Journal of Mathematical Extension
Vol. 17, No. 8, (2023) (4)1-11
URL: https://doi.org/10.30495/JME.2023.2542
ISSN: 1735-8299
Original Research Paper

Weighted Composition, Volterra and Integral Operators on Hardy Zygmund-Type Spaces

M. Hassanlou*
Urmia University of Technology

E. Abbasi
Mahabad Branch, Islamic Azad University

Abstract. In this paper, we investigate weighted composition, Volterra and Integral operators on second derivative Hardy spaces. Some equivalent conditions for boundedness of the operators will be given using the boundedness on the Hardy spaces. Also we give a criteria for compactness of weighted composition operators.

AMS Subject Classification: Primary 47B38; Secondary 47B33, 46E15.
Keywords and Phrases: derivative Hardy spaces, bounded operator, compact operator, weighted composition operators

1 Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plan $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic function on $\mathbb{D}$. For $\psi, \varphi \in H(\mathbb{D})$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$ the weighted composition operators $W_{\varphi,\psi}$ defined by

$$W_{\varphi,\psi} f(z) = \psi(z) f(\varphi(z)) \qquad f \in H(\mathbb{D}), z \in \mathbb{D},$$

Received: October 2022; Accepted: October 2023
*Corresponding Author

which is a generalization of the well-known composition operators C_φ and multiplication operators M_ψ. For $g \in H(\mathbb{D})$, the Volterra type operator T_g is defined by

$$T_g f(z) = \int_0^z f(w)g'(w)dw \qquad f \in H(\mathbb{D}), \qquad z \in \mathbb{D}.$$

Also the integral operator I_g defined as follows

$$I_g f(z) = \int_0^z f'(w)g(w)dw.$$

The space of bounded analytic functions on $\mathbb{D}$ is denoted by H^∞ which is a Banach space with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. For $1 \le p < \infty$, the Hardy space H^p consists of all analytic functions $f \in H(\mathbb{D})$ for which

$$\|f\|_{H^p}^p = \sup_{0<r<1} M_r(f,p) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

These spaces are Banach with the norm $\|\cdot\|_{H^p}$.

Let $1 \le p < \infty$. We denote by S_2^p, the space of analytic functions on $\mathbb{D}$ such that second derivative is in the Hardy space. So,

$$S_2^p = \{f \in H(\mathbb{D}) : \quad f'' \in H^p\}.$$

We define the norm on S_2^p as follows

$$\|f\|_{S_2^p} = |f(0)| + |f'(0)| + \|f''\|_{H^p},$$

and equipped with this norm, S_2^p is a Banach space. These spaces may be called Hardy Zygmund-type spaces. In particular, $S_1^p = S^p$, the space of analytic function with derivative in Hardy space, which is investigated along with weighted composition operators and Volterra operators in [2, 5, 7, 8].

S^p is a Banach space with the norm $\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}$. It is well known by Theorem 3.11 in [4] that, if $f \in S^1$, then f extends continuously to $\overline{\mathbb{D}}$. Thus, the functions in S^p belong to the disc algebra A (the space of analytic functions on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ endowed with the norm $||f||_A = \sup_{z\in\mathbb{D}} |f(z)|$).

It can be proved that $S_2^p \subset S^p \subset H^p$ and there exists a positive constant C_p such that

$$||f||_{S^p} \leq C_p ||f||_{S_2^p}, \qquad f \in S_2^p.$$

For $1 \leq p \leq \infty$ and $f \in H^p$ we have

$$|f(z)| \leq \frac{\|f\|_{H^p}}{(1-|z|^2)^{1/p}}.$$

If $f \in S_2^p$ then there exists a positive constant C such that

$$|f^{(k)}(z)| \leq C \frac{\|f\|_{S_2^p}}{(1-|z|^2)^{k-1+1/p}}, \tag{1}$$

where $k = 0, 1, 2$.

