int_{-1}^{1} |y(s) - x(s)|\, ds. \end{aligned}$$

From the property of Chebyshev norm, we get

$$\begin{aligned} \|Tx - Ty\| & \leq \frac{2}{1800}\|x - y\| \int_{-1}^{1} ds \\ & = \frac{1}{450}\|x - y\|. \end{aligned}$$

Now, consider the following integral equation:

$$u(t) = \frac{1}{180}\int_{-t}^{t} \left[\frac{5s - 2x(s)}{10} - s + \frac{1}{10}\right] ds + \left(-\frac{3}{2}t^2 + 3t + \frac{3}{2}\right), \quad t \in [-1,1].$$

Let $S : C[-4,4] \to C[-4,4]$ be as follows:

$$S(u(t)) = \frac{1}{180}\int_{-t}^{t} \left[\frac{5s - 2x(s)}{10} - s + \frac{1}{10}\right] ds + \left(-\frac{3}{2}t^2 + 3t + \frac{3}{2}\right). \tag{18}$$

It can easily be seen that S is the approximation operator of T.

As a result all the conditions of Theorem 2.1 are hold by the equations (17) and (18). Hence, the convergence of the algorithm (3) to x_* and u_*, is obtained in $C[-4,4]$.

Then,

$$|K(t,s,u) - K(t,s,v)| = \left| s - \frac{1}{10} \right| \leq \frac{11}{10} = \varepsilon_1,$$

$$|f(t) - g(t)| = \left| -\frac{1}{2}t^2 + \frac{1}{2} \right| = \frac{1}{2}\left|1 - t^2\right| \leq \frac{1}{2} = \varepsilon_2.$$

From the above processes, all the conditions of Theorem 2.3 are hold. Hence

$$\|x_* - u_*\| \leq 0.502.$$

4 Conclusion

Theorem 2.1 shows that a three-step iterative algorithm can be used effectively for the functional Volterra integral equation of second type (1) in which the existence and uniqueness of its solution are guaranteed. Also, Theorem 2.2 shows that the convergence result of this equation can be obtained without the condition given for the control sequence $\{\upsilon_n\}_{n=0}^{\infty}$ belonging to the iterative algorithm (3). In addition, with Theorem 2.3, the concept of data dependency was obtained for the first time for equation (1), and this result was supported by the numerical example (3.1).

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