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Numerical Solution of Higher-Order Linear and Nonlinear Ordinary Differential Equations with Orthogonal Rational Legendre Functions

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Abstract. In this paper, we describe a method for the solution of linear and nonlinear ordinary differential equations ODE's of arbitrary order with initial or boundary conditions. In this direction we first investigate some properties of orthogonal rational Legendre functions, and then we give the least square method based on these basis functions for the solution of such equations. In this method the solution of an ODE is reduced to a minimization problem, which is then numerically solved via Maple 16. Finally results of this method which are obtained in the form of continuous functions, will be compared with the numerical results in other references.

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1. Introduction

Ordinary differential equations appear in modelling various physical phenomena [10], and, depending on the accompanying side conditions, they are classified under the two main categories of initial and boundary value problems. Moreover, obtaining the analytic solution of these problems

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are difficult or in general impossible, and therefore numerical methods must be utilized. Some of these methods are given in [2,8,16].
Least square methods, which are useful in dealing with ODE problems, have received lots of attention in recent years [1,9,14]. Loghmani and Alavizadeh used the least square method based on B-splines for solution of boundary value problems [18].
In the present paper, we have used the least squares method based on Legendre orthogonal rational functions for solution of higher-order linear ordinary differential equations.
Orthogonal rational functions which are generalizations of orthogonal polynomials were formally introduced by Bultheel [7], and moreover he investigated the properties of these functions in different regions. Legendre rational functions are a class of orthogonal rational functions which generalize Legendre polynomials.
Boyd [4-6] and Christov [11] used some spectral methods for the solution of certain linear problems on infinite intervals by employing systems of orthogonal rational functions. Recently, Guo and others have presented different approximations for solution of differential equations [12,27,30]. The method which is proposed in the present paper is capable of solving ODE's with arbitrary precision, which can be achieved more easily than the previous methods.
This paper is organized as follows: in Section 2, we introduce the rational Legendre functions and also describe some useful properties of these basis functions. In Section 3, we describe the Legendre rational functions approximation. In Section 4, we express a theorem about convergence the series of Legendre rational functions. In Section 5, we propose a least square method based on the rational Legendre functions to solve ODE's with initial and boundary conditions. In Section 6, the proposed method is applied to several numerical examples.

2. Rational Legendre Functions

Let $L_n(x)$ be the Legendre polynomial of degree n. We recall that $L_n(x)$ is the eigenfunction of the singular Sturm-Liouville problem

$$\frac{d}{dx}\left((1-x^2)\frac{d}{dx}L_n(x)\right)+n(n+1)L_n(x)=0,\quad n=0,1,2,\ldots .$$

The Legendre polynomials are orthogonal with respect to the L_2 inner product on the interval [-1,1]:

$$\int_{-1}^{1} L_m(x)L_n(x)dx=\frac{2}{2n+1}\delta_{mn},$$

and satisfy $L_n(1)=1$, where δ_{mn} denotes the Kronecker delta. These polynomials can be determined with the recurrence relation [3]:

$$L_0(x)=1,\quad L_1(x)=x,$$

$$L_{n+1}(x)=\frac{2n+1}{n+1}xL_n(x)-\frac{n}{n+1}L_{n-1}(x),\ n\geqslant 1\ .$$

The rational Legendre function of degree n is defined by

$$R_n(x)=L_n\left(\frac{x-L}{x+L}\right), \tag{1}$$

where the constant parameter L sets the length scale of the mapping. Boyd [4] offered guidelines for optimizing the map parameter L for rational Chebyshev functions, which is useful for rational Legendre functions, too.

The $R_n(x)$ is the nth eigenfunction of the singular Sturm-Liouville problem [13]

$$\frac{(x+L)^2}{2L}\frac{d}{dx}\left(x\frac{d}{dx}R_n(x)\right)+n(n+1)R_n(x)=0,\quad x\in(0,\infty), \tag{2}$$

which satisfies the following recurrence relation for $x\in(0,\infty)$:

$$R_0(x)=1,\ R_1(x)=\frac{x-L}{x+L},$$

$$R_{n+1}(x)=\left(\frac{2n+1}{n+1}\right)\left(\frac{x-L}{x+L}\right)R_n(x)-\left(\frac{n}{n+1}\right)R_{n-1}(x),\quad n\geqslant 1,$$

3. Legendre Rational Functions Approximations

Let $\Lambda = \{x|\ 0 < x < \infty\}$, and for $1 \leqslant P < \infty$ consider

$$L^P_\omega = \{\nu : \Lambda \to \mathbb{R}|\ \nu \text{ is measurable and } \|\nu\|_{L^P_\omega} < \infty\},$$

where

$$\|\nu\|_{L^P_\omega} = \left(\int_0^\infty |\nu(x)|^P \omega(x)dx\right)^{\frac{1}{P}},$$

and $(u,\nu)_\omega$ and $\|\nu\|_\omega$ denote the inner product and the norm of the space $L^2_\omega(\Lambda)$, i.e.:

$$(u,\nu)_\omega = \int_0^\infty u(x)\nu(x)\omega(x)dx, \quad \|\nu\|_\omega = (\nu,\nu)^{\frac{1}{2}}_\omega\ . \tag{3}$$