Roan in [8] characterized bounded, compact and isometric composition operators on S^p. Mac Cluer investigated composition operators on the space S^p in terms of Carleson measure, [7]. A characterization for boundedenss, (weak) compactness and complete continuity of weighted composition operators can be found in [2]. Composition and multiplication operators with some different norms on S^2 were studied in [5]. Also Volterra type operators on S^p spaces studied by authors of [6].

In this paper, noting that the spaces introduced here are not known in the literature, we characterize boundedness of weighted composition operators on S_2^p in terms of such operators on H^p spaces. Also compact weighted composition operators will be studied. Furthermore, we have a brief investigation on the bounded Integral type operators on the S_2^p spaces.

All positive constants are denoted by C which may be varied from one place to another.

2 Boundedness of $W_{\varphi,\psi} : S_2^p \to S_2^p$

In this section, some conditions for boundedenss of weighted composition operators form S_2^p into S_2^p or H^p will be given. Since

$$(W_{\varphi,\psi} f)'' = \psi'' f(\varphi) + (2\psi'\varphi' + \psi\varphi'')f'(\varphi) + \psi\varphi'^2 f''(\varphi)$$

then the study of $W_{\varphi,\psi}$ on S_2^p spaces is related to the study of the following operators

$$\begin{aligned} W_{\varphi,\psi''} :& S_2^p \to H^p \\ W_{\varphi,2\psi'\varphi'+\psi\varphi''} :& S^p \to H^p \\ W_{\varphi,\psi\varphi'^2} :& H^p \to H^p. \end{aligned}$$

Although the above mentioned operators can be used but without loss of generality we replace the operators from S^p or S_2^p by the operators on H^p spaces.

Theorem 2.1. *[3] Let ψ be an analytic function on $\mathbb{D}$, φ be an analytic self-map of $\mathbb{D}$ and $0 < p \leq q < \infty$. Then the weighted composition operator $W_{\varphi,\psi}$ is bounded from H^p into H^q if and only if*

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2}\right)^{q/p} |\psi(w)|^q d\sigma(w) < \infty,$$

where $\partial\mathbb{D}$ is the unit circle and $d\sigma$ is the normalized arc length measure on $\partial\mathbb{D}$.

Lemma 2.2. *Let $1 \leq p < \infty$. Then $W_{\varphi,\psi} : S \to H^p$ is bounded if and only if $\psi \in H^p$, where $S = S_2^p$ or S^p.*

Proof. Suppose that $\psi \in H^p$ and $f \in S$. Then

$$||W_{\varphi,\psi} f||_{H^p} = ||\psi f \circ \varphi||_{H^p} \leq ||f||_\infty ||\psi||_{H^p} \leq C||f||_S ||\psi||_{H^p}.$$

So, $W_{\varphi,\psi}$ is bounded from S into H^p. For converse, using the function $f(z) = 1 \in S$, we have

$$||W_{\varphi,\psi}|| \geq ||W_{\varphi,\psi}(1)||_{H^p} = ||\psi||_{H^p}.$$

□

Theorem 2.3. *Let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi} : S_2^p \to S_2^p$ is bounded.

(b) $\psi \in S_2^p$ *and the operators* $W_{\varphi,2\psi'\varphi'+\psi\varphi''} : H^p \to H^p$ *and* $W_{\varphi,\psi\varphi'^2} : H^p \to H^p$ *are bounded.*

(c)

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |2\psi'(w)\varphi'(w) + \psi(w)\varphi''(w)|^p d\sigma(w) < \infty,$$
$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |\psi(w)\varphi'^2(w)|^p d\sigma(w) < \infty.$$

