Journal of Mathematical Extension
Vol. 17, No. 7, (2023) (1)1-22
URL: https://doi.org/10.30495/JME.2023.2548
ISSN: 1735-8299
Original Research Paper

On Similarity Reductions and Conservation Laws of the Two Non-linearity Terms Benjamin-Bona-Mahoney Equation

M. Jafari*
Payame Noor University

A. Zaeim
Payame Noor University

M. Gandom
Payame Noor University

Abstract. In this paper, the Lie group of point symmetries for a kind of Benjamin-Bona-Mahoney (BBM) equation is obtained by applying the classical Lie symmetry method. An optimal system of sub-algebras of dimension one for the BBM equation is deduced by classifying the adjoint representation orbits of the Lie symmetry group. Then, for any infinitesimal symmetry generators of the Lie group, the related similarity reductions are generated and new group invariant solutions for the BBM equation are obtained. Also, new conservation laws for this equation are constructed by the method of scaling. The conservation laws densities are calculated by using the concept of variables weight, scaling symmetry and Euler operator and their fluxes are computed by applying the homotopy operator.

AMS Subject Classification: 68W30; 70S10; 70G65

Received: October 2022; Accepted: September 2023
*Corresponding Author

Keywords and Phrases: Benjamin-Bona-Mahoney equation, Lie symmetry group, Scaling symmetry, Optimal system, Homotopy operator, Conservation law.

1 Introduction

The well known Korteweg-de Vries equation (KdV equation) was applied by Benjamin, Bona and Mahony for modeling long surface gravitational waves of small amplitude that propagate uni-directionally in $1+1$ dimensions [4]. This equation which is known as Benjamin-Bona-Mahony equation (or BBM equation), is also called the regularized long-wave equation (RLWE) in literature. In contrary to the KdV equation, which is unstable in its high wave number components, the solutions of the BBM equation are unique and stable. Furthermore, the KdV equation has an infinite number of integrals of motion, while the BBM equation just admits three of them [9, 27]. Before, in 1966, this equation was appeared in the work by Peregrine, for study of undular bores [29]. Some other mathematicians have also studied different types of this equation and have obtained useful solutions for this equation [3, 14]. In this paper, we take a kind of the BBM equation under attention, which has two nonlinear terms as

$$u_t + pu_x + quu_x + ru^2u_x + su_{xxx} = 0, \tag{1}$$

for arbitrary nonzero constants p, q, r and s [20]. Biswas utilized the solitary wave ansatz and computed an exact 1-soliton solution of (1) In [5]. Also, Weierstrass elliptic function was instructed for obtaining additional exact solutions to some nonlinear evolutionary equations. For instance, in [17, 18], Kuru examined the BBM-like equation, and another kind of generalized BBM equation was investigated by Estevez et al. [12]. Furthermore, by using an equivalent way, Deng et al. analyze a nonlinear variant of the PHI-four equation [11]. A new modified BBM local fractional equation based on the local fractional derivative was recently investigated in [32] and practical and interesting results were obtained from it. In the present work, we obtain similarity reductions for (1) by the method of Lie classical symmetry and we compute its conservation laws by the scaling method. As we know, analysis of differential equations is interwoven with these algebraic methods. Reductions of the order, computation of invariant solutions and determining new ones from the know solutions of ODEs are nice applications of the Lie symmetry theory which was founded by Sophus Lie in the late of nineteenth century [19]. In order to see more applications of Lie's symmetry method we refer to [8] and references therein. Based on the Lie symmetry group of an ODE, an algorithm is developed to find particular solutions of the differential equation, which are famous as group invariant solutions, by the Lie's symmetry method. The number of independent variables of the differential equation is decreased in the reduced system and group invariant solutions are in fact the solutions of this system [2, 21, 22, 23, 33]. Lie's symmetry method was generalized by Bluman and Cole, in order to compute the group invariant solutions [7].

A substantial concept in physics are conservation laws, which show the invariance of a physical quantity during time of occurrence a physical process. Conservation laws, which are actually the mathematical model of physical laws related to the constancy of energy and mass, play an essential role in the study of differential equations and the analysis of their solutions. The basic laws of fluid mechanics dictate that mass is conserved in a control volume for fluids of constant density. Therefore, the total mass entering the control volume must be equal to the total mass leaving the control volume plus the mass accumulated in the control volume. In other words, the total energy of an isolated system remains constant. The same principle applies to fluid mechanics. There are many ways to find the conservation laws of a differential equation [6, 13, 24, 25]. One of the common procedure is Noether's method, in which Noether's theorem comes into use. This theorem expresses the connection between the variational symmetries of a Lagrangian and the conservation laws of the corresponding Euler-Lagrange equations [1, 15, 26]. Another method that is discussed in the present study is the scaling method. In this algorithmic method, concepts of calculus of variations and linear algebra are used [30]. We consider a linear combination of differential expressions with arbitrary coefficients which are invariant under the dilation symmetry of the differential equation. Then we take the derivative of the initial density with respect to time variable and apply the equation to it. Next, we apply the Euler operator on the obtained expression and set the result equal to zero. By solving the resulting linear system, the indeterminate coefficients of the density are obtained. After the initial density is calculated, the corresponding flux can be obtained by applying the homotopy operator. The present paper is organized in the following way. In order to keep the paper self-contained, section 2 consists of some recent studies and prerequisite which we will use them later. In section 3, we obtain the classical symmetries of BBM equation from Lie's method. Section 4 is devoted to finding the one-parameter optimal system of subalgebras and reducing the equation by the generators of these subalgebras. Finally, the conservation laws of the BBM equation are calculated by the scaling method in section 5.

