Journal of Mathematical Extension
Vol. 17, No. 10, (2023) (3)1-18
URL: https://doi.org/10.30495/JME.2023.2566
ISSN: 1735-8299
Original Research Paper

Further Results on Hilbert-Schmidt Numerical Radius Inequalities

M. Guesba
El Oued University

Abstract. In this paper, several inequalities involving the Hilbert-Schmidt numerical radius inequalities for 2×2 operator matrices operators are established. In particular, we obtain some generalizations and refinements of earlier inequalities. Some upper and lower bounds for the Hilbert-Schmidt numerical radius inequalities for 2×2 operator matrices operators is also given.

AMS Subject Classification: 47A05, 47A55, 47B15.
Keywords and Phrases: Numerical radius, Hilbert-Schmidt, Operator matrix, Inequality.

1 Introduction

Throughout this paper, $\mathcal{H}$ denotes a non trivial separable complex Hilbert space, with norm $\|\cdot\|$, induced by the inner product $\langle \cdot, \cdot \rangle$. The algebra of linear bounded operators acting on $\mathcal{H}$ will be denoted by $\mathcal{B}(\mathcal{H})$. Among the most interesting scalar quantities associated with $T \in \mathcal{B}(\mathcal{H})$ is the so called numerical radius, defined by

$$\omega(T) = \sup \{ |\langle Tx, x \rangle| : x \in \mathcal{H}, \|x\| = 1 \}.$$

Received: November 2022; Accepted: January 2024

This quantity has received a considerable attention in the literature due to its importance in operator theory and matrix analysis. However, due to the difficulty of computing the exact value of $\omega(T)$, a considerable attention has been put towards finding upper and lower bounds of this quantity, see [4, 9, 11, 12], and the references therein for example.

Since the operator norm $\|T\|$ of an operator T is easier to compute than $\omega(T)$, the following inequality [11]

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, T \in \mathcal{B}(\mathcal{H}), \tag{1}$$

had been useful in the literature because of the easier lower and upper bounds of $\omega(\cdot)$.

Attempts to sharpen (1) have been made by numerous authors, as one can find in [1, 10] to mention a few.

Another formula for $\omega(T)$ in terms of the operator norm $\|\cdot\|$ is the following useful identity that has been used extensively in the literature

$$\omega(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re\left(e^{i\theta}T\right) \right\|. \tag{2}$$

Motivated by this formula, the so called Hilbert-Schmidt numerical radius has been recently defined in [2] as follows

$$\omega_2(T) = \sup_{\theta \in \mathbb{R}} \left\| \Re\left(e^{i\theta}T\right) \right\|_2,$$

for any operator $T \in C_2(\mathcal{H})$; the Hilbert-Schmidt class. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is said to belong to the Hilbert–Schmidt class $C_2(\mathcal{H})$ if

$$\sum_{i,j=1}^{\infty} |\langle Te_i, e_j \rangle|^2 = \sum_{i=1}^{\infty} \|Te_i\|^2 < \infty,$$

for any orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\mathcal{H}$.

Further, we recall the notation $\|T\|_2$ for $T \in C_2(\mathcal{H})$ as the Hilbert-Schmidt norm of T, defined via

$$\|T\|_2 = \sum_{i=1}^{\infty} \|Te_i\|^2 = (tr(T^*T))^{\frac{1}{2}} < \infty.$$

Note that $\|T\|_2$ is unitarily invariant, in the sense that

$$\|UTV\|_2 = \|T\|_2 \,,$$

for every $T \in C_2(\mathcal{H})$ and unitary operators $U, V \in \mathcal{B}(\mathcal{H})$. We refer the reader to [2, 13] for further details.

In addition, for every $T \in C_2(\mathcal{H})$, the Hilbert-Schmidt numerical radius $\omega_2(\cdot)$ satisfies

$$\frac{1}{\sqrt{2}} \|T\|_2 \leq \omega_2(T) \leq \|T\|_2 \,, \tag{3}$$

similar to (1).

Moreover, if $T \in C_2(\mathcal{H})$ is self-adjoint (or normal), then

$$\omega_2(T) = \|T\|_2 \,, \tag{4}$$

and

$$\omega_2(T) = \frac{1}{\sqrt{2}} \|T\|_2 \,, \tag{5}$$

if and only if $T^2 = 0$. For proofs and more facts, we refer the reader to [2, 3].