Proof. $(b) \to (a)$**:** Suppose that $\psi \in S_2^p$ and $W_{\varphi,2\psi'\varphi'+\psi\varphi''} : H^p \to H^p$ and $W_{\varphi,\psi\varphi'^2} : H^p \to H^p$ are bounded. Let $f \in S_2^p$. Then using the proof of Lemma 2.2, we obtain

$$\begin{aligned} ||(\psi f\circ\varphi)''||_{H^p} \le& ||\psi'' f\circ\varphi||_{H^p} + ||(2\psi'\varphi' \\ &+ \psi\varphi'')f'\circ\varphi||_{H^p} + ||\psi\varphi'^2 f''\circ\varphi||_{H^p} \\ \le& C_1||\psi''||_{H^p}||f||_{S_2^p} + C_2||f'||_{H^p} + C_3||f''||_{H^p} \\ \le& C_1||\psi''||_{H^p}||f||_{S_2^p} + C_2||f||_{S^p} + C_3||f||_{S_2^p} \\ \le& C_1||\psi''||_{H^p}||f||_{S_2^p} + C_2C_p||f||_{S_2^p} + C_3||f||_{S_2^p}. \end{aligned} \tag{2, 3}$$

Also, we have

$$\begin{aligned} &|(\psi f\circ\varphi)(0)| + |(\psi f\circ\varphi)'(0)| \\ &= |\psi(0)f(\varphi(0))| + |\psi'(0)f(\varphi(0)) + \psi(0)\varphi'(0)f'(\varphi(0))| \\ &\le C\frac{|\psi(0)|\|f\|_{S_2^p}}{(1-|\varphi(0)|^2)^{-1+1/p}} + C\frac{|\psi'(0)|\|f\|_{S_2^p}}{(1-|\varphi(0)|^2)^{-1+1/p}} \\ &\quad + C\frac{|\psi(0)\varphi'(0)|\|f\|_{S_2^p}}{(1-|\varphi(0)|^2)^{1/p}}. \end{aligned} \tag{4}$$

From (2) and (4), we get the desired result.

$(a) \to (b)$**:** Suppose that $W_{\varphi,\psi} : S_2^p \to S_2^p$ is a bounded operator. Consider the function $f(z) = 1 \in S_2^p$. Then

$$||W_{\varphi,\psi}|| \ge ||W_{\varphi,\psi}(1)||_{S_2^p} = ||\psi||_{S_2^p}.$$

So $\psi \in S_2^p$. Consider the function $f(z) = z \in S_2^p$. It can be easily obtained that $\psi\varphi \in S_2^p$ or $\psi''\varphi + 2\psi'\varphi' + \psi\varphi'' \in H^p$. Therefore $2\psi'\varphi' +$

$\psi\varphi'' \in H^p$. Also replacing the function $f(z) = z^2$, one can get $\psi\varphi'^2 \in H^p$.

Now we show that $W_{\varphi,\psi\varphi'^2} : H^p \to H^p$ is bounded. Let $f \in H^p$. There exists a function $g \in S_2^p$ such that $f = g''$. Hence,

$$\begin{aligned}
||W_{\varphi,\psi\varphi'^2} f||_{H^p} = ||\psi\varphi'^2 f \circ \varphi||_{H^p} \\
&= ||\psi\varphi'^2 f \circ \varphi + (2\psi'\varphi' + \psi\varphi'')g' \circ \varphi + \psi'' g \circ \varphi \\
&\quad - (2\psi'\varphi' + \psi\varphi'')g' \circ \varphi - \psi'' g \circ \varphi||_{H^p} \\
&\leq ||W_{\varphi,\psi} g||_{S_2^p} + ||(2\psi'\varphi' + \psi\varphi'')g' \circ \varphi||_{H^p} + ||\psi'' g \circ \varphi||_{H^p} \\
&\leq C_1 ||g||_{S_2^p} + C_2 ||2\psi'\varphi' + \psi\varphi''||_{H^p} ||g'||_\infty + C_3 ||\psi''||_{H^p} ||g||_\infty \\
&\leq C_1 ||f||_{H^p} + C_2 ||2\psi'\varphi' + \psi\varphi''||_{H^p} ||f||_{H^p} + C_3 ||\psi''||_{H^p} ||f||_{H^p}
\end{aligned}$$

In a similar way we can prove that $W_{\varphi,2\psi'\varphi'+\psi\varphi''} : H^p \to H^p$ is a bounded operator.