2 Preliminaries

In this section, some prerequisites and recent studies on classical symmetries, and the conservation laws of differential equations, which are used later are stated (see [26] and [30]). Suppose that Δ be a system of PDEs,

$$\Delta \equiv \Delta_l(x, u^{(M)}) = 0, \qquad l = 1, .., v,$$

where $u = (u^1, ..., u^s)$ and $x = (x^1, ..., x^r)$ are dependent and independent variables respectively and M is the order of Δ. Also, $u^{(M)}$ means the u derivatives from order 0 to M and v is the number of system equations. Let

$$Y = \sum_{k=1}^{r} \eta^k(x, u)\partial_{x^k} + \sum_{j=1}^{s} \varphi_j(x, u)\partial_{u^j},$$

be a general vector filed in the space of system variables and $Y^{(M)}$ represent the prolongation of Y of order M. In order that Y be a symmetry group generator for Δ, the invariant criterion must be applied as follows, (Theorem 2.36 of [26])

$$Y^{(M)}[\Delta_l(x, u^{(M)})] = 0, \quad l = 1, ..., v, \quad \text{whenever} \quad \Delta_l(x, u^{(M)}) = 0. \tag{2}$$

A conservation law for a PDE is a divergence term that holds on solutions of the desired PDE. In the equations that are in the evolution form, one of the independent variables is recognized as the time t and the residual variables $x = (x^1, ..., x^r)$ show the spatial variables,

$$u_t = P(x, u^{(M)}),$$

and all the x-derivatives of u denotes by $u^{(M)}$. In this case, the conservation law becomes as follows

$$D_t \rho + \mathrm{Div} J = 0, \tag{3}$$

and D_t is the total derivative related to t, J is the flux and $\mathrm{Div}J$ is the total divergence of J with respect to $x^1, ..., x^r$ and ρ is density map. The map of density ρ, and its corresponding flux $J = (J_1, ..., J_r)$, are operations of u, t, x and the u derivatives related to t and x [26].

Definition 2.1. *Suppose $h = h(x, u^{(M)}(x))$ be a scaler differential function. We define the zeroth-Euler operator as follows*

$$\mathcal{L}_{u(x)} h = (\mathcal{L}_{u^1(x)} h, \mathcal{L}_{u^2(x)} h, ..., \mathcal{L}_{u^s(x)} h)$$

with

$$\mathcal{L}_{u^j(x)} h = \sum_{k_1=0}^{M_1^j} ... \sum_{k_r=0}^{M_r^j} (-D_{x^1})^{k_1} ... (-D_{x^r})^{k_r} \frac{\partial h}{\partial u^j_{k_1 x^1 ... k_r x^r}}, \ j = 1, ..., s, \tag{4}$$

where M_i^j are the degrees of h for the components u^j related to x^i, $i = 1, ..., r$ [30].

Since BBM equation has x as its spatial variable, let focus on the one dimensional case.

Definition 2.2. *Let $h = h(x, u^{(M)}(x))$ be a M-order map. We say that h is exact if a map like $H(x, u^{(M-1)}(x))$ is found so that $h = \mathrm{Div}H = D_x H$.*

In the following theorem, we demonstrate a test for the differential function exactness. For this test, we use the zeroth order Euler operator, which plays a fundamental role in finding conservation laws.

Theorem 2.3. *h is an exact differential function iff $\mathcal{L}_{u(x)}h = 0$, where 0 is the vector $(0,0,...,0)$ with s components [30].*

Definition 2.4. *For a differential function $h = h(x, u^{(M)}(x))$, the 1D homotopy operator is expressed as*

$$\mathcal{H}_{u(x)}h = \int_0^1 (\sum_{j=1}^{s} \mathcal{I}_{u^j(x)}h)[\lambda u]\frac{d\lambda}{\lambda}, \tag{5}$$

where $\mathcal{I}_{u^j(x)}h$ is considered by

$$\mathcal{I}_{u^j(x)}h = \sum_{k=1}^{M_1^j} \left(\sum_{i=0}^{k-1} u^j_{ix}(-D_x)^{k-(i+1)} \right) \frac{\partial h}{\partial u^j_{kx}}. \tag{6}$$

By applying this operator, Div^{-1} is constructed by the subsequent theorem.

Theorem 2.5. *Consider an exact differential mapping such as $h = h(x, u^{(M)}(x))$, that means, there is a function $H(x, u^{(M-1)}(x))$ so that $h = D_x H$. Then we have*

$$H = D_x^{-1}h = \mathcal{H}_{u(x)}h.$$

For more details and the proof of above theorems refer to [30].

3 Classical Symmetry of the BBM Equation

In this section, the classical symmetries of equation (1) from Lie's method are obtained. Suppose,

$$\begin{cases} \hat{t} &= t + \epsilon\tau(t,x,u) + o(\epsilon^2), \\ \hat{x} &= x + \epsilon\eta(t,x,u) + o(\epsilon^2), \\ \hat{u} &= u + \epsilon\varphi(t,x,u) + o(\epsilon^2), \end{cases}$$

be the Lie group of one-dimensional point transformations. The corresponding generator for this transformation symmetry group is generally as follows,

$$Y = \tau(t,x,u)\partial_t + \eta(t,x,u)\partial_x + \varphi(t,x,u)\partial_u. \tag{7}$$

