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Characterization of Best Approximation Points with Lattice Homomorphisms

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Abstract. In this paper we prove some characterization theorems in the theory of best approximation in Banach lattices. We use a new idea for finding the best approximation points in an ideal. We find the distance between an ideal I and an element x by using lattice homomorphisms. We introduce maximal ideals of an AM space and characterize other ideals by the maximal ideals. We also give a new representation for principle ideals in Banach lattices that is a majorizing subspace and we show that these principle ideals are proximinal. The role of lattice homomorphisms in this paper is very important.

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1. Introduction

The theory of best approximation by elements of convex sets and reverses convex sets (i.e., complements of convex sets) in normed linear spaces has many important applications in mathematics and other sciences. This theory is well developed (see, e.g., [10, 11] and the references therein). However, convexity is sometimes a very restrictive assumption,

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so there is a clear need to study the best approximation by not necessarily convex sets. In this direction, Rubinov and Singer [9] developed a theory of best approximation by elements of so-called normal sets in the non-negative orthant R^I_+ of a finite-dimensional coordinate space R^I endowed with the max-norm. Martinez-Legaz, Rubinov and Singer have developed a theory of best approximation by downward subsets of the space R^I [6]. Also, Mohebi and Rubinov [8] generalized these concepts and developed the theory of best approximation by closed normal and downward subsets of Banach lattices X with a strong unit. If we compare the definition of a downward subset with a solid subset we can say that a solid subset is an absolutely downward subset.

The aim of this paper is to examine a theory of best approximation by element of solid subsets in Banach lattices with an order unit. This problem is related to the monotonicity theory which is studied in [2, 4, 5]. In [4, 5] the dominated approximation problem is considered, but we want to examine the problem of best approximation without the dominated restriction. We examine this problem for ideals and some special subsets in Banach lattices, that some of them are true for solid subsets. An ideal is a convex subset but a solid subset is not in general a convex subset. The structure of the paper is as follows. In the Section 1.2 we recall that main definitions and some results on best approximation in Banach lattices. The best approximation problem in AM spaces and their ideals are discussed in section 2.1. In section 2.2 we examine the proximinality problem for ideals in Banach lattices with an order unit **1**. We also show that

$$d(x, I) = \inf\{\lambda \geqslant 0 : \quad x - \lambda \mathbf{1} \in I\}$$

where I is an ideal in a Banach lattice X and $x \in X^+$. In the final section we give a new representations for ideals in Banach lattices and obtain some more results about their best approximations. We show that every principle ideal that is a majorizing subspace in a Banach lattice is a proximinal subset.

1.1 Preliminaries

Let W be a non-empty subset of the normed linear space X. For any $x \in X$, define

$$d(x, W) = \inf\{\|x - y\| : y \in W\}.$$

Recall (see e.g [10]) that a point $w_0 \in W$ is called a best approximation for $x \in X$ if

$$\|w_0 - x\| = d(x, W).$$

If each $x \in X$ has at least one best approximation $w_0 \in W$, then W is called a proximinal subset of X. The (possibly empty) set of best approximations x from W is defined by

$$P_W(x) = \{y \in W : \|x - y\| = d(x, W)\}.$$

We recall some definitions from lattice theory (see e.g [1, 7]).
A real vector space X is said to be an ordered vector space whenever it is equipped with an order relation $\geqslant$ (i.e., $\geqslant$ is a reflexive, antisymmetric, and transitive binary relation on X) that is compatible with the algebraic structure of X in the sense that it satisfies the following two axioms:

i) If $x \geqslant y$, then $x + z \geqslant y + z$ holds for all $z \in X$,

ii) If $x \geqslant y$, than $\alpha x \geqslant \alpha y$ holds for all $\alpha \geqslant 0$.

A vector x in an ordered vector space X is called positive whenever $x \geqslant 0$ holds. The set of all positive vectors of X will be denoted by X^+, i.e.,

$$X^+ = \{x \in X : x \geqslant 0\}.$$

The set X^+ of positive vectors is called the positive cone of X. A Riesz space is an ordered vector space X with the additional property that for each pair of vectors $x, y \in X$ the supremum and the infimum of the set $\{x, y\}$ both exist in X. Following the classical notation, we shall write

$$x \vee y = sup\{x, y\} \quad and \quad x \wedge y = inf\{x, y\}.$$

For every $x \in X$ let

$$x^+ = x \vee 0 \quad , \quad x^- = (-x) \vee 0 \quad , \quad and \quad |x| = x \vee (-x).$$