The equivalency of (b) and (c) comes from Theorem 2.1. □

The following corollaries can be obtained from Theorem 2.3.

Corollary 2.4. *Let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(a) $C_\varphi : S_2^p \to S_2^p$ is bounded.

(b) The operators $W_{\varphi,\varphi''} : H^p \to H^p$ and $W_{\varphi,\varphi'^2} : H^p \to H^p$ are bounded.

(c)

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |\varphi''(w)|^p d\sigma(w) < \infty,$$

$$\sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |\varphi'^2(w)|^p d\sigma(w) < \infty.$$

Corollary 2.5. *Let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(a) $M_\psi : S_2^p \to S_2^p$ is bounded.

(b) $\psi \in S_2^p$ and the operators $M_{2\psi'} : H^p \to H^p$ and $M_\psi : H^p \to H^p$ are bounded.

(c)

$$\sup_{a\in\mathbb{D}}\int_{\partial\mathbb{D}}\frac{1-|a|^2}{|1-\overline{a}w|^2}|2\psi'(w)|^p d\sigma(w)<\infty,$$
$$\sup_{a\in\mathbb{D}}\int_{\partial\mathbb{D}}\frac{1-|a|^2}{|1-\overline{a}w|^2}|\psi(w)|^p d\sigma(w)<\infty.$$

3 Compactness of $W_{\varphi,\psi}: S_2^p \to S_2^p$

In this section, we investigate compactness of $W_{\varphi,\psi}: S_2^p \to S_2^p$. Similar to the boundedness, the compactness is also related to the compactness of the operators

$$\begin{aligned} W_{\varphi,\psi''} &: S_2^p \to H^p \\ W_{\varphi,2\psi'\varphi'+\psi\varphi''} &: S^p \to H^p \\ W_{\varphi,\psi\varphi'^2} &: H^p \to H^p, \end{aligned}$$

and again we replace S_2^p and S^p by H^p. Contreras and Hernández-Díaz [1] proved that the inclusion operator $j_p : S^p \hookrightarrow A$ is compact if and only if $1 < p \leq \infty$ and the inclusion operator from S^1 into H^1 is compact. Since S_2^p is a subspace of S^p, the inclusion operator $j_p : S_2^p \hookrightarrow A$ is compact if and only if $1 < p \leq \infty$ and the inclusion operator from S_2^1 into H^1 is compact.

Theorem 3.1. *Let $1 \leq p < \infty$. Then the following conditions are equivalent:*

(a) $W_{\varphi,\psi}: S_2^p \to S_2^p$ is compact.

(b) The operators $W_{\varphi,\psi''}, W_{\varphi,2\psi'\varphi'+\psi\varphi''}, W_{\varphi,\psi\varphi'^2} : H^p \to H^p$ are compact.

Proof. $(b) \to (a)$**:** Assume that the operators in (b) are compact and $\{f_n\}$ is a sequence in the unit ball of S_2^p which converges to zero uniformly on compact subsets of $\mathbb{D}$. We show that $||W_{\varphi,\psi}f_n||_{S_2^p} \to 0$. Then

$$\begin{aligned} ||(\psi f_n\circ\varphi)''||_{H^p} \leq & ||\psi'' f\circ\varphi||_{H^p} + ||(2\psi'\varphi'+\psi\varphi'')f'\circ\varphi||_{H^p} \\ & + ||\psi\varphi'^2 f''\circ\varphi||_{H^p} \\ = & ||W_{\varphi,\psi''}f_n||_{H^p} + ||W_{\varphi,2\psi'\varphi'+\psi\varphi''}f_n||_{H^p} + ||W_{\varphi,\psi\varphi'^2}f_n||_{H^p} \\ & \to 0. \end{aligned}$$