The third order prolongation of this vector field can be calculated as follows,

$$Y^{(3)} = Y + \varphi^t \partial_{u_t} + \cdots + \varphi^{xxx} \partial_{u_{xxx}}, \tag{8}$$

where the coefficients are

$$\begin{cases} \varphi^t &= D_t(\varphi - \tau u_t - \eta u_x) + \eta u_{tx} + \tau u_{tt}, \\ \varphi^x &= D_x(\varphi - \tau u_t - \eta u_x) + \eta u_{xx} + \tau u_{tx}, \\ &\vdots \\ \varphi^{xxx} &= D_x^3(\varphi - \tau u_t - \eta u_x) + \tau u_{txxx} + \eta u_{xxxx}, \end{cases} \tag{9}$$

(For more details, refer to [26].) By using the invariance criterion (2) we obtain,

$$Y^{(3)}(u_t + p u_x + q u u_x + r u^2 u_x + s u_{xxx}) = 0, \tag{10}$$

whenever

$$u_t + p u_x + q u u_x + r u^2 u_x + s u_{xxx} = 0.$$

By substituting (8) along with coefficients (9) in relation (10) we have,

$$\varphi_t - u_x \eta_t + 3 s u_x^2 u_{xx} \tau_{uu} q u + \cdots + 2\varphi u u_x = 0.$$

After collecting the similar terms and setting their coefficients equal to zero, the system of determining equations is deduced as

$$\begin{array}{lllll} \varphi_x = 0, & \varphi_t = 0, & \varphi_u = \frac{4\varphi r}{q+2ru}, & \tau_x = 0, & \tau_u = 0, \\ \tau_t = \frac{-6\varphi r}{q+2ru}, & \eta_t = \frac{\varphi(q^2-4rp)}{q+2ru}, & \eta_x = \frac{-2\varphi r}{q+2ru}, & \eta_u = 0. & \end{array}$$

By solving this system, we have:

Theorem 3.1. *The Lie group of classical symmetry of the BBM equation has a Lie algebra which is generated by (7) with the coefficients,*

$$\eta = c_1(q^2 t - 2rx - 4prt) + c_3, \quad \tau = -6c_1 r t + c_2, \quad \varphi = c_1(q + 2ru),$$

where c_1, c_2 and c_3 are real arbitrary constants.

Corollary 3.2. *The generators of the symmetry Lie algebra of (1) are*

$$Y_1 = \partial_t, \quad Y_2 = \partial_x, \quad Y_3 = (q^2 t - 4prt - 2rx)\partial_x - 6rt\partial_t + (q + 2ru)\partial_u.$$

The one-parameter group $H_i(\varepsilon)$ can be obtained by exponentiating of the generator Y_i. $(i = 1, \cdots, 3)$

$$\begin{aligned}
&H_1(\varepsilon) : (t, x, u) \longrightarrow (t + \varepsilon, x, u),\\
&H_2(\varepsilon) : (t, x, u) \longrightarrow (t, x + \varepsilon, u),\\
&H_3(\varepsilon) : (t, x, u) \longrightarrow (te^{-6r\varepsilon}, \frac{e^{-2r\varepsilon}}{4r}(q^2 t - 4rpt + 4rx) +\\
&\qquad\qquad\qquad e^{-6r\varepsilon} t(p - \frac{q^2}{4r}), \frac{1}{2r}(e^{2r\varepsilon}(q + 2ru) - q)).
\end{aligned}$$

Corollary 3.3. *If $u = f(t, x)$ is a solution of the Eq. (1), then the following functions will also be solutions of the equation,*

$$\begin{aligned}
&H_1(\varepsilon).f(t, x) = f(t - \varepsilon, x),\\
&H_2(\varepsilon).f(t, x) = f(t, x - \varepsilon),\\
&H_3(\varepsilon).f(t, x) = \frac{q}{2r}(e^{2r\varepsilon} - 1) +\\
&\qquad\qquad f\left(e^{6r\varepsilon} t, e^{2r\varepsilon}(x + t(e^{4r\varepsilon} - 1)(p - \frac{q^2}{4r}))\right) e^{2r\varepsilon}.
\end{aligned}$$

Also, if functions H_1, H_2 and H_3 are combined arbitrarily, the result will be a solution to the equation.

4 Reductions and Optimal System of the BBM Equation

In this section, by using the group of symmetry for BBM equation, we compute the optimal system of one-dimensional subalgebras. It is clear that an infinite number of infinitesimal generators exist for the group G of symmetries, because any linear combination of infinitesimal generators is itself an infinitesimal generator. So an important tip is to specify, which generators give different types of solutions. Thus, we need to determine those invariant solutions, such that no transformations in the symmetry group may convert them to each other. It is the principle of the subalgebras optimal system concept. On the other hand, the classification of one-dimensional subalgebras is equivalent to classifying and identifying the adjoint representation orbits [27]. The subalgebras optimal system arises from the selection of a representative from each equivalence class. The classification of orbits can be handled by taking any member in Lie algebra and then simplifying it as much as possible by influencing different adjoint transformations on it [28, 27]. For each $Y_i,\ i=1,...,3$, the adjoint representations are defined as follows

$$\mathrm{Ad}(\exp(s.Y_i).Y_j) = Y_j - s.[Y_i, Y_j] + \frac{s^2}{2}.[Y_i, [Y_i, Y_j]] - \cdots, \tag{11}$$

where s is a parameter and $[Y_i, Y_j]$ is the Lie algebra commutator for $i, j = 1, \cdots, 3$ [27]. We can compute all of the adjoint representations of the Lie symmetry group for the BBM equation by (11). They are shown in Table 1. The (i, j) entry express $\mathrm{Ad}(\exp(s.Y_i).Y_j)$.