We say that $x, y \in X$ are disjoint (denoted by $x \perp y$) if $|x| \wedge |y| = 0$. For each $x \in X$ we define $x^{\perp} = \{y \in X : |x| \wedge |y| = 0\}$.
Let X be an ordered set and $x, y \in X$ such that $x \leqslant y$, we denote the order interval in X by $[x, y]$ and,

$$[x, y] = \{z \in X : x \leqslant z \leqslant y\}.$$

A subset in a Riesz space is called order bounded if it is included in an order interval. Let X be a Riesz space, a linear functional $f : X \to \mathbb{R}$ is said to be an order bounded functional if f maps order bounded subsets of X to bounded subsets of $\mathbb{R}$. The vector space $X^{\sim}$ of all order bounded linear functionals on X is called the order dual of X. Garrett Birkhoff shows that the norm dual of a Banach lattice X coincides with its order dual, i.e., $X' = X^{\sim}$; see [1].

Definition 1.1.1. *A norm, $\|.\|$, on X is called a lattice norm if for each $x, y \in X$, such that $|x| \leqslant |y|$, then $\|x\| \leqslant \|y\|$. If X is a Riesz space and $\|.\|$ a norm on X, then $(X, \|.\|)$ is called a normed Riesz space. A normed Riesz space which is complete with respect to the norm is called a Banach lattice.*

Definition 1.1.2. *An operator $T : E \to F$ between two Riesz spaces is said to be a lattice homomorphism whenever it preserves the lattice operations, that is, whenever $T(x \vee y) = T(x) \vee T(y)$ holds for all $x, y \in E$. A lattice homomorphism which is in addition one-to-one and isometry is referred to as a isometric lattice isomorphism.*

Definition 1.1.3. i) *A subspace U of X is called a sublattice of X if $x \vee y \in U$ and $x \wedge y \in U$ for all $x, y \in U$.*
ii) *A subset A of X is called solid if $|x| < |y|$ for some $y \in A$ implies that $x \in A$.*
iii) *Every solid subspace I of X is called an ideal in X.*
Let X be a Riesz space. If S is a solid subset of X, then S is not necessary a convex subset.

Example 1.1.4. Let $(\mathbb{R}^2, \leqslant)$ be a Riesz space such that $(x_1, y_1) \leqslant (x_2, y_2)$ if and only if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$, then $S = \{(x, 0), (0, y) : -1 \leqslant x, y \leqslant 1\}$ is a solid subset of $\mathbb{R}^2$ that is not a convex subset of $\mathbb{R}^2$.

Let X be a Riesz space and $x \in X$. It is clear that $x^{\perp}$ is a closed convex solid subset of X. Also an easy verification shows that $d(x, x^{\perp}) = \|x\|$. Therefore $0 \in P_{x^{\perp}}(x)$.

Lemma 1.1.5. *Let X be a Riesz space and $y, z \in X^{+}$ such that $z \leqslant y$. If $x \in [z, y]^{\perp}$ and $x \geqslant 0$, then $d(x, [z, y]) = \|x \vee z\|$.*

Proof. For each $t \in [z, y]$ we have $\|x - t\| = \|x \vee t\|$ and so $\|x \vee z\| \leqslant \|x \vee t\| \leqslant \|x \vee y\|$. Therefore $d(x, [z, y]) = \|x \vee z\|$.
If X is a Riesz space and A is a subset of X^{+} such that $\inf(A) = y$ and if $x \in A^{\perp}$ and $x \geqslant 0$, then $d(x, A) = \|x \vee y\|$. Because for each $t \in A$ we have $\|x - t\| = \|x \vee t\|$ and since $\inf(A) = y$ so for each $z \in A$, $\|x \vee y\| \leqslant \|x \vee z\|$. Therefore $d(x, A) = \|x \vee y\|$. $\square$

Definition 1.1.6. *A Banach lattice X is said to be an abstract M-space, whenever its norm is an M-norm, i.e., if $x \wedge y = 0$ in X implies*

$$\|x \vee y\| = max\{\|x\|, \|y\|\}.$$

An abstract M-space is known as an AM-space.