Note that the function $\omega(x) = \frac{2L}{(x+L)^2}$ is a non-negative, integrable, real-valued function on Λ, The orthogonality of rational Legendre functions leads to

$$(R_m, R_n)_\omega = \int_0^\infty R_m(x)R_n(x)\omega(x)dx = \frac{2}{2n+1}\delta_{mn},$$

where δ_{mn} denotes the Kronecker delta. Thus for any $\nu \in L^2_\omega(\Lambda)$ we have

$$\nu(x) = \sum_{j=0}^{+\infty} c_j R_j(x), \tag{4}$$

where

$$c_j = \frac{(\nu, R_j)_\omega}{\|R_j\|^2_\omega} = \frac{2j+1}{2}\int_\Lambda \nu(x)R_j(x)\omega(x)dx.$$

4. Orthogonal Projection

Let N be any positive integer, and $\Re_N = span\{R_0(x), R_1(x), ..., R_N(x)\}$, where R_j's are defined in (1). We define the orthogonal projection $P_N : L^2_\omega(\Lambda) \to \Re_N$ by

$$[P_N u](x) = \sum_{j=0}^{N} c_j R_j(x),$$

where $u \in L^2_\omega(\Lambda)$ and c_j's are the rational Legendre-Fourier coefficients of u. Then $P_N u$ is the orthogonal projection of u on $\Re_N$ with respect to inner product (3). Hence orthogonality of rational Legendre functions leads to

$$(P_N u - u, \phi)_\omega = 0, \quad \forall \phi \in \Re_N.$$

To obtain the order of the error $\|P_N u - u\|_\omega$, we consider the space

$$H^r_{\omega,T}(\Lambda) = \{\nu | \ \nu \text{ is a measurable and } \|\nu\|_{r,\omega,T} < \infty\},$$

where for non-negative integer r, the norm $\|\nu\|_{r,\omega,T}$ is defined by [3]:

$$\|\nu\|_{r,\omega,T} = \left(\sum_{j=0}^{2r} \left\| (x+1)^{r+j} \frac{d^j}{dx^j} \nu \right\|^2_\omega \right)^{\frac{1}{2}}$$

and T is the Sturm-Liouville operator in (2),i.e.,

$$[T\nu](x) = -\frac{1}{\omega(x)} \frac{d}{dx} \left(x \frac{d}{dx} \nu(x) \right) .$$

By induction, we have

$$[T^r \nu](x) = \sum_{j=0}^{2r} (x+1)^{r+j} p_j(x) \frac{d^j}{dx^j} \nu(x)$$

where $p_j(x)$'s are rational functions which are bounded uniformly on the interval Λ. Therefore, T^r is continuous mapping from $H^r_{\omega,T}(\Lambda)$ to $L^2_\omega(\Lambda)$.

Theorem 4.1. *For any $u \in H^r_{\omega,T}(\Lambda)$ and any $r \geqslant 0$, there is a positive constant C such that*

$$\|P_N u - u\|_\omega \leqslant C N^{-r} \|u\|_{r,\omega,T}$$

Proof. See [3]. $\square$

5. Explanation of the Method

Suppose U is an open subset of $\mathbb{R}^{m+2}$ and $g : U \to \mathbb{R}$ is a continuous function, and consider the differential equation

$$g(x, y(x), y'(x), ..., y^{(m)}(x)) = 0, \quad a < x < b, \tag{5}$$

with the general separated boundary conditions

$$\sum_{j=0}^{m-1} \alpha_{i,j} y^{(j)}(a_i) = A_i \ \ , \ 0 \leqslant i \leqslant m-1 \ . \tag{6}$$

We convert problem (5) to an optimal control problem

$$\min_{y \in C[a,b]} \int_a^b [g(x, y(x), y'(x), ..., y^{(m)}(x))]^2 \omega(x) dx$$

under the separated boundary conditions

$$\sum_{j=0}^{m-1} \alpha_{i,j} y^{(j)}(a_i) = A_i \ \ , \ 0 \leqslant i \leqslant m-1,$$

The actual solution of (5)-(6) is a function v such that

$$\begin{cases} \|g(x, v(x), v'(x), ..., v^{(m)}(x))\|_\omega^2 = 0 \\ \sum_{j=0}^{m-1} \alpha_{i,j} v^{(j)}(a_i) = A_i \ \ , \ 0 \leqslant i \leqslant m-1 \ . \end{cases}$$