Table 1: Adjoint action of the generators Y_i.

Ad	Y_1	Y_2	Y_3
Y_1	Y_1	Y_2	$sBY_2 - 6srY_1 + Y_3$
Y_2	Y_1	Y_2	$-2srY_2 + Y_3$
Y_3	$\frac{e^{2sr}-e^{6sr}}{4r}BY_2 + e^{6sr}Y_1$	$e^{2sr}Y_2$	Y_3

where $B = (q^2 - 4rp)$. Now the following theorem can be expressed.

Theorem 4.1. *The one-dimensional optimal system for symmetry Lie algebra of the equation (1) is produced by the following generators*

$$1)\ Y_3, \qquad 2)\ Y_1, \qquad 3)\ Y_2.$$

Proof. Suppose that $F_i^{\varepsilon} : \mathfrak{g} \to \mathfrak{g}$ be the adjoint transformation $Y \mapsto \mathrm{Ad}(\exp(\varepsilon.Y_i).Y)$. Considering the basis $Y_i,\ i = 1, ..., 3$, the matrices corresponding to F_i^{ε} are obtained as follows,

$$M_1^{\varepsilon} = \begin{bmatrix} 1 & 0 & 6r\varepsilon \\ 0 & 1 & -\varepsilon B \\ 0 & 0 & 1 \end{bmatrix}, M_2^{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2r\varepsilon \\ 0 & 0 & 1 \end{bmatrix}, M_3^{\varepsilon} = \begin{bmatrix} e^{-6r\varepsilon} & 0 & 0 \\ \frac{e^{-2\varepsilon r}-e^{-6\varepsilon r}}{4r}B & e^{-2r\varepsilon} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If $Y=\sum_{i=1}^{3} a_iY_i$ be an element of $\mathfrak{g}$ in the general form, then

$$\begin{aligned} F_3^{\varepsilon_3}\circ F_2^{\varepsilon_2}\circ F_1^{\varepsilon_1}: Y \mapsto {} & (e^{-6r\varepsilon}a_1+6r\varepsilon a_3)Y_1+(\cdots \\ & +(\cdots+2e^{-2r\varepsilon}r\varepsilon)a_3)Y_2+a_3Y_3. \end{aligned}$$

Now, by influencing the appropriate adjoint representation $F_i^{\varepsilon_i}$ on Y, we simplify Y as follows.

- If $a_3\neq 0$, then we can make the coefficients of Y_2 and Y_1 zero by setting $\varepsilon_2=\frac{-a_2}{2ra_3}$ and $\varepsilon_1=\frac{-a_1}{6r}$ in $F_2^{\varepsilon_2}$ and $F_1^{\varepsilon_1}$ respectively. With scaling of Y if needed, it can be assumed that $a_3=1$. Therefore, Y is simplified to the case (1).
- If $a_3=0$ and $a_1\neq 0$, then we can make the coefficients of Y_2 zero by setting $\varepsilon_3=\frac{-1}{4r}ln(\frac{4a_1pr-a_1q^2-4a_2r}{a_1(4pr-q^2)})$ in $F_3^{\varepsilon_3}$. With scaling of Y if needed, it can be assumed that $a_1=1$. Therefore, Y is simplified to the case (2).
- If $a_1=a_3=0$, then with scaling of Y if needed, it can be assumed that $a_2=1$. So Y is simplified to the case (3).

In this way, there is no other case to check and the proof is complete. $\square$

Corollary 4.2. *The generators of the optimal system for the one-parameter subalgebras of the symmetry algebra of two non-linearity terms Benjamin-Bona-Mahoney equation are equal to the generators of the symmetry algebra of this equation.*

Equation (1) has (t,x,u) as coordinates. For reducing this equation, we need to introduce new coordinates (z,f), based on independent differential invariants. Then, the reduced equations are obtained by replacing new coordinates and using the chain rule. Consider the infinitesimal generator $Y_1=\partial_t$. For obtaining the independent invariants I, we must solve $(\partial_t)I=0$. To solve this equation, the corresponding characteristic ordinary differential equation can be solved

$$\frac{dt}{1}=\frac{dx}{0}=\frac{du}{0}.$$

By solving this system, $z = x$ and $f = u$ are obtained as the functionally independent invariants. The derivatives of u with respect to t and x can be obtained using the chain rule in terms of the new variables z and f. By replacing the new variables and their derivatives in equation (1), the reduced equation is obtained as

$$pf' + qff' + rf^2f' + sf''' = 0.$$

where the prime in the above and all reduced equations denotes differentiation with respect to z. For solving this equation we differentiate it twice with respect to z. By placing the same values of the equation itself in the resulting expression and simplifying it, the following first-order equation is obtained

$$(f')^2 = -\frac{r}{6s}f^4 - \frac{q}{3s}f^3 - \frac{p}{s}f^2 + c_1 f + c_2.$$

By solving this ordinary differential equation, we get three sets of solutions

$$u(t,x) = \frac{-6p}{q + i\sqrt{q^2 - 6pr}\,\sinh(\sqrt{\frac{p}{s}}(x - \frac{pt}{s}) + N)},$$

$$u(t,x) = \frac{-6pq\ \mathrm{sech}^2(\sqrt{\frac{p}{4s}}(x - \frac{pt}{s}))}{2q^2 - 3pr(1 - \tanh(\sqrt{\frac{p}{4s}}(x - \frac{pt}{s})))^2},$$

$$u(t,x) = \frac{-6p\ \mathrm{sech}(\sqrt{\frac{p}{s}}(x - \frac{pt}{s}))}{\sqrt{q^2 - 6pr} + q\ \mathrm{sech}(\sqrt{\frac{p}{s}}(x - \frac{pt}{s}))},$$

where N is a constant value. In a similar way, reduced equations can be obtained for all the generators in the optimal system.