Two important properties of $\omega_2(\cdot)$ are that it is weakly unitarily invariant in the sense

$$\omega_2(UTU^*) = \omega_2(T) \,, \tag{6}$$

for every $T, U \in \mathcal{B}(\mathcal{H})$ such that U is unitary operator, and that it is self-adjoint in the sense

$$\omega_2(T^*) = \omega_2(T) \,,$$

for every $T \in C_2(\mathcal{H})$.

A considerable attention in the literature has been directed to the study of numerical radius inequalities for 2×2 block operators [5, 7].

Extending these block results, inequalities of ω_2 for block operators have been studied recently, as one can see in [2, 3, 6, 8].
For example, in [3] the following inequalities were shown:

$$\frac{\max\{\omega_2(A+B), \omega_2(A-B)\}}{\sqrt{2}} \leq \omega_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$$

$$\leq \frac{\omega_2(A+B)+\omega_2(A-B)}{\sqrt{2}}, \tag{7}$$

for $A, B \in C_2(\mathcal{H})$.

Our aim in this paper is to give several Hilbert-Schmidt numerical radius inequalities of 2×2 block operators. These inequalities improve and extend some earlier related inequalities. More precisely, we derive new upper and lower bounds for the Hilbert-Schmidt numerical radius inequalities of 2×2 blocks, and some applications are obtained.

The following lemmas will be needed in our analysis.

Lemma 1.1. *([6]) Let $A, X \in C_2(\mathcal{H})$. Then*

$$\omega_2(AXA^*) \leq \|A\|_2^2\,\omega_2(X).$$

Lemma 1.2. *[3] Let $A, B \in C_2(\mathcal{H})$. Then*

(a) $\omega_2\left(\begin{bmatrix} 0 & A \\ e^{i\theta}B & 0 \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right)$ *for every $\theta \in \mathbb{R}$.*

(b) $\omega_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix}\right)$.

(c) $\omega_2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \sqrt{2}\omega_2(A)$.

(d) $\omega_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) \leq \sqrt{\omega_2^2(A)+\omega_2^2(B)}$. *In particular, if A, B are self-adjoint, then*

$$\omega_2\left(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right) = \sqrt{\omega_2^2(A)+\omega_2^2(B)}.$$

Lemma 1.3 ([3], Theorem 1 (b))**.** *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}\right) = \sqrt{\omega_2^2(A)+\frac{1}{2}\|B\|_2^2}.$$

Lemma 1.4. *[3] If $A, B \in C_2(\mathcal{H})$ are positive operators, then*

$$\omega_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) = \frac{\|A+B\|_2}{\sqrt{2}}.$$

Lemma 1.5. *Let $A, B, C, D \in C_2(\mathcal{H})$. Then*

(i) $\omega_2\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right) \leq \omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

(ii) $\omega_2\left(\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}\right) \leq \omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)$.

Proof. (i) Clearly we have

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}. \tag{8}$$

Let $U = \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}$. Then, U is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$ and

$$\begin{aligned} \omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) &= \omega_2\left(U^*\begin{bmatrix} A & B \\ C & D \end{bmatrix}U\right) \\ &= \omega_2\left(\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}\right). \end{aligned}$$

So, by (8) and the triangle inequality we get the desired result.
(ii) In view of Lemma 1.2 (a,b) and the assertion (i), we deduce that

$$\omega_2\left(\begin{bmatrix} A & -B \\ -C & D \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} -A & B \\ C & -D \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right).$$

Moreover, by using the fact that

$$\begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A & B \\ C & D \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -A & B \\ C & -D \end{bmatrix},$$

and the subadditivity property of $\omega_2(\cdot)$, we get the required result.
□

Lemma 1.6. *Let $A \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) = \omega_2\left(\begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix}\right).$$

Proof. Let $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$. Then, U is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$ and we have

$$\begin{aligned}
\omega_2\left(\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}\right) &= \omega_2\left(U^*\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}U\right) \\
&= \frac{1}{2}\omega_2\left(\begin{bmatrix} -2A & 0 \\ 0 & 2A \end{bmatrix}\right) \\
&= \omega_2\left(\begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix}\right).
\end{aligned}$$