For approximating the solution of (5) and (6) by the elements of $\Re_N$ we consider

$$v_k(x) = \sum_{i=0}^{k} c_i R_i(x), \quad k \in \mathbb{N}, \tag{7}$$

where the coefficients $\{c_i\}$ are determined from the least square problem

$$\min_{c_i \in \mathbb{R}} \|g(x, v_k(x), v_k'(x), \ldots, v_k^{(m)}(x))\|_\omega^2,$$

subject to constraints:

$$\sum_{j=0}^{m-1} \alpha_{i,j} v_k^{(j)}(a_i) = A_i \ \ , \ 0 \leqslant i \leqslant m-1, \tag{8}$$

Note that the above minimization problem is equivalent to the following system:

$$\begin{cases} \frac{\partial}{\partial c_i}||g(x, v_k(x), v_k'(x), \ldots, v_k^{(m)}(x))||_\omega^2 = 0, \quad (i = 0, 1, \ldots, n-1), \\ \sum_{j=0}^{m-1} \alpha_{i,j} v_k^{(j)}(a_i) = A_i \quad , \ 0 \leqslant i \leqslant m-1 \ . \end{cases}$$

In [18], the convergence of this method with B-spline functions has been investigated for ODE's.

6. Numerical Results

In this section, seven linear and nonlinear test problems are solved by using the above method. The rational Legendre approximate solution of the system of equations (5) and (6) is obtained by using a linear combination (7), and the resulting minimization problem is solved by Maple 16 with 80 digits precision and with Optimization package. Absolute errors and the least square errors(LSE),i.e.,

$$\mathrm{LSE}(y) = \int_a^b [y(x) - y^*(x)]^2 \omega(x) dx$$

where $y(x)$ is the exact solution and $y^*(x)$ is the approximate solution, for these test problems [1-7] are calculated and depicted in tables and figures.

6.1 Initial value problems

Test problem 1. ([25]) Consider the linear initial value problem

$$\begin{aligned} (x+1)y'(x) + y(x) = 1, &\qquad x \in [0, 1], \\ y(0) = 0, & \end{aligned}$$

with the exact solution

$$y(x) = \frac{x}{x+1} \ .$$

By using the method we have obtained the following results:

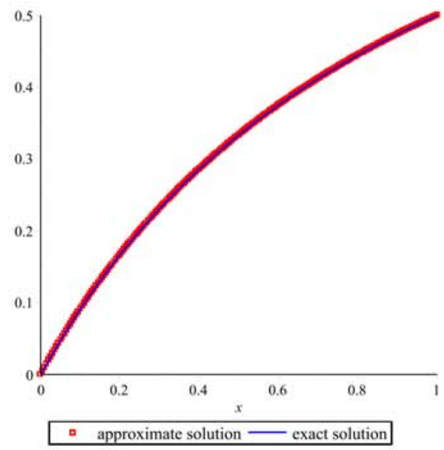


Figure 1. Approximate and exact solutions for TP1 with N=5 LSE=0.0

The approximate solution which is obtained by using this method with N=5 is:

$$y^*(x) = 0.5R_0(x) + 0.5R_1(x) = 0.5 + 0.5\,\frac{x-1}{x+1} = \frac{x}{x+1}$$

Test problem 2. ([25]) Consider the linear initial value problem

$$(x+1)y'''(x) + y''(x) - \frac{1}{1+x}\,y'(x) + xy(x) = x\ln(x+1), \qquad x \in [0,1],$$
$$y(0) = 0, \quad y'(0) = 1, \quad y''(0) = -1,$$

with the exact solution

$$y(x) = \ln(x+1).$$

By using the presented method we obtaine the following results:
The approximate solution which is obtained by using this method with N=5 is:

$$y^*(x) = 1.454R_0(x) + 1.671R_1(x) + 0.3422R_2(x) + 0.1518R_3(x)$$
$$+0.0315R_4(x) + 0.0057R_5(x),$$

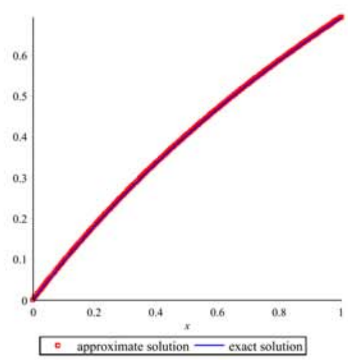
In Tables 1 and 2 the obtained values using the present method together with the results obtained from rational Chebyshev Collocation method and also the exact values of $y(x)$ are tabulated.