Example 1.1.7. If K is a Hausdorff compact topological space, then $C(K)$ the set of all f such that

$$\|f\| = \sup\{|f(k)| : k \in K\},$$

is an AM space having unit the constant function one.
The ideal generated by a singleton $\{e\}$ in X is called a principal ideal and is denoted by X_e. If for some $e \in X$ we have $X_e = X$, then $e \in X$ is called an order unit or strong unit and we denoted the order unit of X by $\mathbf{1}$. The order unit norm on X is defined as follow

$$\|x\| = \inf\{\lambda > 0 : |x| \leqslant \lambda \mathbf{1}\}.$$

Lemma 1.1.8. ([7]) *Assume that X is an AM space with an order unit $\mathbf{1}$ and the order unit norm $\|.\|_{\mathbf{1}}$. Let*

$$B = \{x' \in X'_{+} :< x', \mathbf{1} >= 1\},$$

and $K = ex(B)$ the set of all extreme points of B, then K is $\sigma(X', X)$-compact and the mapping

$$X \ni x \to f_x \in C(K) \quad defined \quad by \quad f_x(x') =< x', x >$$

for all $x' \in K$ is an isometric lattice isomorphism.
Recall that $X^\sim$ denote the order dual space of X and for each Banach lattice X we have $X' = X^\sim$.

Lemma 1.1.9. ([7]) *For every $0 \neq x' \in X^\sim$ the following assertions are equivalent.*

i) *x' is a lattice homomorphism,*

ii) *$x' \in X^+$ and $(x')^{-1}(0)$ is an ideal in X.*

Lemma 1.1.10. ([7]) *For every closed subspace J of $C(K)$ the following assertions are equivalent.*

i) *J is an algebraic ideal in $C(K)$,*

ii) *J is an ideal in $C(K)$,*

iii) *There exists a closed set K_0 satisfying$J = \{f \in C(K) : f(K_o) = 0\}$.*

Let I be any ideal in $C(X)$ and let $Z[f] = \{x \in X : f(x) = 0\}$ for each $f \in C(X)$. If $\cap_{f \in I} Z[f]$ is nonempty, we call I a fixed ideal.

Lemma 1.1.11. ([3]) *The fixed maximal ideals in C(X) are precisely the sets*

$$M_p = \{f \in C(X) : f(p) = 0\},$$

for $p \in X$. The ideals M_p are distinct for distinct p.

Lemma 1.1.12. ([3]) *If X is compact, then every ideal I in C(X) is fixed.*

Theorem 1.1.13. *Let J be a maximal ideal in $C(K)$ such that K is a compact set. Let $f \in C(K)$, then $d(f, J) = |f(p)|$ for some $p \in K$, and J is a proximinal subset of $C(K)$.*

Proof. Since K is a compact set and J is a maximal ideal in $C(K)$, by Lemma 1.1.11 and Lemma 1.1.12, $J = \{f \in C(K) : f(p) = 0\}$ for some $p \in K$. Suppose that $f \in C(K)$, and $g \in J$. Then $\|f - g\| = sup\{|f(t)-g(t)| : t \in K\}$. Since $p \in K$ and $g \in J$ we have $\|f-g\| \geqslant |f(p)|$ and since g was arbitrary, $d(f, J) \geqslant |f(p)|$.
On the other hand if we put $g_0 = f - f(p)$, then $g_0(p) = 0$ and so $g_o \in J$, and $\|f - g_0\| = |f(p)|$, hence $d(f, J) = |f(p)|$. For each $f \in C(K)$, we have $f - f(p) \in P_J(f)$. So J is a proximinal subset of $C(K)$. □

2. Characterization of Nearest Points and Proximinality of Ideals

In this section we examine ideals in AM space and in general Banach lattices and give some applications in best approximation.

2.1 Ideals in AM spaces and some application to best approximation

In this subsection we characterize maximal ideals in an AM space X and some application of this characterization to best approximation. Also we characterize other ideals in AM spaces with maximal ideals. We will show that a maximal ideal I in AM space X is of the form $I = \{x :< x', x >= 0\}$ for a lattice homomorphism $x' \in X'$. We also show that other ideals are intersection of some maximal ideals.