□

Lemma 1.7. *Let $X \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} -X & -X \\ X & X \end{bmatrix}\right) = \sqrt{2}\,\|X\|_2\,.$$

Proof. Notice that for $X \in C_2(\mathcal{H})$, we have

$$\begin{bmatrix} -X & -X \\ X & X \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Let U be the unitary operator as in proof of Lemma 1.6. and by (5), we have

$$\begin{aligned}
\omega_2\left(\begin{bmatrix} -X & -X \\ X & X \end{bmatrix}\right) &= \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} -X & -X \\ X & X \end{bmatrix}\right\|_2 \\
&= \frac{1}{\sqrt{2}}\left\|U^*\begin{bmatrix} -X & -X \\ X & X \end{bmatrix}U\right\|_2 \\
&= \frac{1}{2\sqrt{2}}\left\|\begin{pmatrix} 0 & -4X \\ 0 & 0 \end{pmatrix}\right\|_2 \\
&= \sqrt{2}\,\|X\|_2\,.
\end{aligned}$$

□

2 Main Results

In this section, we present our results.

The following is the extension of Lemma 1.1 to 2×2 blocks.

Theorem 2.1. *Let $A, B, C, D \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} 0 & ACB^* \\ BDA^* & 0 \end{bmatrix}\right) \le 2\|A\|_2 \|B\|_2 \omega_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right).$$

Proof. Let $T = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $S = \begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}$. Then, we have

$$TST^* = \begin{bmatrix} 0 & ACB^* \\ BDA^* & 0 \end{bmatrix}.$$

Noting that

$$\|T\|_2^2 = \left\|\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}\right\|_2^2 = \|A\|_2^2 + \|B\|_2^2,$$

and using Lemma 1.1, we get

$$\begin{aligned} \omega_2\left(\begin{bmatrix} 0 & ACB^* \\ BDA^* & 0 \end{bmatrix}\right) &= \omega_2(TST^*) \\ &\le \|T\|_2^2 \omega_2(S) \\ &\le \left(\|A\|_2^2 + \|B\|_2^2\right) \omega_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right). \end{aligned}$$

Thus,

$$\omega_2\left(\begin{bmatrix} 0 & ACB^* \\ BDA^* & 0 \end{bmatrix}\right) \le \left(\|A\|_2^2 + \|B\|_2^2\right) \omega_2\left(\begin{bmatrix} 0 & C \\ D & 0 \end{bmatrix}\right). \tag{9}$$

In that case $A = 0$, the result is clear.

If $A \ne 0$, then by replacing A and B by tA and $\frac{1}{t}B$, respectively in (9), where $t = \sqrt{\frac{\|B\|_2}{\|A\|_2}}$, we get the desired inequality $\square$

Now, using (7), we have the following result.

Corollary 2.2. *Let $A, B, C, D \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\left[\begin{array}{cc} 0 & ACB^* \\ BDA^* & 0 \end{array}\right]\right) \leq \sqrt{2}\, \|A\|_2 \, \|B\|_2 \left(\omega_2(C+D) + \omega_2(C-D)\right).$$

In [6], commutator inequalities were shown for operators of the form $AXB^* + BX^*A^*$. In the the following, we present commutator inequality for ω_2 for operators of the form $AXB^* + BYA^*$.

Theorem 2.3. *Let $A, B, X, Y \in C_2(\mathcal{H})$. Then*

$$\omega_2(AXB^* \pm BYA^*) \leq 2\, \|A\|_2 \, \|B\|_2 \, \omega_2\left(\left[\begin{array}{cc} 0 & X \\ Y & 0 \end{array}\right]\right). \tag{10}$$

Proof. Let $T = \left[\begin{array}{cc} A & B \\ 0 & 0 \end{array}\right]$ and $S = \left[\begin{array}{cc} 0 & X \\ Y & 0 \end{array}\right]$. Then, we have

$$TST^* = \left[\begin{array}{cc} AXB^* + BYA^* & 0 \\ 0 & 0 \end{array}\right].$$