Table 1: Approximates and exact values for test problem 2 for N=6

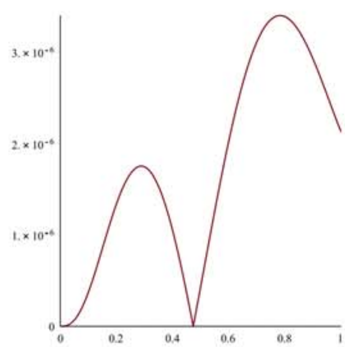
x	Exact solution	RC method	Abs Error	Present method	Abs Error
0.0	0.0000000000	0.00000000	0.00000	0.0000000000	0.00000
0.1	0.0953101798	0.09518697	1.23E-4	0.0953101093	7.05E-8
0.2	0.1823215567	0.18167362	6.48E-4	0.1823214070	1.50E-7
0.3	0.2623642644	0.26081935	1.54E-3	0.2623642480	1.64E-8
0.4	0.3364722366	0.33372802	2.74E-3	0.3364724663	2.30E-7
0.5	0.4054651081	0.40125453	4.21E-3	0.4054654768	3.69E-7
0.6	0.4700036292	0.46407087	5.93E-3	0.4700039187	2.90E-7
0.7	0.5306282510	0.52271881	7.91E-3	0.5306283003	4.93E-8
0.8	0.5877866649	0.57764525	1.01E-4	0.5877864739	1.91E-7
0.9	0.6418538861	0.62922562	1.26E-2	0.6418536027	2.83E-7
1.0	0.6931471805	0.67777984	1.54E-2	0.6931469709	2.10E-7

Table 2: Approximates and exact values for test problem 2 for N=7

x	Exact solution	RC method	Abs Error	Present method	Abs Error
0.0	0.0000000000	0.00000000	0.00000	0.0000000000	0.00000
0.1	0.0953101798	0.09518485	1.25E-4	0.0953101760	3.71E-9
0.2	0.1823215567	0.18173604	5.86E-4	0.1823215559	8.52E-10
0.3	0.2623642644	0.26105289	1.31E-3	0.2623642735	9.05E-9
0.4	0.3364722366	0.33419588	2.28E-3	0.3364722451	8.53E-9
0.5	0.4054651081	0.40197707	3.49E-3	0.4054651043	3.76E-9
0.6	0.4700036292	0.46505273	4.95E-3	0.4700036141	1.51E-8
0.7	0.5306282510	0.52396995	6.66E-3	0.5306282357	1.54E-8
0.8	0.5877866649	0.57919041	8.60E-3	0.5877866586	6.26E-9
0.9	0.6418538861	0.63110549	1.07E-2	0.6418538880	1.92E-9
1.0	0.6931471805	0.68004804	1.31E-2	0.6931471827	2.15E-9



(a) Approximate and exact solutions LSE=4.278 E-12



(b) Absolute error

Figure 2. Errors for test problem 3 with N=6

Test problem 3. ([24,25]) Consider the linear initial value problem

$$y''(x) + 2xy'(x) = 0, \qquad x \in [0,1],$$
$$y(0) = 0 \ \ , y'(0) = \frac{2}{\sqrt{\pi}},$$

with the exact solution

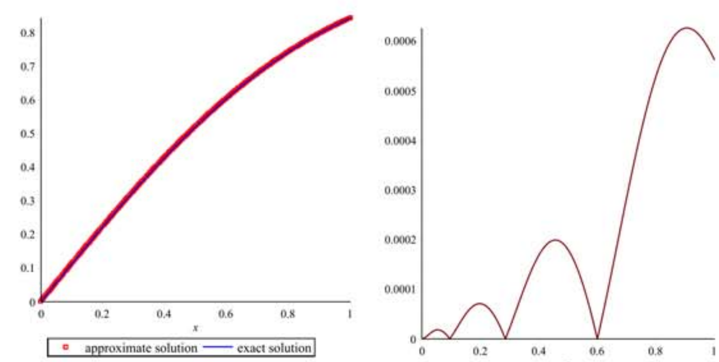
$$y(x) = \frac{2}{\sqrt{\pi}} \int_0^x exp(-t^2)dt.$$

By using the method we have obtained the following results:

The approximate solution which is obtained by using this method with N=5 is:

$$y^*(x) = 0.3713R_0(x) - 0.2075R_1(x) - 1.113R_2(x) - .7316R_3(x)$$
$$-.2281R_4(x) - 0.0305R_5(x),$$

In Table 3 the resulting values using the present method together with the Rational Chebyshev Collocation method and also the exact values of $y(x)$ are tabulated.



(a) Approximate and exact solutions LSE=3.990 E-8

(b) Absolute error

Figure 3. Errors for test problem 3 with N=5

Table 3: Approximates and exact values for test problem 3 for N=5

x	Exact solution	RC method	Abs Error	Present method	Abs Error
0.0	0.0000000000	0.0000000	0.00000	0.0000000000	0.00000
0.1	0.1124629160	0.1124386	2.43E-5	0.1124622250	6.91E-7
0.2	0.2227025892	0.2228901	1.88E-4	0.2227044042	1.82E-6
0.3	0.3286267594	0.3285654	6.14E-5	0.3286237932	2.97E-6
0.4	0.4283923550	0.4283688	2.36E-5	0.4283900363	2.32E-6
0.5	0.5204998778	0.5204235	7.64E-5	0.5205059668	6.09E-6
0.6	0.6038560908	0.6038157	4.04E-5	0.6038630643	6.97E-6
0.7	0.6778011938	0.6776712	1.30E-4	0.6777968757	4.32E-6
0.8	0.7421009647	0.7422375	1.37E-4	0.7420872860	1.37E-5
0.9	0.7969082124	0.7968211	8.71E-5	0.7969014534	6.76E-6
1.0	0.8427007929	———	——	0.8427053088	4.52E-6