Definition 2.1.1. *Let X be a Banach lattice and $x' \in X'$. The zero set of $x' \in X'$ is denoted by $N(x')$ and defined as follow,*

$$N(x') = \{x \in X : \quad < x', x >= 0\}.$$

Proposition 2.1.2. *Let X be an AM space with an order unit $\mathbf{1}$ and $B = \{x' \in X'_+ :< x', \mathbf{1} >= 1\}$. I, is a maximal ideal in X if and only if $I = N(x')$ for some $x' \in ex(B)$.*

Proof. Put $K = ex(B)$, then by Lemma 1.1.8, K is $\sigma(X, X')$-compact and X is isometrically lattice isomorphic to $C(K)$. By Lemma 1.1.11, fixed maximal ideals of $C(K)$ are precisely $J_p = \{f \in C(K) : f(p) = 0\}$

and since K is compact, all ideals of $C(K)$ are fixed. This complete the proof. $\square$

We note that by Lemma 1.1.9, x' is a lattice homomorphism.

Theorem 2.1.3. *Let X be an AM space with an order unit $\mathbf{1}$. If I is a maximal ideal in X and $x \in X$, then $x- < x', x > \mathbf{1} \in P_I(x)$ and $d(x, I) = | < x', x > |$ for some $x' \in X'$.*

Proof. Suppose that I is a maximal ideal in X. By Proposition 2.1.2, there exists an $x' \in ex(B)$ such that $I = \{x :< x', x >= 0\}$, where $B = \{x' \in X'_+ :< x', \mathbf{1} >= 1\}$. So $x- < x', x > \mathbf{1} \in I$ and hence $d(x, I) \leqslant | < x', x > |$. For each $y \in I$,

$$\|x - y\| = sup\{| < y', x - y > | : \|y'\| \leqslant 1\}.$$

Since $\|x'\| \leqslant 1$ and $< x', y >= 0$ we get $\|x - y\| \geqslant | < x', x > |$. Therefore $d(x, I) = | < x', x > |$. In the end, $z = x- < x', x > \mathbf{1} \in I$ and $\|x - z\| = d(x, I)$. Therefore $z \in P_I(x)$.

In Proposition 2.1.2, we show that if X is an AM space with an order unit $\mathbf{1}$, then $N(x')$ is a maximal ideal in X for each $x' \in ex(B)$. In continuation we will show that if I is an ideal in X and X is an AM space with an order unit $\mathbf{1}$, then there exists a subset K of $ex(B)$ such that I is the intersection of zero sets in K. $\square$

Theorem 2.1.4. *Let X be an AM space with an order unit $\mathbf{1}$. If I is a closed ideal in X, Then there exists a closed subset K of $ex(B)$ such that $I = \cap_{x' \in K} N(x')$.*

Proof. We have only to combine Lemma 1.1.5 and Lemma 1.1.8. $\square$

Theorem 2.1.5. *Let X be an AM space with an order unit $\mathbf{1}$. If I is a closed ideal in X and $x \in X^+ \setminus I$, then $d(x, I) = sup_{x' \in K} d(N(x'), x)$, for a subset K of $ex(B)$ and $P_I(x) \neq \emptyset$*

Proof. By Theorem 2.1.4, there exists a closed subset K of $ex(B)$ such that $I = \cap_{x' \in K} N(x')$. Since $N(x')$ for each $x' \in K$ is a maximal ideal by Theorem 2.1.3, $d(x, N(x')) =< x', x >$, for each $x' \in K$ and also $x- < x', x > \mathbf{1} \in N(x')$. We define $t = sup\{< x', x >: x' \in K\}$. Since

$I \subset N(x')$ for each $x' \in K$, it follows that $d(x, I) \geqslant d(x, N(x'))$. So $d(x, I) \geqslant t$. Also since $\|x\| \geqslant t$, we have $x - t\mathbf{1} \geqslant 0$. Therefore $0 \leqslant x - t\mathbf{1} \leqslant x - < x', x > \mathbf{1}$. Since $N(x')$ is an ideal, $x - t\mathbf{1} \in N(x')$ for each $x' \in K$ and so $x - t\mathbf{1} \in I$. Now

$$d(x, I) \leqslant \|x - (x - t\mathbf{1})\| = \|t\mathbf{1}\| = t \leqslant d(x, I).$$

This shows that $d(x, I) = sup_{x' \in K} d(N(x'), x)$ and $x - t\mathbf{1} \in P_I(x)$. $\square$

Theorem 2.1.6. *Let X be a Banach lattice with an order unit $\mathbf{1}$ and $K = [-r, r]$ a closed interval in $\mathbb{R}$. Suppose that $x' \geqslant 0$ is an element in X' such that $< x', \mathbf{1} >= 1$. If $I = \{x : \quad < x', x > \in K\}$ and $x \in X^+ \setminus I$, then $d(x, I) =< x', x > -r$ and $x - (< x', x > -r)\mathbf{1} \in P_I(x)$.*

