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Extension of Shift-Invariant Frames for Locally Compact Abelian Groups

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Abstract. Let G be a locally compact abelian group with a uniform lattice subgroup. In this paper, we verify extension of shift-invariant systems in $L^2(G)$ to tight frames. We show that any shift-invariant Bessel sequence with an at most countable number of generators in $L^2(G)$ can be extended to a tight frame for its closed linear span by adding another shift-invariant system with at most the same number of generators in $L^2(G)$. Also, we yield an extension of the given Bessel sequence to a pair of dual frame sequences.

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1. Introduction

A shift-invariant space (SIS) is a closed subspace V of $L^2(\mathbb{R})$ that is invariant under translations by integers i.e. if $f \in V$, then $f(.+k) \in V$ for all $k \in \mathbb{Z}$. Shift-invariant spaces have many applications in frame theory, numerical analysis, multiresolution analysis(MRA) and wavelet theory [2, 3, 10, 11, 18] The theory of shift-invariant spaces in $L^2(\mathbb{R})$ was presented by Helson [14] and characterized in $L^2(\mathbb{R}^n)$ by Bownik in [4]. In [16, 17] principal shift-invariant spaces, that is, shift-invariant spaces generated by one single function, was studied in the context of locally compact abelian groups. They studied LCA groups with a uniform lattice (i.e. a discrete and co-compact subgroup). The authors in [1]

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presented and investigated shift-invariant spaces for local fields. They studied shift-invariant spaces of $L^2(G)$, where G is a locally compact abelian group, or in general a local field, with a compact open subgroup.
In this paper, we assume G is a locally compact abelian group with uniform lattice H. Then G/H is compact, $H^\perp \approx \widehat{G/H}$ and $\widehat{G}/H^\perp \approx \widehat{H}$, where $H^\perp = \{\xi \in \widehat{G} :< \xi, h >= 1, \forall h \in H\}$ is called the annihilator of H and $\widehat{G}$ is dual group of G. In the same manner, all the result of this paper satisfy for local fields.
On the other hand, frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [13] for study of nonharmonic Fourier series. They were reintroduced and development in 1986 by Daubechies et al. [12] and popularized from then on. The theory of frames plays an important role in signal processing, sampling theory and neural networks [6, 7, 8]. For basic results on frames, see [9].
Let H be a Hilbert space and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist the constants $A, B > 0$ such that

$$A\|f\|^2 \leqslant \sum_{i\in I} |< f, f_i >|^2 \leqslant B\|f\|^2$$

for all $f \in H$. The system $\{f_i\}_{i\in I} \subseteq H$ is a Bessel sequence if at least the right inequality is satisfied. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame if $A = B$ it is called a Parseval frame. A dull frame for the frame $\{f_i\}_{i\in I}$ is a Bessel sequence $\{g_i\}_{i\in I} \subseteq H$ such that

$$f = \sum_{i\in I} < f, g_i > f_i$$

for all $f \in H$.
In [5] was shown that any shift-invariant Bessel sequence in $L^2(\mathbb{R})$ with an at most countable number of generators can be extended to a tight frame by adding another shift-invariant system with at most the same number of generators. Here, we generalize this approach to locally compact abelian groups with a uniform lattice. Similarly, we can do it for local fields.

2. Main Results

First we need the following lemma.

Lemma 2.1. [16] *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(G)$ and $V = \overline{span}\{T_k g_n\}_{k\in H, n\in I}$. Then there exists a collection of functions $\{\varphi_n\}_{n\in I}$ in $L^2(G)$ such that $\{T_k\varphi_n\}_{k\in H, n\in I}$ is a Parseval frame for V.*

Proposition 2.2. *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(G)$ such that $\{T_k g_n\}_{k\in H,n\in I}$ is a Bessel sequence with bound $B > 0$. Then there exists a collection of functions $\{h_n\}_{n\in I}$ in V such that*

$$\{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k h_n\}_{k\in H,n\in I}$$

is a tight frame for V with bound B.