Test problem 4. ([23]) Consider a special case of Lane-Emden equation which is named after astrophysicists Jonathan Homer Lane and Robert Emden [15] :

$$y''(x) + \frac{2}{x}y'(x) + y^2(x) = \frac{9(x^4 + 3x^3 + 6)}{x(x+3)^3}, \qquad x \in [0, 1],$$

$$y(0) = 1\ ,\ y'(0) = 0,$$

with the exact solution

$$y(x) = \frac{3x}{x+3}\ .$$

By using the method we have obtained the following results:
The approximate solution which is obtained by using this method with N=10 is:

$$y^*(x) = 1.0962R_0(x)+1.3887R_1(x)+3.7001\times10^{-1}R_2(x)+9.7651\times10^{-2}R_3(x)$$

$$+2.5226\times10^{-2}R_4(x)+6.2451\times10^{-3}R_5(x)+1.4327\times10^{-3}R_6(x)+2.8965\times10^{-4}R_7(x)$$

$$+4.7853\times10^{-5}R_8(x) + 5.6817\times10^{-6}R_9(x) + 3.6021\times10^{-7}R_{10}(x)$$

6.2 Boundary value problems

Test problem 5. ([25]) Consider the linear boundary value problem

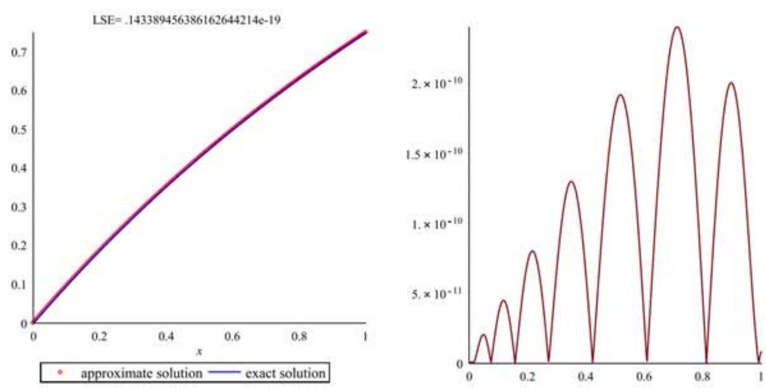
$$y''(x) - \frac{1-x}{(1+x)^2}\, y(x) = \frac{1}{(1+x)^2}, \qquad x \in [0,1]$$
$$y(0) = 1 \quad , y(1) = \frac{1}{2},$$

with the exact solution

$$y(x) = \frac{1}{1+x}.$$

By using the method we have obtained the following results:
The approximate solution which is obtained by using this method with N=5 is:

$$y^*(x) = 0.5R_0(x) - 0.5R_1(x) - 1.023 \times 10^{-78} R_2(x) - 2.814 \times 10^{-78} R_3(x)$$
$$-2.815 \times 10^{-78} R_4(x) - 9.659 \times 10^{-79} R_5(x).$$



(a) Approximate and exact solutions LSE=1.434E-20

(b) Absolute error

Figure 4. Errors for test problem 4 with N=10

since the coefficients of $R_2(x)$,..., $R_5(x)$ are negligible, by ignoring them we obtain the exact solution, i.e.:

$$y^*(x) = 0.5R_0(x) - 0.5R_1(x) = 0.5 - 0.5\, \frac{x-1}{x+1} = \frac{1}{x+1}$$

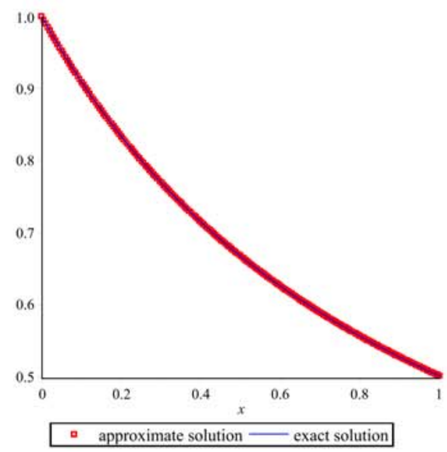


Figure 5. Approximate and exact solutions for TP5 with N=5
LSE=6.773 E-161

Test problem 6. ([19,20,21,22,29]) Consider the following nonlinear boundary value problem of twelfth-order