By Lemma 1.3, it can be observed that

$$\begin{aligned} \omega_2(AXB^* + BYA^*) &= \omega_2\left(\left[\begin{array}{cc} AXB^* + BYA^* & 0 \\ 0 & 0 \end{array}\right]\right) \\ &= \omega_2(TST^*) \\ &\leq \|T\|_2^2 \, \omega_2(S) \\ &= \left(\|A\|_2^2 + \|B\|_2^2\right) \omega_2\left(\left[\begin{array}{cc} 0 & X \\ Y & 0 \end{array}\right]\right). \end{aligned}$$

Consequently,

$$\omega_2(AXB^* + BYA^*) \leq \left(\|A\|_2^2 + \|B\|_2^2\right) \omega_2\left(\left[\begin{array}{cc} 0 & X \\ Y & 0 \end{array}\right]\right). \tag{11}$$

For $A = 0$ the result is clear.

If $A \neq 0$ then, by replacing A and B by tA and $\frac{1}{t}B$, respectively in (11), where $t = \sqrt{\frac{\|B\|_2}{\|A\|_2}}$, we get

$$\omega_2 (AXB^* + BYA^*) \leq 2 \|A\|_2 \|B\|_2 \, \omega_2 \left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right). \tag{12}$$

Replacing A by iA in the inequality (12), we have

$$\omega_2 (AXB^* - BYA^*) \leq 2 \|A\|_2 \|B\|_2 \, \omega_2 \left(\begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} \right). \tag{13}$$

From (12) and (13) we get the required inequality. This completes the proof. □

As an immediate consequence of Theorem 2.3, we have the following inequality, stated without using blocks and obtained earlier in [6, Remark 2.5].

Corollary 2.4. *Let $A, B, X \in C_2(\mathcal{H})$. Then,*

$$\omega_2 (AXB^* \pm BXA^*) \leq 2\sqrt{2} \|A\|_2 \|B\|_2 \, \omega_2 (X) .$$

Proof. Letting $X = Y$ in Theorem 2.3, we get

$$\omega_2 (AXB^* \pm BXA^*) \leq 2 \|A\|_2 \|B\|_2 \, \omega_2 \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right).$$

Using Lemma 1.2 (c), we have

$$\omega_2 \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) = \sqrt{2} \omega_2 (X) .$$

Therefore, we obtain

$$\omega_2 (AXB^* \pm BXA^*) \leq 2\sqrt{2} \|A\|_2 \|B\|_2 \, \omega_2 (X) ,$$

as required. □

The following result is another consequence of Theorem 2.3, obtained earlier in [6, Remark 2.4].

Corollary 2.5. *Let $A, X \in C_2(\mathcal{H})$. Then*

$$\omega_2(AX \pm XA^*) \leq 2\sqrt{2}\,\|A\|_2\,\omega_2(X).$$

Proof. By putting $X = Y$ and $B = I$ in Theorem 10 we get the desired result. $\square$

Remark 2.6. Let $A, B, X, Y \in C_2(\mathcal{H})$. Then, using (7), we have the following inequality:

$$\omega_2(AXB^* \pm BYA^*) \leq \sqrt{2}\,\|A\|_2\,\|B\|_2\,(\omega_2(X+Y) + \omega_2(X-Y)).$$

In the case $Y = X^*$, we get

$$\omega_2(AXB^* \pm BX^*A^*) \leq \sqrt{2}\,\|A\|_2\,\|B\|_2\,(\omega_2(X+X^*) + \omega_2(X-X^*)),$$

which has recently proven in [6, Theorem 2.3].

In the following, we obtain an upper bound for a general 2×2 block operator. It should be noted that in [6, Theorem 2.6], it was shown that

$$\omega_2\left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right) \leq \frac{\|A\|_2 + \|B\|_2}{2}.$$

The following result extends this inequality to any block $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$. When A, D are self-adjoint, [6, Theorem 2.11] presented some upper and lower bounds for the Hilbert-Schmidt numerical radius of such blocks.