Proof. For any $\alpha \in [1 - \frac{r}{<x', x>}, 1 + \frac{r}{<x', x>}]$, we have $x - \alpha < x', x > \mathbf{1} \in I$, especially $z = x - (< x', x > -r)\mathbf{1} \in I$. Hence $d(x, I) \leqslant \|x - z\|$ or equivalently $d(x, I) \leqslant < x', x > -r$. For any $y \in I$

$$\|x - y\| = sup\{| < y', x - y > | : \|y'\| \leqslant 1\}.$$

Since $\|x'\| \leqslant 1$ and $< x', x >> r$, we get $\|x - y\| \geqslant < x', x > -r$ and therefore $d(x, I) =< x', x > -r$. Since $z \in I$ and $\|x - z\| = d(x, I)$, we have $z \in P_I(x)$. $\square$

2.2 Proximinality of ideals in Banach lattices

Let X be a Banach lattice and I a closed ideal in X. We show that I is a proximinal subset of X for every $x \in X^+$ and in continuation we will show that $d(x, I) = \inf\{\lambda \geqslant 0 : \quad x - \lambda\mathbf{1} \in I\}$.

Theorem 2.2.1 *Let X be a Banach lattice with an order unit $\mathbf{1}$. If I is a closed subset in X and $x \in X^+$, then $P_I(x) \neq \emptyset$.*

Proof. If $x \in I$ then $x \in P_I(x)$. Suppose that $x \in X^+ \setminus I$ and $d(x, I) = r$. For each $r_0 > r$, $s(x, r_0) \cap I \neq \emptyset$, especially $s(x, r + \frac{2n+1}{2n(n+1)}) \cap I \neq \emptyset$. Suppose that $y_n \in s(x, r + \frac{2n+1}{2n(n+1)}) \cap I$ So for this y_n we have $r + \frac{1}{n+1} \leqslant \|y_n - x\| \leqslant r + \frac{1}{n}$. So

$$x + (r + \frac{1}{n+1})\mathbf{1} \leqslant y_n \leqslant x + (r + \frac{1}{n})\mathbf{1}.$$

For each $n \leqslant m$, we have $x + (r + \frac{1}{n})\mathbf{1} \geqslant x + (r + \frac{1}{m})\mathbf{1}$. Since $x \geqslant 0$ we have $|y_n| \leqslant x + (r + \frac{1}{n})\mathbf{1}$ and $y_n \geqslant y_m$. By the relation $|y_n - y_m| = 2|y_n| \vee |y_m| - |y_n + y_m|$,

$$\begin{aligned} |y_n - y_m| &\leqslant 2(x + (r + \frac{1}{n})\mathbf{1}) - (2x + (2r + \frac{1}{n+1} + \frac{1}{m+1})\mathbf{1}) \\ &= (\frac{1}{n} - \frac{1}{n+1})\mathbf{1} + (\frac{1}{n} - \frac{1}{m+1})\mathbf{1}. \end{aligned}$$

Thus $\|y_n - y_m\| \leqslant \|\frac{1}{n} - \frac{1}{n+1}\| + \|\frac{1}{n} - \frac{1}{m+1}\|$, and this tend to zero as $n, m \to \infty$. So $\{y_n\}$ is a cauchy sequence in I and since I is a closed subspace, there exists a $y_0 \in I$ such that $y_n \to y_0$ and $\|y_0 - x\| = r$. Therefore $P_I(x) \neq \emptyset$. □

Theorem 2.2.2. *Let X be a Banach lattice with an order unit $\mathbf{1}$ and order unit norm, I a closed ideal in X and $x \in X^+$. Suppose that $d(x, I) = r$ and $0 \notin P_I(x)$, then* $\inf P_I(x) = x - r\mathbf{1}$.