$$y^{(12)}(x) = 2e^{x}y^{2}(x) + y'''(x), \qquad x \in [0,1]$$

$$y(0) = y''(0) = y^{(4)}(0) = y^{(6)}(0) = y^{(8)}(0) = y^{(10)}(0) = 1$$

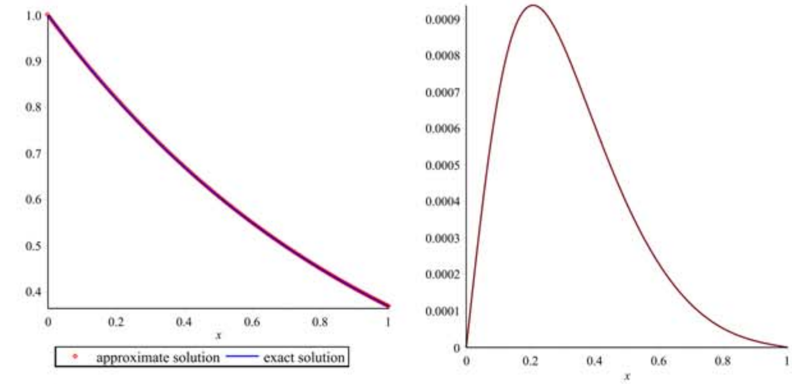
$$y(1) = y''(1) = y^{(4)}(1) = y^{(6)}(1) = y^{(8)}(1) = y^{(10)}(1) = e^{-1}$$

with the exact solution

$$y(x) = e^{-x} \ .$$

By using the method we have obtained the following results:
The approximate solution which is obtained by using this method with N=10 is:

$$y^{*}(x) = 0.5360R_0(x) - 0.5840R_1(x) - 0.1024R_2(x) + 0.0570R_3(x) + 0.0657R_4(x)$$
$$+0.0387R_5(x) + 0.0167R_6(x) + 0.0054R_7(x) + 0.0012R_8(x)$$
$$+0.0002R_9(x) + 0.0000R_{10}(x)$$



(a) Approximate and exact solutions LSE=2.639E-7

(b) Absolute error

Figure 6. Errors for test problem 6 with N=10

Test problem 7. ([17,26,28]) Consider the following an eighth-order boundary value problem:

$$y^{(8)}(x) + \phi(x)y(x) = \psi(x), \quad x \in [a,b]$$
$$y(a) = A_0, y''(a) = A_2, y^{(4)}(a) = A_4, y^{(6)}(a) = A_6$$
$$y(b) = B_0, y''(b) = B_2, y^{(4)}(b) = B_4, y^{(6)}(b) = B_6$$

where $y(x)$ and $\phi(x)$ and $\psi(x)$ are continuous functions defined in the interval [a,b]. A_i and B_i (i =0,2,4,6), are finite real constants.
Exact solutions with various constants and functions for test problem 7 are listed in Table 4.

Table 4: Variables for differential equations and boundary conditions in test problem 7

Example	7.1	7.2	7.3	7.4
[a,b]	[0,1]	[-1,1]	[-1,1]	[-1,1]
$\phi(x)$	x	$-x$	-1	-1
$\psi(x)$	$-(48+15x+x^3)e^x$	$-(55+17x+x^2-x^3)e^x$	$-8[2x\cos(x)+7\sin(x)]$	$8[2x\sin(x)-7\cos(x)]$
A_0	0	0	0	0
A_2	0	$\frac{2}{e}$	$-4\cos(1)-2\sin(1)$	$-4\sin(1)+2\cos(1)$
A_4	-8	$-\frac{4}{e}$	$8\cos(1)+12\sin(1)$	$8\sin(1)-12\cos(1)$
A_6	-24	$-\frac{18}{e}$	$-12\cos(1)-30\sin(1)$	$-12\sin(1)+30\cos(1)$
B_0	0	0	0	0
B_2	$-4e$	$-6e$	$4\cos(1)+2\sin(1)$	$-4\sin(1)+2\cos(1)$
B_4	$-16e$	$-20e$	$-8\cos(1)-12\sin(1)$	$8\sin(1)-12\cos(1)$
B_6	$-36e$	$-42e$	$12\cos(1)+30\sin(1)$	$-12\sin(1)+30\cos(1)$
Solution	$x(1-x)e^x$	$(1-x^2)e^x$	$(x^2-1)\sin(x)$	$(x^2-1)\cos(x)$

By using the method we have obtained the following results:
The approximate solution of example 7.1 which is obtained by using this method with N=20 is:

$$y^*(x) = -2.7957 \times 10^{10} R_0(x) - 7.6249 \times 10^{10} R_1(x) - 1.0500 \times 10^{11} R_2(x)$$
$$-1.1029 \times 10^{11} R_3(x) - 9.6510 \times 10^{10} R_4(x) - 7.2702 \times 10^{10} R_5(x)$$
$$-4.7855 \times 10^{10} R_6(x) - 2.7714 \times 10^{10} R_7(x) - 1.4156 \times 10^{10} R_8(x)$$
$$-6.3751 \times 10^{9} R_9(x) - 2.5241 \times 10^{9} R_{10}(x) - 8.7432 \times 10^{8} R_{11}(x)$$
$$-2.6303 \times 10^{8} R_{12}(x) - 6.8038 \times 10^{7} R_{13}(x) - 1.4926 \times 10^{7} R_{14}(x)$$
$$-2.7250 \times 10^{6} R_{15}(x) - 4.0316 \times 10^{5} R_{16}(x) - 4.6470 \times 10^{4} R_{17}(x)$$
$$-3.9169 \times 10^{3} R_{18}(x) - 2.1485 \times 10^{2} R_{19}(x) - 5.7581 R_{20}(x)$$