Theorem 2.7. *Let $A, B, C, D \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \sqrt{\omega_2^2(A) + \omega_2^2(D)} + \frac{\|C\|_2 + \|B\|_2}{\sqrt{2}}.$$

Proof. Notice that $\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So, by (5) we have

$$\omega_2\left(\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}\left\|\begin{bmatrix} 0 & 0 \\ C & 0 \end{bmatrix}\right\|_2 = \frac{1}{\sqrt{2}}\|C\|_2.$$

Similarly, we have

$$\omega_2\left(\left[\begin{array}{cc} 0 & B \\ 0 & 0 \end{array}\right]\right) = \frac{1}{\sqrt{2}}\left\|B\right\|_2.$$

Now, by applying the properties of $\omega_2(\cdot)$ with Lemma 1.2 (d), we get

$$\begin{aligned}
\omega_2\left(\left[\begin{array}{cc} A & B \\ C & D \end{array}\right]\right) &= \omega_2\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right] + \left[\begin{array}{cc} 0 & B \\ 0 & 0 \end{array}\right] + \left[\begin{array}{cc} 0 & 0 \\ C & 0 \end{array}\right]\right) \\
&\leq \omega_2\left(\left[\begin{array}{cc} A & 0 \\ 0 & D \end{array}\right]\right) + \omega_2\left(\left[\begin{array}{cc} 0 & B \\ 0 & 0 \end{array}\right]\right) \\
&+ \omega_2\left(\left[\begin{array}{cc} 0 & 0 \\ C & 0 \end{array}\right]\right) \\
&\leq \sqrt{\omega_2^2(A) + \omega_2^2(D)} + \frac{\left\|B\right\|_2 + \left\|C\right\|_2}{\sqrt{2}}.
\end{aligned}$$

This completes the proof. $\square$

In [6], some upper bounds for the Hilbert-Schmidt numerical radius of the block operator $\left[\begin{array}{cc} A & B \\ -A & -B \end{array}\right]$ were obtained. Now, we find some upper and lower bounds for the Hilbert-Schmidt numerical radius of $\left[\begin{array}{cc} -A & -B \\ B & A \end{array}\right]$.

Theorem 2.8. *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\sqrt{2}\max\left\{\omega_2(A), \omega_2(B)\right\} \leq \omega_2\left(\left[\begin{array}{cc} -A & -B \\ B & A \end{array}\right]\right) \leq \sqrt{2}\left(\omega_2(A) + \omega_2(B)\right). \tag{14}$$

Proof. By Lemma 1.6 and Lemma 1.2, we have

$$\begin{aligned}
\omega_2\left(\left[\begin{array}{cc} -A & -B \\ B & A \end{array}\right]\right) &\leq \omega_2\left(\left[\begin{array}{cc} -A & 0 \\ 0 & A \end{array}\right]\right) + \omega_2\left(\left[\begin{array}{cc} 0 & -B \\ B & 0 \end{array}\right]\right) \\
&= \omega_2\left(\left[\begin{array}{cc} 0 & A \\ A & 0 \end{array}\right]\right) + \omega_2\left(\left[\begin{array}{cc} 0 & B \\ B & 0 \end{array}\right]\right) \\
&= \sqrt{2}\left(\omega_2(A) + \omega_2(B)\right).
\end{aligned}$$

On the other hand, by Lemma 1.5, it follows that

$$\begin{aligned}\omega_2\left(\begin{bmatrix} -A & -B \\ B & A \end{bmatrix}\right) &\geq \max\left\{\omega_2\left(\begin{bmatrix} -A & 0 \\ 0 & A \end{bmatrix}\right), \omega_2\left(\begin{bmatrix} 0 & -B \\ B & 0 \end{bmatrix}\right)\right\} \\ &= \sqrt{2}\max\left\{\omega_2(A), \omega_2(B)\right\}.\end{aligned}$$

□

In the following we obtain an upper bound for the Hilbert-Schmidt numerical radius of operator matrix $\begin{bmatrix} B & -A \\ A & B \end{bmatrix}$.