Proof. If $x \in I$, then $d(x, I) = 0$ and $P_I(x) = \{x\}$, so the proof is complete. Suppose that $x \in X^+ \setminus I$, since $0 \notin P_I(x)$ so $\|x\| > r$, by the definition of order unit norm we have $x - r\mathbf{1} > 0$. So $x - (r + \varepsilon)\mathbf{1} \geqslant 0$, for each $\varepsilon \in [0\ ,\ \|x\| - r]$. For all such $\varepsilon > 0$, there exits a $y_\varepsilon \in I$, such that $\|y_\varepsilon - x\| < r + \varepsilon$, or equivalently $x - (r + \varepsilon)\mathbf{1} < y_\varepsilon < x + (r + \varepsilon)\mathbf{1}$. Since $x - (r + \varepsilon)\mathbf{1} \geqslant 0$ and $y_\varepsilon \in I$, we have $x - (r + \varepsilon)\mathbf{1} \in I$. Tend ε to zero, also I is a closed ideal, therefore $x - r\mathbf{1} \in I$ and clearly $x - r\mathbf{1} \in P_I(x)$. Now if $y \in P_I(x)$, since $P_I(x) \subset B(x, r) \cap I$, we have $x - r\mathbf{1} \leqslant y \leqslant x + r\mathbf{1}$. Thus $\inf P_I(x) = x - r\mathbf{1}$. □

Theorem 2.2.3. *Let X be a Banach lattice with an order unit $\mathbf{1}$ and order unit norm, I a closed ideal in X. If $x \in X^+$ such that $0 \notin P_I(x)$, then $d(x, I) = inf\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in I\}$.*

Proof. Suppose that $A = \{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in I\}$. If $x \in I$, then $x - 0\mathbf{1} = x \in I$, and so $inf A = 0 = d(x, I)$. Suppose that $x \notin I$, then $r = d(x, I) > 0$. If $\lambda > 0$ is arbitrary such that $x - \lambda\mathbf{1} \in I$, then

$$\lambda = \|\lambda\mathbf{1}\| = \|x - (x - \lambda\mathbf{1})\| \geqslant d(x, I) = r.$$

On the other hand by Theorem 2.2.2, $x - r\mathbf{1} \in I$. It follows that $r \in A$. Hence $inf A = r$. $\square$

Theorem 2.2.4. *Let X be a Banach lattice with an order unit $\mathbf{1}$ and order unit norm, $\{I_k\}_{k\in J}$ a family of closed ideals in X. If $x \in X^+$ such that $0 \notin P_{I_k}(x)$ for each $k \in J$, then $d(x, \cap I_k) = sup_{k\in J} d(x, I_k)$.*

Proof. If $x \in \cap_{k\in J} I_k$, then $d(x, I_k) = 0$ for each $k \in J$ and the proof is complete. Suppose that $x \in X^+ \setminus \cap_{k\in J} I_k$. Put $I = \cap_{k\in J} I_k$ and $r_k = d(x, I_k)$. Since $I \subset I_k$ for all $k \in J$, we have $r_k \leqslant d(x, I)$ for all $k \in J$. Define $t = sup_{k\in J} r_k$. Then $t \leqslant d(x, I)$. If $t = \infty$, then $d(x, I) = \infty$, and hence the result holds. Suppose that $t < \infty$. The inequality $t \geqslant r_k$ for all $k \in J$ implies that $x - t\mathbf{1} \leqslant x - r_k\mathbf{1}$. Since $x - r_k\mathbf{1} \geqslant 0$ for each $k \in J$, $x - t\mathbf{1} \geqslant 0$ and therefore $|x - t\mathbf{1}| \leqslant |x - r_k\mathbf{1}|$.
Since $x - r_k\mathbf{1} \in I_k$ and I_k is an ideal in X, we get $x - t\mathbf{1} \in I_k$, for each $k \in J$. Therefore $x - t\mathbf{1} \in \cap_{k\in J} I_k$. In view of Theorem 2.2.3,

$$d(x, I) = inf\{\lambda \geqslant 0 : x - \lambda\mathbf{1} \in I\}.$$

Since $x - t\mathbf{1} \in I$, it follows that $d(x, I) \leqslant t$. Therefore, $d(x, I) = t$.
We recall that a subset S is a solid subset in Banach lattice space X if $|x| \leqslant |y|$ and $y \in S$ implies that $x \in S$. $\square$

Remark 2.2.5. *In the above Theorems we can use a solid subset instead of an ideal.*

3. A New Representation of Ideals in Banach Lattices

In this section we introduce a new representation of ideals in a Banach lattice and with this new idea we give some applications to the best approximation problem.

Definition 3.1. *Suppose that X is a Banach lattice with an order unit $\mathbf{1}$ and $x \in X$. The zero set of $x \in X$ is denoted by $Nall(x)$ and define*

$$Nall(x) = \{x' \in X' : x' \text{ is a lattic homomorphism and } < x', x >= 0\}.$$

By the definition of $Nall(x)$, it is clear that $Nall(x) = Nall(|x|)$.

