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Original Research Paper

On the Associated Prime Ideals and Support of Local Cohomology Modules on Systems of Ideals

M.Y. Sadeghi*
Payame Noor University

Z. Habibi
Payame Noor University

F. Shahid
Payame Noor University

Abstract. Let R be a commutative Noetherian ring and M an R-module. We begin by some results about the connection between the general local cohomology modules with respect to a system of ideals of R and an arbitrary Serre subcategory of R-modules.
Let $n = \dim R$ and $\mathfrak{a}$ be an ideal of R. We show that $\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a}) \bigcup (\bigcup_{i=1}^{n-j} \mathrm{Supp}_R^{j+i}(H^j_{\mathfrak{a}}(M)))$ for all $j \geq 0$. As a consequence, if R is a semi-local ring and M is a minimax R-module of dimension at most three, then the R-modules $H^j_{\mathfrak{a}}(M)$ have finite sets of associated prime ideals for all $j \geq 0$. Moreover, we give some results on the finiteness of the Bass numbers and the Betti numbers and cofiniteness of the ordinary local cohomology modules.

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*Corresponding Author

1 Introduction

Throughout this article, R is a commutative Noetherian ring with non-zero identity and M is an R-module. By $\mathbb{N}_0$, we mean the set of non-negative integers. Moreover, we use $\mathrm{Mod}(R)$ to denote the category of all R-modules and R-homomorphisms and $\mathrm{Max}(R)$ to denote the set of maximal ideals of R. Let Φ be a non-empty set of ideals of R. Recall that Φ is a system of ideals of R if, for any $\mathfrak{a}, \mathfrak{b} \in \Phi$, there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. For an R-module M and an ideal $\mathfrak{a}$ of R, the i-th local cohomology module of M with respect to $\mathfrak{a}$ is defined as

$$H^i_{\mathfrak{a}}(M) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{a}^n, M\right).$$

As a generalization of these modules, for a system of ideals Φ of R, Bijanzadeh [6] defined the submodule $\Gamma_\Phi(M)$ of M as follows:

$$\Gamma_\Phi(M) = \{x \in M \,|\, \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is a covariant, R-linear and left exact functor from $\mathrm{Mod}(R)$ to itself. The author [6] denoted the functor $\Gamma_\Phi(-)$ by $\mathrm{L}_\Phi(-)$ and called it as the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $H^i_\Phi(-)$. For an ideal $\mathfrak{a}$ of R, if $\Phi = \{\mathfrak{a}^i | i > 0\}$, then the functor $H^i_\Phi(-)$ coincides with the ordinary local cohomology functor $H^i_{\mathfrak{a}}(-)$. From now on, we refer to $H^i_\Phi(M)$ as the general local cohomology module. For any unexplained notation and terminology, we refer the reader to [5, 6, 7].

Grothendieck [10], proposed the following conjecture.
Grothendieck's Conjecture. If R is a Noetherian ring, then for any ideal $\mathfrak{a}$ of R and any finitely generated R-module M, the module $\mathrm{Hom}_R\big(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)\big)$ is finitely generated for all $i \geq 0$.

Hartshorne [11], showed that this conjecture is not true in general. Furthermore, he defined an R-module M to be $\mathfrak{a}$-cofinite if $\mathrm{Supp}(M) \subseteq \mathrm{Var}(\mathfrak{a})$ and $\mathrm{Ext}^j_R(R/\mathfrak{a}, M)$ is finitely generated for all $j \geq 0$.

Hartshorne refined Grothendieck's conjecture and he asked:
Hartshorne's Question. When the R-module $H^i_{\mathfrak{a}}(M)$ ia $\mathfrak{a}$-cofinite for all $i \geq 0$?

Among the basic problems which is discussed by Huneke [13], we are interested in determining the Artinianness and the finiteness of the set of associated primes of R-module $H^i_{\mathfrak{a}}(M)$. It is well known that if M is a finitely generated R-module of Krull dimension d, then $H^d_{\mathfrak{a}}(M)$ is Artinian for any ideal $\mathfrak{a}$ of R, (see [16, Proposition 5.1]). Moreover, in the case that R is local, Marley [15] showed that $\mathrm{Supp}_R\big(H^{d-1}_{\mathfrak{a}}(M)\big)$ is a finite set.

Concerning cofiniteness of local cohomology modules, Melkersson [16, Theorem 2.1] showed that if $(R, \mathfrak{m})$ is a local ring of dimension at most 2, then for every finitely generated R-module M, all local cohomology modules $H^i_{\mathfrak{a}}(M)$ are $\mathfrak{a}$-cofinite.