Proposition 2.9. *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix}\right) \leq \sqrt{\omega_2^2(A+iB) + \omega_2^2(A-iB)}.$$

Proof. Let $T = \begin{bmatrix} iB & -A \\ A & iB \end{bmatrix}$ and $U = \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}$. Then, U is a unitary operator on $\mathcal{H} \oplus \mathcal{H}$. Using (6), we get

$$\begin{aligned}\omega_2(T) &= \omega_2(U^*TU) \\ &= \omega_2\left(\begin{bmatrix} -i(A-B) & 0 \\ 0 & i(A+B) \end{bmatrix}\right) \\ &\leq \sqrt{\omega_2^2(-i(A-B)) + \omega_2^2(i(A+B))} \text{ (Lemma 1.2(d))} \\ &= \sqrt{\omega_2^2(A-B) + \omega_2^2(A+B)}.\end{aligned}$$

Hence

$$\omega_2\left(\begin{bmatrix} iB & -A \\ A & iB \end{bmatrix}\right) \leq \sqrt{\omega_2^2(A-B) + \omega_2^2(A+B)}.$$

Replacing B by $-iB$, we obtain

$$\omega_2\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix}\right) \leq \sqrt{\omega_2^2(A+iB) + \omega_2^2(A-iB)}.$$

□

Corollary 2.10. *If $A, B \in C_2(\mathcal{H})$ are self-adjoint operators. Then*

$$\omega_2\left(\begin{bmatrix} B & -A \\ A & B \end{bmatrix}\right) \leq \sqrt{2}\omega_2(A+iB).$$

Now, we are in a position to prove the following result.

Theorem 2.11. *Let $A, B, C, D \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leq \frac{\omega_2(C+B)+\omega_2(D-A)}{\sqrt{2}}$$
$$+\tfrac{1}{2}\sqrt{\omega_2^2((C-B)+i(A+D))+\omega_2^2((C-B)-i(A+D))}.$$

Proof. Let $U=\frac{1}{\sqrt{2}}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$ be a unitary operator on $\mathcal{H}\oplus\mathcal{H}$. Then, using (6), Proposition 2.9 and Lemma 1.2 (c,d). We see that

$$\begin{aligned}
&\omega_2\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right)\\
&= \omega_2\left(U^*\begin{bmatrix} A & B \\ C & D \end{bmatrix}U\right)\\
&= \frac{1}{2}\omega_2\left(\begin{bmatrix} A+B+C+D & -A+B-C+D \\ -A-B+C+D & A-B-C+D \end{bmatrix}\right)\\
&= \frac{1}{2}\omega_2\left(\begin{bmatrix} A+D & B-C \\ C-B & A+D \end{bmatrix}+\begin{bmatrix} B+C & D-A \\ D-A & -B-C \end{bmatrix}\right)\\
&= \frac{1}{2}\omega_2\left(\begin{bmatrix} A+D & B-C \\ C-B & A+D \end{bmatrix}+\begin{bmatrix} B+C & 0 \\ 0 & -B-C \end{bmatrix}\right.\\
&+ \left.\begin{bmatrix} 0 & D-A \\ D-A & 0 \end{bmatrix}\right)\\
&\leq \frac{1}{2}\omega_2\left(\begin{bmatrix} A+D & B-C \\ C-B & A+D \end{bmatrix}\right)+\frac{1}{2}\omega_2\left(\begin{bmatrix} B+C & 0 \\ 0 & -(B+C) \end{bmatrix}\right)\\
&+ \frac{1}{2}\omega_2\left(\begin{bmatrix} 0 & D-A \\ D-A & 0 \end{bmatrix}\right)\\
&\leq \frac{1}{2}\sqrt{\omega_2^2((C-B)+i(A+D))+\omega_2^2((C-B)-i(A+D))}\\
&\quad +\frac{\sqrt{2}}{2}(\omega_2(B+C)+\omega_2(D-A)).
\end{aligned}$$

□

As a consequence of Theorem 2.11, we have the following results.

Corollary 2.12. *Let $A, B, C \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\left[\begin{array}{cc} A & B \\ C & -A \end{array}\right]\right) \leq \frac{1}{\sqrt{2}}\left(\omega_2(C-B)+\omega_2(C+B)+2\omega_2(A)\right).$$

Corollary 2.13. *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\left[\begin{array}{cc} A & B \\ B & A \end{array}\right]\right) \leq \omega_2(A)+\sqrt{2}\omega_2(B).$$

Corollary 2.14. *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) \leq \frac{\omega_2(A+B)+\omega_2(A-B)}{\sqrt{2}}.$$

In the next theorem we obtain a new upper bound for a 2×2 off-diagonal block operators. For the usual numerical radius, related result have been given in [7].