The reduced form equation for case (1): $Y_3 = (q^2t - 4prt - 2rx)\partial_x - 6rt\partial_t + (q+2ru)\partial_u$. Functionally independent invariants are $z = \frac{tq^2+4r(x-tp)}{4r\sqrt[3]{t}}$ and $f = \frac{\sqrt[3]{t}(q+2ru)}{2r}$ and by using these new variables the corresponding reduced equation is obtained as follows.

$$f + f'n - 3rf^2f' - 3sf''' = 0.$$

The reduced equation for case (2): $Y_2 = \partial_x$. Functionally independent invariants are $z = t$ and $f = u$ and by using these new variables the corresponding reduced equation is obtained as

$$f' = 0.$$

Solving this reduced equation we get the trivial solution $f = c$, that c is an optional scaler. In this case, trivial group invariant solution $u(t, x) = c$ is deduced..

It is also possible to determine other group invariant solutions for the equation. For example the functionally independent invariants for $Y2 + aY_1 = \partial_x + a\partial_t$ are $z = -ax + t$ and $f = u$ and by using these new variables the corresponding reduced equation is obtained as

$$(1 - ap)f' - aqff' - raf^2 f' - sa^3 f''' = 0.$$

The similarity solutions can be obtained via solving these reduced equations, Some implicit form for solutions of the above equations can be found in [10]

5 Conservation Laws for the BBM Equation

In this section, the conservation laws of BBM equation are computed by applying the method of scaling. To calculate the conservation laws from this method, the first step is to deduce the density ρ from the scaling or dilation symmetry. Then calculate the corresponding flux J by applying the operator of homotopy. For constructing the incipient density, we consider a linear compound of invariant component under dilation symmetry with unspecified coefficients. Now, we get the total derivative of the initial density related to time and in the resulting expression we replace wherever there was a derivative with respect to time from the BBM equation. Therefore, there is no derivative with respect to time in the obtained expression. Also this expression must be exact by relation (3). So, unspecified coefficients will be determined as the solutions of the equations, that is constructed by using the theorem of exactness 2.3,

$$\mathcal{L}_{u(x)}(D_t \rho) = 0.$$

After solving this system and specifying the coefficients, the real density is obtained. Now, by using the homotopy operator, that is inverse of the divergence, the flux can also be obtained.

$$J = -\mathrm{Div}^{-1}(D_t\rho).$$

5.1 Dilation symmetry of the BBM equation

Some algorithms exist for calculating scaling symmetries [6, 26]. In this work, we apply the notion of weights of variables to compute the dilation symmetries for the BBM equation [16, 30].

Definition 5.1. *The weight of variable x in an equation is a number such as p that does not change the equation if we replace the term x with $\lambda^p x$. We show the weight for the dilation symmetry $x \to \lambda^{-p}x$ with $\mathrm{w}(x) = -p$. Also, the weight for derivatives D_x is determined according to the weight of x. Actually, $\mathrm{w}(D_x) = p$ whenever $\mathrm{w}(x) = -p$.*

Definition 5.2. *The rank of a monomial is defined as the summation of its variables weight. Also, we say that a differential function h is uniform in rank if all its monomials have the same rank.*

Since u_x, uu_x and u_{xxx} make the weight equations incompatible, the rank uniformity of BBM equation is violated. To solve this problem, we assume that parameters p, q, r and s have weight. So the weight-balance equations for (1) are

$$\begin{aligned}
\mathrm{w}(u) + \mathrm{w}(D_t) = \quad & \mathrm{w}(p) + \mathrm{w}(u) + \mathrm{w}(D_x) = \\
& \mathrm{w}(q) + 2\mathrm{w}(u) + \mathrm{w}(D_x) = \\
& \mathrm{w}(r) + 3\mathrm{w}(u) + \mathrm{w}(D_x) = \\
& \mathrm{w}(s) + \mathrm{w}(u) + 3\mathrm{w}(D_x).
\end{aligned}$$

For solving this system, we set $\mathrm{w}(D_x) = 1 = \mathrm{w}(u)$. Therefore, a solution for the weight-balance equations system is obtained as

$$\mathrm{w}(D_t) = 3, \qquad \mathrm{w}(p) = 2, \qquad \mathrm{w}(q) = 1, \qquad \mathrm{w}(r) = 0, \qquad \mathrm{w}(s) = 0.$$

As a result, the dilation symmetry for the BBM equation is

$$(t, x, u, p, q) \to (\lambda^{-3}t, \lambda^{-1}x, \lambda u, \lambda^2 p, \lambda q).$$

Since the conservation laws are established on the solutions of the equation, so the conservation laws must also have the property of rank uniformity. That is, both the density and the corresponding flux must be uniform in rank. Based on this fact, we try to find the real density by considering a preselected rank for the initial density and also considering the scaling symmetry (see [31] for more information).