Definition 3.2. *The ideal generated by x is called a principal ideal and is denoted by E_x. We know that*

$$E_x = \cup_{n\in\mathbb{N}}[-n|x|, n|x|],$$

for every $x \in X$ without loss of generality we can assume that $x \geqslant 0$.
A vector subspace G of an ordered vector space X is a majorizing subspace whenever for each $x \in X$ there exists some $y \in G$ with $x \leqslant y$ (or, equivalently, if for each $x \in X$ there exists some $y \in G$ with $y \leqslant x$).

In the following Proposition, we use the theorem known as Liecki-Luxemburg-Schep Theorem ([1]) Which stated that every lattice homomorphism whose domain is a majorizing subspace has always a lattice homomorphic extension to the whole space.

Proposition 3.3. *Let X be a Banach lattice with an order unit $\mathbf{1}$. If $x \in X^+$, such that the principle ideal generated by x is a majorizing subspace, then*

$$E_x = \{y \in X : Nall(x) \subseteq Nall(y)\}.$$

Proof. Put $J = \{y \in X : Nall(x) \subseteq Nall(y)\}$. Suppose that $y \in E_x$, there exits an $n \in \mathbb{N}$ such that $|y| \leqslant nx$. If $x' \in Nall(x)$, then

$$\begin{aligned} |<x', y>| &= <x', |y|> \\ &\leqslant <x', nx> \\ &= n<x', x> \\ &= 0. \end{aligned}$$

Thus $x' \in Nall(y)$. Since x' was arbitrary in $Nall(x)$, we have $Nall(x) \subseteq Nall(y)$ and $y \in J$. Therefore $E_x \subseteq J$.
Suppose that $t \notin E_x$. Since E_x is a principle ideal that is a majorizing subspace by Lipecki-Luxemburg-Schep theorem there exists a lattice homomorphism $x' \in X'$ such that $<x', t> \neq 0$ and $<x', y> = 0$ for each

$y \in E_x$. So $x' \notin Nall(t)$ and $x' \in Nall(x)$. Therefore $Nall(x) \nsubseteq Nall(t)$, and hence $t \notin J$. Therefore $J \subseteq E_x$, this complete the proof. $\square$

Theorem 3.4. *Let X be a Banach lattice with an order unit $\mathbf{1}$. If E_x is a majorizing subspace, then E_x is a proximinal subset of X.*

Proof. By Proposition 3.3, $E_x = \{y \in X : Nall(x) \subseteq Nall(y)\}$. For each $w \in X$, if $w \in E_x$, then $d(w, E_x) = 0$ and $w \in P_{E_x}(w)$. Suppose that $w \notin E_x$. For any $y \in E_x$,

$$\begin{aligned} \|y-w\| &= sup\{| < x', y-w > | : \|x'\| = 1\} \\ &\geqslant sup\{| < x', y-w > | : x' \in Nall(x), \quad < x', \mathbf{1} >= 1\} \\ &= sup\{| < x', w > | : x' \in Nall(x), \quad < x', \mathbf{1} >= 1\}. \end{aligned}$$

Put $K = \{x' \in Nall(x) :< x', \mathbf{1} >= 1\}$. Therefore

$$d(w, E_x) \geqslant \sup_{x' \in K} | < x', w > |.$$

We define $u = w - (\sup_{x' \in K} | < x', w > |)\mathbf{1} \wedge w$. If $y' \in Nall(x)$, then

$$\begin{aligned} < y', u > &= < y', w > -(\sup_{x' \in K} | < x', w > |) < y', \mathbf{1} > \wedge < y', w > \\ &= < y', w > - < y', w > \\ &= 0. \end{aligned}$$

Therefore $y' \in Nall(u)$ and so $Nall(x) \subseteq Nall(u)$. Thus $u \in E_x$. Also

$$\begin{aligned} d(w, E_x) &\leqslant \|w-u\| \\ &= \|\sup_{x' \in K} | < x', w > |\mathbf{1}\| \\ &= \sup_{x' \in K} | < x', w > |\|\mathbf{1}\|. \end{aligned}$$

Thus $d(w, E_x) = \sup_{x' \in K} | < x', w > |$. Since $u \in E_x$ and $\|w-u\| = d(w, E_x)$, E_x is a proximinal subset of X. $\square$

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