Proof. Consider the linear operator

$$S_1 : L^2(G) \to L^2(G), S_1 f := \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n.$$

Then S_1 is bounded and the operator

$$S_2 : V \to V, S_2 := BI - S_1$$

is positive. Thus, S_2 has a unique positive square root, to be denoted by $S_2^{1/2}$. We can write any $f \in V$ as

$$f = \frac{1}{B}[S_1 f + (BI - S_1)f] = \frac{1}{B}[S_1 f + S_2^{1/2} S_2^{1/2} f].$$

Using the frame decomposition associated with the frame sequence $\{T_k\varphi_n\}_{k\in H,n\in I}$ from Lemma 1.1(on the element $S_2^{1/2} f$) we arrive at

$$f = \frac{1}{B}\left[S_1 f + S_2^{1/2} \sum_{k\in H,n\in I} \langle S_2^{1/2} f, T_k\varphi_n\rangle T_k\varphi_n\right]$$
$$= \frac{1}{B}\left[\sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, S_2^{1/2} T_k\varphi_n\rangle S_2^{1/2} T_k\varphi_n\right].$$

It is a standard result that the operator S_2 commutes with the operators T_k. Since $S_2^{1/2}$ is a limit of polynomials in S_2 in the strong operator topology, it follows that also $S_2^{1/2}$ commutes with T_k. Thus, for any $f \in V$ we arrive at

$$f = \frac{1}{B}\left[\sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, T_k S_2^{1/2}\varphi_n\rangle T_k S_2^{1/2}\varphi_n\right]$$

Letting $h_n := S_2^{1/2}\varphi_n \in V$, it follows that

$$\|f\|^2 = \langle f, f\rangle = \frac{1}{B}\left[\sum_{k\in H,n\in I} |\langle f, T_k g_n\rangle|^2 + \sum_{k\in H,n\in I} |\langle f, T_k h_n\rangle|^2\right].$$

This completes the proof. □

Example 2.3. Let

$$G = \{(x_j)_{j\in\mathbb{Z}} : x_j \in \{0,1\}\ for\ all\ j\ and\ x_j = 0\ for\ all\ j > n,\ for\ some\ n \in \mathbb{Z}\}.$$

If addition on G is the coordinate-wise addition modulo 2 i.e.

$$(z_j) = (x_j) \oplus (y_j) \Leftrightarrow z_j = x_j + y_j\ (mod\ 2),$$

for all $(x_j), (y_j) \in G$, then G is a locally compact abelian group under Cartesian product topology. The locally compact abelian group G is called Cantor dyadic group (see [15]).
The group G is identified with the half real line $[0, \infty)$ under the mapping $x \to |x| = \sum_{j\in\mathbb{Z}} x_j 2^j$.

Let

$$H = \{x \in G : x_j = 0,\ for\ j < 0\}\ and\ D = \{x \in G : x_j = 0,\ for\ j \geqslant 0\}.$$

Then H is a uniform lattice, D is compact and $G/H = D$. Under the map $x \to |x|$, H is identified with the nonnegative integers and D is identified with the interval $[0, 1]$.
Now, let the system $\{T_k\chi_D : k \in H\}$. Then

1. The system $\{T_k\chi_D + T_{k\oplus1}\chi_D : k \in H\}$ is a Bessel sequence in $L^2(G)$ with bound $B = 4$;
2. $\{T_k\chi_D + T_{k\oplus1}\chi_D : k \in H\}$ is not a frame sequence;
3. $V := \overline{span}\{T_k\chi_D + T_{k\oplus1}\chi_D : k \in H\} = \overline{span}\{T_k\chi_D : k \in H\}$

In the same manner of Example 2.3 in [7], if $h = \chi_D - T_1\chi_D$, then

$$\{T_k\chi_D + T_{k\oplus1}\chi_D : k \in H\} \cup \{T_k h : k \in H\}$$

is a tight frame for V with bound 4.

Note: Similar to Example 2.5 in [7] we can show the generators $\{h_n\}_{n\in\mathbb{Z}}$ in proposition 2.1 is optimal.

Corollary 2.4. *Let $\{g_n\}_{n\in I}$ be an at most countable family of functions in $L^2(G)$ such that $\{T_k g_n\}_{k\in H,n\in I}$ is a Bessel sequence with bound $B > 0$. Then there exists functions $\{\varphi_n\}_{n\in I}$ and $\{\psi_n\}_{n\in I}$ in V such that*

$$\{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k\varphi_n\}_{k\in H,n\in I}, \{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k\psi_n\}_{k\in H,n\in I}$$

form a pair of dual frames for V.