5.2 Computing the real density

We describe how to construct the density of a conservation law in this subsection. For this purpose, we first consider a fixed rank for the initial density. For example, for equation (1), we start with rank 4. Consider the set $\mathcal{P}$ of all monomials that consist of dependent variables and their derivatives of rank 4. The method of constructing the mentioned set is that we consider the set

$$\mathcal{Q} = \{u^4, u^3, u^2, u, qu^3, pu^2, q^2u^2, qu^2, q^3u, q^2u, qu, pu\}$$

of dependent variables and their powers up to the rank 4 and then by taking total derivation of them, we raise the rank of its sentences to 4. So we have

$$\begin{aligned}&\mathcal{P} =\\ &\{u^4, u^2u_x, u_x^2, uu_{xx}, u_{xxx}, qu^3, pu^2, q^2u^2, quu_x, q^3u, q^2u_x, qu_{xx}, pu_x\}.\end{aligned} \tag{12}$$

To have a non-trivial density, all sentences that are total divergence should be removed from the list. Also, only one of the equivalent divergence sentences should be kept in the list and the rest should be deleted. In this case, for the convenience of calculations, we keep the sentence that has a lower order of derivation and remove the rest. By applying the Euler operator (4) to the members of the set (12), the following set is obtained.

$$\mathcal{L}_{u(x)}\mathcal{P} = \{4u^3, 0, -2u_{xx}, 2u_{xx}, 0, 3qu^2, 2pu, 2q^2u, 0, q^3, 0, 0, 0\}.$$

According to the Theorem 2.3, u^2u_x, u_{xxx}, quu_x, q^2u_x, qu_{xx} and pu_x are divergences and must be eliminated from the list. Also, u_x^2 and uu_{xx} are divergence-equivalent. Therefore, we eliminate uu_{xx}, which has a higher order. So, the new list after removing redundant terms is as follows

$$\mathcal{P} = \{u^4, u_x^2, qu^3, pu^2, q^2u^2\}.$$

Now, by combining the members of $\mathcal{P}$ with unspecified coefficients c_i, the incipient density in rank 4 correspond to the BBM equation is obtained as

$$\rho = c_1 u^4 + c_2 u_x^2 + c_3 q u^3 + c_4 p u^2 + c_5 q^2 u^2. \tag{13}$$

In order to calculate the real density, the coefficients of ρ must be specified. For this purpose, we first take the total derivation of ρ with respect to the time variable.

$$D_t \rho = (4c_1 u^3 + 3c_3 q u^2 + 2c_4 p u + 2c_5 q^2 u) u_t + 2c_2 u_x u_{xt}.$$

Then, we substitute u_t and u_{xt} with their equal values from equation (1). If we denote $-D_t\rho$ by E, we have

$$E = -(4c_1 u^3 + 3c_3 q u^2 + 2c_4 p u + 2c_5 q^2 u)(-p u_x - q u u_x - r u^2 u_x - \\ s u_{xxx}) - 2c_2 u_x((-q u_x - 2r u u_x) u_x + (-p - q u - r u^2) u_{xx} - s u_{xxxx}).$$

Based on the definition of conservation law, i.e. relation (3), the above statement must be exact. As a result, the effect of the zeroth-Euler operator on it should be vanish. So

$$\mathcal{L}_{u(x)} E = -6c_2 q u_x u_{xx} - 72 c_1 s u u_x u_{xx} - 18 c_3 s q u_x u_{xx} - 12 c_2 r u u_x u_{xx} - \\ 24 c_1 s u_x^3 - 4 c_2 r u_x^3 = 0.$$

A system of linear equations is produced by setting the similar terms equal to zero. The coefficients can be determined by solving these equations

$$c_1 = 1, \;\; c_2 = -\tfrac{6s}{r}, \;\; c_3 = \tfrac{2}{r}, \;\; c_4 = c_5 = 0. \tag{14}$$

Therefore, the real density is obtained by placing the above coefficients in the initial density (13).

$$\rho = u^4 + \frac{2q}{r} u^3 - \frac{6s}{r} u_x^2.$$

5.3 Constructing the flux of conservation laws

In this subsection, we explain how to calculate the flux corresponding to the density of a conservation law. We explain this work for the obtained density in the previous subsection. The key point of this problem is the fact that $J = \mathrm{Div}^{-1}(E)$. Therefore, J can be obtained by using theorem 2.5. After placing (14) in E we have,

$$\begin{aligned}
E &= \tfrac{2}{r}(2rpu^3u_x + 5rqu^4u_x + 2r^2u^5u_x + 2rsu^3u_{xxx} + 3pqu^2u_x + \\
&\quad 3q^2u^3u_x - 6squ_x^3 + 3qsu^2u_{xxx} - 12rsuu_x^3 - 6spu_xu_{xx} - \\
&\quad 6squu_xu_{xx} - 6sru^2u_xu_{xx} - 6s^2u_xu_{xxxx}).
\end{aligned}$$

By using relation (6), the integrand $\mathcal{I}_{u(x)}E$ can be calculated as

$$\begin{aligned}
\mathcal{I}_{u(x)}E &= \tfrac{2}{r}(2rpu^4 + 5rqu^5 + 2r^2u^6 + 3pqu^3 + 3q^2u^4 - 18squu_x^2 - \\
&\quad 24rsuu_x^2 + 9squ^2u_{xx} + 8rsu^3u_{xx} - 6spu_x^2 - 12s^2u_xu_{xxx} + \\
&\quad 6s^2u_{xx}^2).
\end{aligned}$$