Theorem 2.15. *Let $A, B \in C_2(\mathcal{H})$. Then*

$$\omega_2\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) \leq \sqrt{2}\left(\omega_2(A)+\omega_2(B)\right)-\frac{\left|\omega_2(A+B)-\omega_2(A-B)\right|}{\sqrt{2}}. \tag{15}$$

Proof. Recall that for any two real numbers α and β, we have

$$\frac{\alpha+\beta}{2}=\max\{\alpha,\beta\}-\frac{|\alpha-\beta|}{2}. \tag{16}$$

Using the identities (16) and (7), we have

$$\begin{aligned}
\omega_2\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) &\leq \frac{\omega_2(A+B)+\omega_2(A-B)}{\sqrt{2}} \\
&= \sqrt{2}\max\left\{\omega_2(A+B),\omega_2(A-B)\right\} \\
&- \frac{\left|\omega_2(A+B)-\omega_2(A-B)\right|}{\sqrt{2}} \\
&\leq \sqrt{2}\left(\omega_2(A)+\omega_2(B)\right)-\frac{\left|\omega_2(A+B)-\omega_2(A-B)\right|}{\sqrt{2}}.
\end{aligned}$$

Thus,

$$\omega_2\left(\left[\begin{array}{cc} 0 & A \\ B & 0 \end{array}\right]\right) \leq \sqrt{2}\left(\omega_2(A)+\omega_2(B)\right)-\frac{\left|\omega_2(A+B)-\omega_2(A-B)\right|}{\sqrt{2}},$$

as required. $\square$

Applying Theorem 2.15 we have the following result, which relates the Hilbert-Schmidt numerical radius with the Hilbert-Schmidt norm of the real and imaginary parts of the operator.

Corollary 2.16. *Let $A \in C_2(\mathcal{H})$ with $A = \Re A + i\ \Im A$. Then,*

$$\|A\|_2 + \left|\|\Re A\|_2 - \|\Im A\|_2\right| \leq 2\omega_2(A).$$

Proof. Putting $B = A^*$ in the inquality (15), we have

$$\begin{aligned} \omega_2\left(\left[\begin{array}{cc} 0 & A \\ A^* & 0 \end{array}\right]\right) &\leq \sqrt{2}\left(\omega_2(A)+\omega_2(A^*)\right) \\ &- \frac{\left|\omega_2(A+A^*)-\omega_2(A-A^*)\right|}{\sqrt{2}} \\ &= 2\sqrt{2}\omega_2(A) - \frac{\left|\omega_2(2\Re A)-\omega_2(2i\Im A)\right|}{\sqrt{2}}. \end{aligned}$$

Since $\Re A$ and $\Im A$ are self-adjoint operators. So, by (4) we get

$$\omega_2\left(\left[\begin{array}{cc} 0 & A \\ A^* & 0 \end{array}\right]\right) \leq 2\sqrt{2}\omega_2(A) - \frac{2\left|\|\Re A\|_2 - \|\Im A\|_2\right|}{\sqrt{2}}. \tag{17}$$

Further, clearly $\left[\begin{array}{cc} 0 & A \\ A^* & 0 \end{array}\right]$ is a self-adjoint operator. Again by (4)we have

$$\begin{aligned} \omega_2\left(\left[\begin{array}{cc} 0 & A \\ A^* & 0 \end{array}\right]\right) &= \left\|\left[\begin{array}{cc} 0 & A \\ A^* & 0 \end{array}\right]\right\|_2 \\ &= \sqrt{\|A\|_2 + \|A^*\|_2} \\ &= \sqrt{2}\,\|A\|_2. \end{aligned}$$

Hence, (17) implies that

$$\|A\|_2 \leq 2\omega_2(A) - \left|\|\Re A\|_2 - \|\Im A\|_2\right|.$$

Therefore

$$\|A\|_2 + |\|\Re A\|_2 - \|\Im A\|_2| \leq 2\omega_2(A).$$

$\square$

Our final result can be stated as follows.

Theorem 2.17. *Let $A, B, X \in C_2(\mathcal{H})$ be such that A and B are self-adjoin. Then*

$$\|X\|_2 \leq \omega_2(X + A) + \omega_2(X + iB). \tag{18}$$

Proof. We first prove that

$$\frac{\|S + T^*\|_2}{\sqrt{2}} \leq \omega_2\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right) \leq \frac{\omega_2(T + S) + \omega_2(T - S)}{\sqrt{2}}, \tag{19}$$

for any $T, S \in C_2(\mathcal{H})$.