Proof. Let

$$S_1 : L^2(G) \to L^2(G), S_1 f := \sum_{k\in H, n\in I} \langle f, T_k g_n\rangle T_k g_n.$$

For any $f \in V$, applying the frame decomposition associated with the frame sequence $\{T_k\varphi_n\}_{k\in H,n\in I}\}$ from Lemma 1.1, we can write f as follows:

$$\begin{aligned} f &= S_1 f + (I - S_1)f & (1)\\ &= \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle (I - S_1)f, T_k\varphi_n\rangle T_k\varphi_n & (2)\\ &= \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, (I - S_1)^* T_k\varphi_n\rangle T_k\varphi_n. & (3)\end{aligned}$$

It's clear that $(I - S_1)^* = (I - S_1)$. Now, for any $h \in V$, from Lemma 7.21 in [9] we have

$$(I - S_1)T_k h = T_k h - S_1 T_k h = T_k h - T_k S_1 h = T_k(I - S_1)h.$$

Hence

$$(I - S_1)T_k = T_k(I - S_1).$$

Put $\psi_n = (I - S_1)\varphi_n$, Then (1.1) becomes

$$\begin{aligned} f &= \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, (I - S_1)T_k\varphi_n\rangle T_k\varphi_n\\ &= \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, T_k(I - S_1)\varphi_n\rangle T_k\varphi_n\\ &= \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n + \sum_{k\in H,n\in I} \langle f, T_k\psi_n\rangle T_k\varphi_n.\end{aligned}$$

This proves that

$\{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k\varphi_n\}_{k\in H,n\in I}$ and $\{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k\psi_n\}_{k\in H,n\in I}$

form a pair of dual frames for V. □

Theorem 2.5. *Suppose that $\{T_k g_n\}_{k\in H,n\in I}$ is a Bessel sequence in $L^2(G)$ with the upper bound B. For any $\lambda \geqslant B$, there is a sequence $\{h_m\}_{m\in\mathbb{Z}}$ such that $\{T_k g_n\}_{k\in H,n\in I} \bigcup \{T_k h_m\}_{k\in H,m\in\mathbb{Z}}$ is a tight frame for $L^2(G)$ with the bound λ.*

Proof. Let

$$S_1 f = \sum_{k\in H,n\in I} \langle f, T_k g_n\rangle T_k g_n, \ \ \forall f \in L^2(G).$$

Since $\{T_k g_n\}_{k\in H, n\in I}$ is a Bessel sequence in $L^2(G)$ with the upper bound B, S_1 is self-adjoint, positive and satisfies

$$\langle Sf, f\rangle \leqslant B\|f\|^2, \ \ \forall f \in L^2(G).$$

For any $\lambda \geqslant B$, define $S_2 : L^2(G) \to L^2(G)$ by

$$S_2 = \lambda I - S_1.$$

Then S_2 is positive and commutes with T_n. From Lemma 2.4.4 in [9] we know that $S_2^{1/2}$ exists, is self-adjoint and can be expressed as a limit of a sequence of polynomials in S_2. Accordingly $S_2^{1/2}$ commutes with T_n.
Take $\{T_k\psi_m\}_{k\in H, m\in\mathbb{Z}}$ to be a Parseval frame for $L^2(G)$. Like in the proof of Proposition 2.2 we see that

$$\begin{aligned} S_2 f = S_2^{1/2} S_2^{1/2} f &= S_2^{1/2} \sum_{k\in H, m\in\mathbb{Z}} \langle S_2^{1/2} f, T_k\psi_m\rangle T_k\psi_m \\ &= \sum_{k\in H, m\in\mathbb{Z}} \langle f, S_2^{1/2} T_k\psi_m\rangle S_2^{1/2} T_k\psi_m \\ &= \sum_{k\in H, m\in\mathbb{Z}} \langle f, T_k S_2^{1/2}\psi_m\rangle T_k S_2^{1/2}\psi_m. \end{aligned}$$

Put $h_m = S_2^{1/2}\psi_m$. Since $S_1 + S_2 = \lambda I$, it follows that

$$\{T_k g_n\}_{k\in H, n\in I} \bigcup \{T_k h_m\}_{k\in H, m\in\mathbb{Z}}$$

is a tight frame for $L^2(G)$ with frame bound λ. □

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