The approximate solution of example 7.2 which is obtained by using this method with N=5 is:

$$y^*(x) = -4.5236 \times 10^{14} R_)(x) - 1.0219 \times 10^{15} R_1(x) - 8.6467 \times 10^{14} R_2(x)$$
$$-1.5982 \times 10^{14} R_3(x) + 5.6238 \times 10^{14} R_4(x) + 8.8260 \times 10^{14} R_5(x)$$
$$+7.5425 \times 10^{14} R_6(x) + 4.0897 \times 10^{14} R_7(x) + 1.0012 \times 10^{14} R_8(x)$$
$$-5.9698 \times 10^{13} R_9(x) - 9.2210 \times 10^{13} R_{10}(x) - 6.6618 \times 10^{13} R_{11}(x)$$
$$-3.4137 \times 10^{13} R_{12}(x) - 1.3491 \times 10^{13} R_{13}(x) - 4.2117 \times 10^{12} R_{14}(x)$$
$$-1.0390 \times 10^{12} R_{15}(x) - 1.9952 \times 10^{11} R_{16}(x) - 2.8881 \times 10^{10} R_{17}(x)$$
$$-2.9733 \times 10^{9} R_{18}(x) - 1.9446 \times 10^{8} R_{19}(x) - 6.0828 \times 10^{6} R_{20}(x)$$

The approximate solution of example 7.3 which is obtained by using this method with N=20 is:

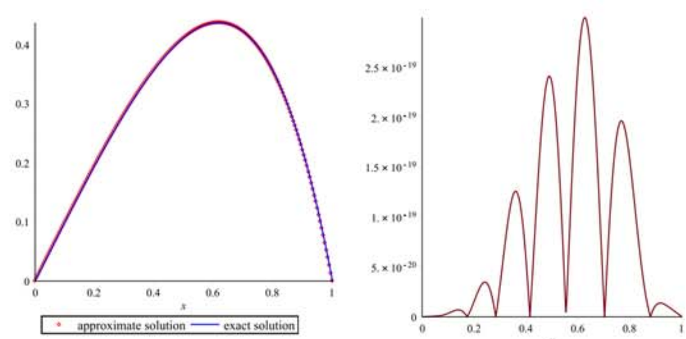
$$y^*(x) = -4.5959 \times 10^{11} R_0(x) - 1.2522 \times 10^{12} R_1(x) - 1.7208 \times 10^{12} R_2(x)$$
$$-1.8013 \times 10^{12} R_3(x) - 1.5680 \times 10^{12} R_4(x) - 1.1724 \times 10^{12} R_5(x)$$
$$-7.638410^{11} R_6(x) - 4.3638 \times 10^{11} R_7(x) - 2.1902 \times 10^{11} R_8(x)$$

$$-9.6469 \times 10^{10} R_9(x) - 3.7166 \times 10^{10} R_{10}(x) - 1.2456 \times 10^{10} R_{11}(x)$$
$$-3.6037 \times 10^{9} R_{12}(x) - 8.9063 \times 10^{8} R_{13}(x) - 1.8542 \times 10^{8} R_{14}(x)$$
$$-3.1903 \times 10^{7} R_{15}(x) - 4.4172 \times 10^{6} R_{16}(x) - 4.7306 \times 10^{5} R_{17}(x)$$
$$-3.6786 \times 10^{4} R_{18}(x) - 1.8484 \times 10^{3} R_{19}(x) - 4.5062 \times 10^{1} R_{20}(x)$$

The approximate solution of example 7.4 which is obtained by using this method with N=20 is:

$$y^*(x) = 1.3059 \times 10^{8} R_0(x) + 3.5404 \times 10^{8} R_1(x) + 4.8158 \times 10^{8} R_2(x)$$
$$+4.9640 \times 10^{8} R_3(x) + 4.2320 \times 10^{8} R_4(x) + 3.0817 \times 10^{8} R_5(x)$$
$$+1.9443 \times 10^{8} R_6(x) + 1.0694 \times 10^{8} R_7(x) + 5.1377 \times 10^{7} R_8(x)$$
$$+2.1544 \times 10^{7} R_9(x) + 7.8620 \times 10^{6} R_{10}(x) + 2.4844 \times 10^{6} R_{11}(x)$$
$$+6.7495 \times 10^{5} R_{12}(x) + 1.5611 \times 10^{5} R_{13}(x) + 3.0327 \times 10^{4} R_{14}(x)$$
$$+4.8576 \times 10^{3} R_{15}(x) + 6.2490 \times 10^{2} R_{16}(x) + 6.2084 \times 10^{1} R_{17}(x)$$
$$+4.4727 R_{18}(x) + 2.0798 \times 10^{-1} R_{19}(x) + 4.6873 \times 10^{-3} R_{20}(x)$$