Let $Q = \begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}$. Then

$$\begin{aligned} 2\|S + T^*\|_2^2 &= \left\|\begin{bmatrix} 0 & S + T^* \\ S^* + T & 0 \end{bmatrix}\right\|_2^2 \\ &= \left\|\begin{bmatrix} 0 & (S^* + T)^* \\ S^* + T & 0 \end{bmatrix}\right\|_2^2 \\ &= \|Q + Q^*\|_2^2 \\ &= \omega_2^2(Q + Q^*) \ \text{(by (4))} \\ &\leq 4\omega_2^2(Q) \\ &= 4\omega_2^2\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right). \end{aligned}$$

Hence,

$$\frac{\|S + T^*\|_2}{\sqrt{2}} \leq \omega_2\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right).$$

Moreover, by (7) we have

$$\omega_2\left(\begin{bmatrix} 0 & S \\ T & 0 \end{bmatrix}\right) \leq \frac{\omega_2(T + S) + \omega_2(T - S)}{\sqrt{2}}.$$

Hence, we obtain the desired inequality (19). So, we conclude that

$$\|S+T^*\|_2 \leq \omega_2\left(T+S\right)+\omega_2\left(T-S\right) \tag{20}$$

Now, the desired inequality (18) follows from the inequality (20) applied to the operators $S=X+\frac{A+iB}{2}$ and $T=\frac{-A+iB}{2}$. This completes the proof. $\square$

Acknowledgements
The author would like to thank the referee for careful reading and kind suggestions.

References

[1] A. Abu-Omar, F. Kittaneh, Upper and lower bounds for the numerical radius with an application to involution operators, Rocky Mountain J. Math. 45 (4) (2015) 1055–1064.

[2] A. Abu Omar and F. Kittaneh, A generalization of the numerical radius, Linear Algebra Appl. 569 (2019), 323–334.

[3] A. Aldalabih and F. Kittaneh, Hilbert-Schmidt numerical radius inequalities for operator matrices, Linear Algebra Appl. 581(2019), 72–84.

[4] Y. Al-manasrah and F. Kittaneh, A generalization of two refined Young inequalities, Positivity 19 (2015), no. 4, 757–768.

[5] M. Bakherad and Kh. Shebrawi, Upper bounds for numerical radius inequalities involving off-diagonal operator matrices, Ann. Funct. Anal. 9 (2018), no. 3, 297–309.

[6] M. Hajmohamadi, R. Lashkaripour, Some Inequalities Involving Hilbert-Schmidt Numerical Radius on 2×2 Operator Matrices. Filomat 34:14 (2020), 4649–4657.

[7] O. Hirzallah, F. Kittaneh, K. Shebrawi, Numerical radius inequalities for certain 2×2 operator matrices, Integral Equations Operator Theory 71 (2011) 129–147.

[8] S. Sahooa, M. Sababheh, Hilbert-Schmidt Numerical Radius of Block Operators,Filomat 35:8 (2021), 2663–2678.

[9] M. Sattari, M. S. Moslehian and T. Yamazaki, Some genaralized numerical radius inequalities for Hilbert space operators, Linear Algebra Appl. 470 (2014), 1–12.

[10] F. Kittaneh, M.S. Moslehian and T. Yamazaki, Cartesian decomposition and numerical radius inequalities, Linear Algebra Appl. 471 (2015), 46–53.

[11] K.E. Gustafson and D.K.M. Rao, Numerical Range, The Field of Values of Linear Operators and Matrices, Springer, New York, 1997.

[12] Kh. Shebrawi, M. Bakherad, Generalizations of the Aluthge transform of operators, Filomat, 32 (2018), no. 18, 6465–6474.

[13] J.L. van Hemmen and T. Ando, An inequality for trace ideals, Comm. Math. Phys. 76 (1980), no. 2, 143–148.

Messaoud Guesba
Assistant Professor of Mathematics
Department of Mathematics
University of El Oued
El Oued, Algeria
E-mail: guesbamessaoud2@gmail.com, guesba-messaoud@univ-eloued.dz