On the other hand, since uniformly convergence on compact subsets implies pointwise convergence, so

$$|(\psi f_n\circ\varphi)(0)|+|(\psi f_n\circ\varphi)'(0)|\to 0.$$

So,

$$||W_{\varphi,\psi}f_n||_{S_2^p}=|(\psi f_n\circ\varphi)(0)|+|(\psi f_n\circ\varphi)'(0)|+||(\psi f_n\circ\varphi)''||_{H^p}\to 0.$$

$(a)\to(b)$**:** Suppose that $W_{\varphi,\psi}:S_2^p\to S_2^p$ is compact. We just prove that $W_{\varphi,\psi\varphi'^2}:H^p\to H^p$ is compact, the others are similar. Let $\{f_n\}$ be a sequence in the unit ball of H^p which converges to zero uniformly on compact subsets of $\mathbb{D}$. There exists a sequence $\{g_n\}$ in S_2^p with $g_n''=f_n$ and $g_n\to 0$ uniformly on compact subsets of $\mathbb{D}$. So, $||W_{\varphi,\psi}g_n||_{S_2^p}\to 0$. Hence,

$$\begin{aligned}||W_{\varphi,\psi\varphi'^2}f_n||_{H^p}=&||\psi\varphi'^2 f_n\circ\varphi||_{H^p}\\ \leq&||W_{\varphi,\psi}g_n||_{S_2^p}+||(2\psi'\varphi'+\psi\varphi'')g_n'\circ\varphi||_{H^p}+||\psi''g_n\circ\varphi||_{H^p}.\end{aligned}$$

If $p>1$, noting that the inclusion operator $j_p:S_2^p,S^p\hookrightarrow A$ is compact, we have

$$||(2\psi'\varphi'+\psi\varphi'')g_n'\circ\varphi||_{H^p}\leq||2\psi'\varphi'+\psi\varphi''||_{H^p}||g_n'||_A\to 0$$

and also

$$||\psi''g_n\circ\varphi||_{H^p}\leq||\psi''||_{H^p}||g_n||_A\to 0.$$

If $p=1$, the compactness of the inclusion operator from S_2^1 into H^1 implies that $||g_n||_{H^1}\to 0$ and $||g_n'||_{H^1}\to 0$. So there exists a subsequence, say $\{g_n\}$, such that $g_n(z)\to 0$ almost everywhere in $\mathbb{T}$. In particular, $|\psi''(z)g_n(\varphi(z))|\to 0$ almost everywhere in $\mathbb{T}$. On the other hand, for every $z\in\mathbb{T}$

$$\begin{aligned}|\psi''(z)g_n(\varphi(z))|\leq&||g_n||_A|\psi''(z)|\leq||j_1||_{S_2^1\to H^1}||g_n||_{S_2^1}|\psi''(z)|\\ \leq&||j_1||_{S_2^1\to H^1}|\psi''(z)|.\end{aligned}$$

So, by the dominated convergence theorem, $||\psi''g_n\circ\varphi||_{H^1}\to 0$. In a similar way, we can prove that

$$||(2\psi'\varphi'+\psi\varphi'')g_n'\circ\varphi||_{H^1}\to 0.$$

From above equations, we can see that $||W_{\varphi,\psi\varphi'^2} f_n||_{H^p} \to 0$ and so $W_{\varphi,\psi\varphi'^2} : H^p \to H^p$ is compact. □

For $p > 1$, Čučković and Zhao obtained that (among other results) if $1 < p \le q < \infty$, then $W_{\varphi,\psi} : H^p \to H^q$ is compact if and only if

$$\limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} \right)^{q/p} |\psi(w)|^q d\sigma(w) = 0,$$

see [3]. So, the conditions in the previous theorem are equivalent to

$$\limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |\psi''(w)|^q d\sigma(w) = 0,$$
$$\limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |2\psi'\varphi' + \psi\varphi''(w)|^q d\sigma(w) = 0,$$
$$\limsup_{|a|\to 1} \int_{\partial\mathbb{D}} \frac{1-|a|^2}{|1-\overline{a}\varphi(w)|^2} |\psi\varphi'^2(w)|^q d\sigma(w) = 0.$$