The formula for 1D homotopy operator (5) yields,

$$\begin{aligned}
\mathcal{H}_{u(x)}E &= \textstyle\int_0^1 (\mathcal{I}_{u(x)}E)[\lambda u]\frac{d\lambda}{\lambda} = \\
&\quad \textstyle\int_0^1 \frac{2\lambda}{r}((2rp + 3q^2)\lambda^2u^4 + 5rq\lambda^3u^5 + 2r^2\lambda^4u^6 + 3pq\lambda u^3 - \\
&\quad 18sq\lambda uu_x^2 - 24rs\lambda^2uu_x^2 + (9sq\lambda + 8rs\lambda^2u)u^2u_{xx} - 6spu_x^2 - \\
&\quad 12s^2u_xu_{xxx} + 6s^2u_{xx}^2)d\lambda = \tfrac{1}{6r}(4r^2u^6 + 12rqu^5 + 6rpu^4 + \\
&\quad 9q^2u^4 - 72sru^2u_x^2 + 24sru^3u_{xx} - 72squu_x^2 + 12pqu^3 + \\
&\quad 36squ^2u_{xx} - 36spu_x^2 - 72s^2u_xu_{xxx} + 36s^2u_{xx}^2).
\end{aligned}$$

Thus, the density and its flux for BBM equation conservation law of rank 4 is obtain as

$$\begin{aligned}
\rho^{(1)} &= u^4 + \tfrac{2q}{r}u^3 - \tfrac{6s}{r}u_x^2, \\
J^{(1)} &= \tfrac{2}{3}ru^6 + 2qu^5 + pu^4 + \tfrac{3q^2}{2r}u^4 + \tfrac{2pq}{r}u^3 - \tfrac{6ps}{r}u_x^2 - \tfrac{12qs}{r}uu_x^2 - \\
&\quad 12su^2u_x^2 + \tfrac{6qs}{r}u^2u_{xx} + 4su^3u_{xx} + \tfrac{6s^2}{r}u_{xx}^2 - \tfrac{12s^2}{r}u_xu_{xxx}.
\end{aligned}$$

In the following, additional conservation laws for the BBM equation of rank 1 and 2 are presented,

$$
\begin{aligned}
\rho^{(2)} &= u, \\
J^{(2)} &= pu + \tfrac{1}{2}qu^2 + \tfrac{r}{3}u^3 + su_{xx}, \\
\rho^{(3)} &= u^2, \\
J^{(3)} &= pu^2 + \tfrac{2}{3}qu^3 + \tfrac{1}{2}ru^4 - su_x^2 + 2su_xu_{xx}.
\end{aligned}
$$

6 Remark and Discussion

Lie's symmetry method is a powerful method to analyze differential equations. With the help of this method, the equations reduced and the exact solutions of the equations obtained and they can be classified. We have used this method to obtain the symmetry group of the BBM equation, and in addition, using the symmetry group, we have obtained the group invariant solutions and classified them. It should be noted that all equations do not necessarily have a non-trivial symmetry group and it is not necessarily possible to perform the above tasks on every equation. Another application of symmetries is to find the conservation laws of an equation, which has a basic application in differential equation analysis. Also, conservation laws are very important in physics and engineering sciences. There are different methods to find these laws, in this study, we have used the scaling method and using the concept of weight of variables. It should be noted that the obtained conservation laws, besides being new, are non-equivalent and non-trivial. The reason for this is that the effect of the Euler operator on them is non-zero and also the effect of the Euler operator on the difference of their two primitives is also non-zero.

The suggestion that can be made for future works is to obtain the non-classical symmetries, potential symmetries, and generalized symmetries of this equation, as well as obtain new conservation laws from other methods such as the multi-parameter method for this equation.

7 Conclusions

In the present work, we studied the Benjamin-Bona-Mahoney equation that is an evolutionary third order equation which appears for modeling long surface gravity waves. Using the classical Lie method, the Lie group of point symmetries of the equation was obtained. Also, the optimal system of one-parameter subalgebras of the algebra related to the Lie group was built. In the following, we obtained the reduced equations corresponding to the optimal system. The conservation laws, which are important issues in physics and mathematics, have been obtained for BBM equation. These conservation laws are new and are derived from the scaling method. In the scaling method, we utilize the notion of the variable weights to obtain the scaling (dilation) symmetry, and then we make the flux and density of the conservation law by using the Euler and homotopy operator respectively. In the present work, we have obtained the conservation laws of rank 1, 2 and 4.

References

[1] A. Akbulut, M. Kaplan, D. Kumar and F. Tascan, The analysis of conservation laws, symmetries and solitary wave solutions of Burgers–Fisher equation, *International Journal of Modern Physics B*, 35 (2021), no. 22, 2150224.

[2] R. Bakhshandeh-Chamazkoti, M. Alipour, Lie symmetries reduction and spectral methods on the fractional two-dimensional heat equation, *Mathematics and Computers in Simulation*, 200 (2022), 97-107.

[3] R. Bakhshandeh-Chamazkoti, M. Alipour, Lie symmetry analysis for Korteweg-de Vries equation with power-law nonlinearity, *Mathematical Reports*, 22 (2020), no. 72, 163-176.

[4] T. B. Benjamin, J. L. Bona and J. J. Mahony, Model equations for long waves in nonlinear dispersive system, *Philos. Trans. R. Soc. Lond. Ser. A, Mathematical and Physical Sciences*, 272 (1972), no. 1220, 47-78.

[5] A. Biswas, 1-Soliton solution of Benjamin-Bona-Mahoney equation with dual-power law nonlinearity, *Communications in Nonlinear Science and Numerical Simulation*, 15 (2010), 2744-2746.

[6] G. W. Bluman, A. F. Cheviakov and S. C. Anco, *Applications of Symmetry Methods to Partial Differential Equations*, Appl. Math. Sciences, Vol. 168, Springer, New York, (2010).