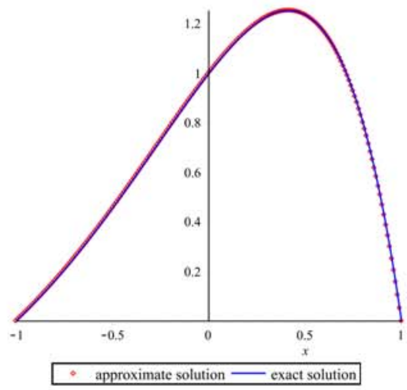
In Figures 7, 8, 9 and 10 exact and approximate solution diagrams for test problem 7 (Examples 7.1, 7.2, 7.3 and 7.4) have been plotted with least square errors and absolute errors.



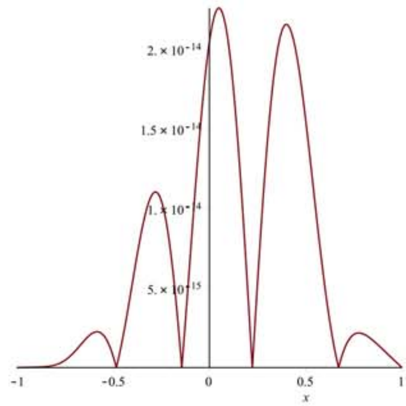
(a) Approximate and exact solutions LSE=-5.625 E-27

(b) Absolute error

Figure 7. Errors for test problem 7, example 7.1 with N=20

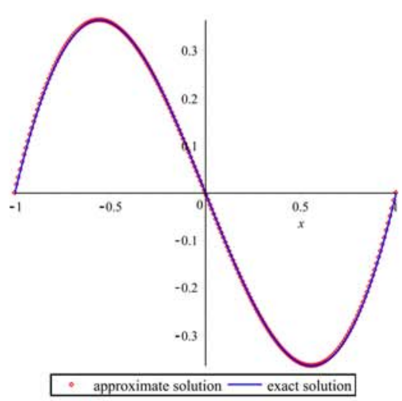


(a) Approximate and exact solutions LSE=3.947E-20

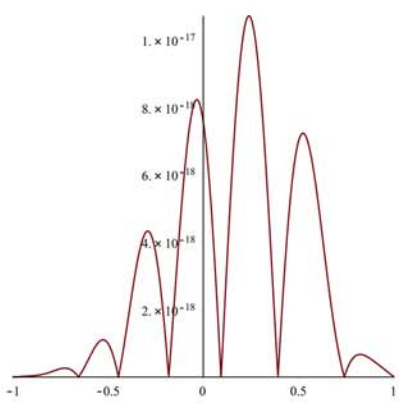


(b) Absolute error

Figure 8. Errors for test problem 7, example 7.2 with N=20

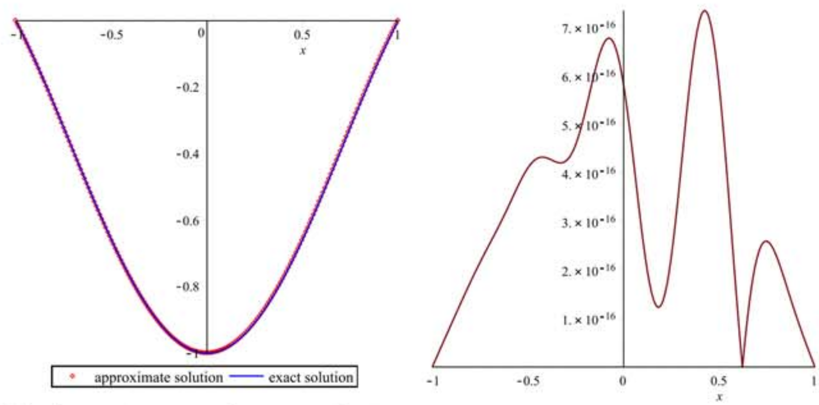


(a) Approximate and exact solutions LSE=1.000E-23



(b) Absolute error

Figure 9. Errors for test problem 7, example 7.3 with N=20



(a) Approximate and exact solutions LSE=2.000E-31

(b) Absolute error

Figure 10. Errors for test problem 7, example 7.4 with N=20

7. Conclusion

In this paper, a new method based on the least square method was given for solution of linear and nonlinear ordinary differential equations. In this method we used orthogonal rational Legendre functions, which are constructed from Legendre orthogonal polynomials, as basis functions. According to theorem 1 in section 4, the approximation of functions in $H^r_{\omega,T}(\Lambda)$ with orthogonal rational Legendre functions has an uniformly bounded error on the interval Λ.

To illustrate the accuracy and efficiency of our method, some well known equations such as Lane-Emden equation are solved. Comparing the numerical results with the results given in other references such as [17-22], [24-26] and [28] shows that the proposed method gives a more accurate solution in the form of a continuous function. Moreover, this method gives an accurate result by using only a few node points.

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