4 Boundedness of $T_g, I_g : S_2^p \to S_2^p$

Theorem 4.1. *Let $1 \le p < \infty$. Then $T_g : S_2^p \to S_2^p$ is bounded if and only if $g \in S_2^p$.*

Proof. Suppose that $g \in S_2^p$. For any $f \in S_2^p$ we have

$$\begin{aligned} ||T_g f||_{S_2^p} &= ||f'g' + fg''||_{H^p} \\ &\le ||f'g'||_{H^p} + ||fg''||_{H^p} \\ &\le ||g'||_{H^p} ||f'||_\infty + ||g''||_{H^p} ||f||_\infty \\ &\le 2C ||g||_{S_2^p} ||f||_{S_2^p}. \end{aligned}$$

For the converse, consider the constant function $f(z) = 1 \in S_2^p$. Then $||T_g|| \ge ||T_g f||_{S_2^p} = ||g||_{S_2^p}$. □

Theorem 4.2. *Let $1 \le p < \infty$. Then $I_g : S_2^p \to S_2^p$ is bounded if and only if $g \in S^p$.*

Proof. Suppose that $g \in S^p$. For any $f \in S_2^p$, we have

$$\begin{aligned} ||I_g f||_{S_2^p} =& ||f''g + f'g'||_{H^p} \\ \leq& ||f''g||_{H^p} + ||f'g'||_{H^p} \\ \leq& ||f''||_{H^p}||g||_\infty + ||f'||_\infty ||g'||_{H^p} \\ \leq& 2C||f||_{S_2^p}||g||_{S^p}. \end{aligned}$$

For the converse, consider the constant function $f(z) = z \in S_2^p$. Then

$$||I_g|| \geq ||I_g f||_{S_2^p} = ||g||_{S^p}.$$

□

We can change the condition $g \in S^p$ with $g \in S_2^p$ in the previous theorem.

References

[1] M.D. Contreras and A.G. Hernandez-Diaz, Weighted composition operators on Hardy spaces, *J. Math. Anal. Appl.* **263**(1) (2001), 224–233.

[2] M.D. Contreras and A.G. Hernandez-Diaz, Weighted composition operators on spaces of functions with derivative in a Hardy space, *J. Oper. Theory.* **52** (2004), 173–184.

[3] Z. Čučković and R. Zhao, Weighted composition operators between different weighted Bergman spaces and different Hardy spaces, *Ill. J. Math.* **51** (2007), 479-498.

[4] P.L. Duren, *Theory of H^p Spaces*, Academic Press, New York, NY, USA, (1970).

[5] C. Gu and Sh. Luo, Composition and multiplication operators on the derivative Hardy space $S^2(\mathbb{D})$, *Complex Var. Elliptic Equ.* **63** (2018), 599-624.

[6] Q. Lin, J. Liu and Y. Wu, Volterra type operators on $S^p(\mathbb{D})$, *J. Math. Anal. Appl.* **461**(2) (2018), 1100–1114.

[7] B. Maccluer, Composition operators on S^p, *Houston J. Math.* **13** (1987), 245–254.

[8] R. Roan, Composition operators on the space of functions with H^p-derivative, *Houston J. Math.* **4** (1978), 423–438.

Mostafa Hassanlou
Assistant Professor of Mathematics
Engineering Faculty of Khoy
Urmia University of Technology,
Urmia, Iran
E-mail: m.hassanlou@uut.ac.ir

Ebrahim Abbasi
Assistant Professor of Mathematics
Department of Mathematics
Mahabad Branch, Islamic Azad University,
Mahabad, Iran
E-mail: ebrahimabbasi81@gmail.com