[7] G. W. Bluman and J. D. Cole, The general similarity solutions of the heat equation, *Journal of Mathematics and Mechanics*, 18 (1969), 1025-1042.

[8] G. W. Bluman and and S. Kumei, *Symmetries and Differential Equations*, Springer, New York, (1989).

[9] J. L. Bona, W. G. Pritchard and L. R. Scott, Solitary wave interaction, *Physics of Fluids*, 23 (1980), no. 3, 438-441.

[10] C. Chen and Y. L. Jiang, Lie group analysis, exact solutions and new conservation laws for combined KdV-mKdV equation. *Differ. Equ. Dyn. Syst.* 28 (2020), 827-840.

[11] X. Deng, M. Zhao and X. Li, Travelling wave solutions for a nonlinear variant of the PHI-four equation, *Mathematical and Computer Modelling*, 49 (2009), no. 3-4, 617-622.

[12] P. G. Estevez, S, Kuru, J. Negro and L. M. Nieto, Travelling wave solutions of the generalized Benjamin-Bona-Mahony equation, *Chaos, Solitons and Fractals*, 40 (2009), no. 4, 2031-2040.

[13] W. Hereman, B. Deconinck and L. D. Poole, Continuous and discrete homotopy operators: a theoretical approach made concrete, *Math. Comput. Simul.,* 74 (2006), 352-360.

[14] M. Inc and A. Akgül, Classifications of soliton solutions of the generalized Benjamin-Bona-Mahony equation with power-law nonlinearity, *Journal of Advanced Physics*, 7 (2018), no. 1, 130-134.

[15] M. Jafari, S. Mahdion, A. Akgül, S. M. Eldin, New conservation laws of the Boussinesq and generalized Kadomtsev-Petviashvili equations via homotopy operator, *Results in Physics*, 47 (2023), 106369.

[16] M. Jafari, A. Zaeim and A. Tanhaeivash, Symmetry group analysis and conservation laws of the potential modified KdV equation using the scaling method, *International Journal of Geometric Methods in Modern Physics*, 19 (2022), no. 7, 2250098.

[17] S. Kuru, Compactons and kink-like solutions of BBM-like equations by means of factorization, *Chaos, Solitons and Fractals*, 42 (2009), no. 1, 626–633.

[18] S. Kuru, Traveling wave solutions of the BBM-like equations, *Journal of Physics A*, 42 (2009), no. 37, 375203.

[19] S. Lie, On integration of a class of linear partial differential equations by means of definite integrals, *Arch. Math.*, 6 (1881), 328-368, Translation by N. H. Inragimov.

[20] M. N. B. Mohamad, Exact solutions to the combined KdV and mKdV equation, *Mathematical Methods of Applied Sciences*, 15 (1992), 73-78.

[21] M. Nadjafikhah, Y. AryaNejad, N. Zandi, Conformal Einstein pp-wave spaces as quantum solutions, *Journal of Mathematical Extension*, 16 (2022), no. 12, 1-19.

[22] M. Nadjafikhah, M. Jafari, Computation of partially invariant solutions for the Einstein Walker manifolds' identifying equations, *Communications in Nonlinear Science and Numerical Simulation,* 18 (2013), no. 12, 3317-3324.

[23] M. Nadjafikhah, M. Jafari, Some general new Einstein Walker manifolds, *Advances in Mathematical Physics,* 2013 (2013), 591852.

[24] R. Naz, *Symmetry Solutions and Conservation Laws for Some Partial Differential Equations in Field Mechanics*, Ph.D. dissertation, University of the Witwatersrand, Johannesburg, (2008).

[25] R. Naz, F. M. Mahomed and D. P. Mason, Comparison of different approaches to conservation laws for some partial differential equations in fluid mechanics, *Appl. Math. Comput.*, 205 (2008), 212-230.

[26] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

[27] P. J. Olver, Euler operators and conservation laws of the BBM equation, *Mathematical Proceedings of the Cambridge Philosophical Society*, 85 (1979), 143-160.

[28] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[29] D. H. Peregrine, Calculations of the development of an undular bore, *Journal of Fluid Mechanics*, 25 (1966), no. 2, 321-330.

[30] D. Poole, *Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations Using Homotopy Operators*, Ph.D. dissertation, Colorado School of Mines, Golden, Colorado, (2009).

[31] D. Poole and W. Hereman, The homotopy operator method for symbolic integration by parts and inversion of divergences with applications, *Applicable Analysis*, 89 (2010), no. 4, 433-455.

[32] K. J. Wang and F. Shi, A new perspective on the exact solutions of the local fractional modified Benjamin-Bona-Mahony equation on Cantor sets, *Fractal and Fractional,* 7 (2023), no. 1, 72.

[33] Y. Zhang, Lie symmetry analysis and exact solutions of the Sawada-Kotera equation, *Turkish Journal of Mathematics*, 41 (2017), no. 1, 158-167.

Mehdi Jafari
Assistant Professor of Mathematics
Department of Mathematics
Payame Noor University (PNU), PO BOX 19395-4697
Tehran, Iran.
E-mail: m.jafarii@pnu.ac.ir

Amirhesam Zaeim
Associate Professor of Mathematics
Department of Mathematics
Payame Noor University (PNU), PO BOX 19395-4697

Tehran, Iran.
E-mail: zaeim@pnu.ac.ir

Mojdeh Gandom
Ph.D. Student
Department of Mathematics
Payame Noor University (PNU), PO BOX 19395-4697
Tehran, Iran.
E-mail: m.gandom@student.pnu.ac.ir