In this paper, we discuss about the Artinianness, finiteness of the support and the associated prime ideals, and cofiniteness of local cohomology modules. The finiteness of the Bass numbers and the Betti numbers will be also studied. This paper is organized as follows:

In Section 2, we give some further contributions to verify the membership of $H^n_{\Phi}(M)$ in an arbitrary Serre subcategory $\mathcal{S}$, for some $n \in \mathbb{N}$, (see for example Proposition 2.2 and Corollary 2.3). As important consequences, we give some results on Artianness of $H^{\dim M-1}_{\Phi}(M)$, when R is a local ring; and $H^{\dim M-1}_{\mathfrak{a}}(M)$, when R is an arbitrary Noetherian ring for an ideal $\mathfrak{a}$ of R (see Corollaries 2.8, 2.12, and Proposition 2.9).

In Section 3, we study finiteness of the support and the associated prime ideals of ordinary local cohomology modules. In Proposition 3.1, as one of the important results of this paper, we prove that if R is an arbitrary Noetherian ring of dimension n, then

$$\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a}) \cup (\cup_{i=1}^{n-j} \mathrm{Supp}_R^{j+i}(H^j_{\mathfrak{a}}(M))),$$

for any R-module M, any ideal $\mathfrak{a}$, and any $j \geq 0$, where $\overline{A}^*(\mathfrak{a})$ denotes $\bigcup_{n\geqslant 0} \mathrm{Ass}_R(R/\ \overline{\mathfrak{a}^n}\ )$ and $\overline{\mathfrak{a}^n}$ denotes the integral closure of $\mathfrak{a}^n$. As some applications of this, we get Corollary 3.3 and Proposition 3.6, which are generalizations of [15, Corollaries 2.4 and 2.7]. We also show that when R is a local ring of dimension n and M is a minimax module such that $H^n_{\mathfrak{a}}(M) \neq 0$, then $\mathrm{Supp}_R(H^{n-1}_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a})$ (see Corollary 3.4). Moreover, if R is a semi-local ring (a non-zero ring having only finitely many maximal ideals) of dimension n and M is a ZD-module, then $\mathrm{Supp}_R(\frac{H^{n-2}_{\mathfrak{a}}(M)}{\mathfrak{a}^j H^{n-2}_{\mathfrak{a}}(M)})$ is a finite set for all $j \geq 0$ (see Corollary 3.5).

In Section 4, we study the cofiniteness and the finiteness of the Bass numbers and the Betti numbers of ordinary local cohomology modules. Let R be a Noetherian ring, M be a finitely generated R-module with $\dim M \leq 3$, and x be a non-zerodivisor on M such that $x^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$. Then Proposition 4.4 shows that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite and consequently the Bass numbers and the Betti numbers of $H^i_{\mathfrak{a}}(M)$ are finite for all $i \geq 0$. In particular, all results hold when $\mathfrak{a}^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$. Corollary 4.5 shows that, the conditions of Proposition 4.4 are available. Finally, Theorem 4.7, as the last result in this paper, shows that if $(R, \mathfrak{m})$ is a regular local ring of dimension $d \leq 3$ and M is a minimax module, then $\operatorname{Ext}^j_R\left(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M)\right)$ is minimax for all $i, j \geq 0$.

2 General Local Cohomology Modules and Serre Subcategories

In this section, we study the membership of the general local cohomology modules in an arbitrary Serre subcategory and their common results for Artinianness of local cohomology modules. Minimax modules have been studied by Zink [23], Zöschinger [24, 25], and Rudlof [19]. Recall that an R-module M is a minimax module if there exists a finitely generated submodule N of M in which M/N is Artinian. According to [21], an R-module M is called an $\mathcal{AF}$ module if there exists an Artinian submodule A of M in which M/A is finitely generated. It is easy to see that the class of minimax modules contains the class of $\mathcal{AF}$ modules and it contains finitely generated and Artinian R-modules, as well. Finally, an R-module M is said to be an $\mathfrak{a}$-cominimax if the support of M is contained in $\operatorname{Var}(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is minimax for all $i \geq 0$. The concept of $\mathfrak{a}$-cominimax modules were introduced in [4] as a generalization of the important notion of $\mathfrak{a}$-cofinite modules.

We start with the following remark, which plays a main role for some results of this paper.

Remark 2.1. Let Φ be a system of ideals of R and let M be an R-module. Then following hold:

(i) If $\dim M = d$, then by Grothendieck's Vanishing Theorem, $H^i_{\mathfrak{a}}(M) = 0$, for all $i > d$ and all $\mathfrak{a} \in \Phi$. Hence, $H^i_{\Phi}(M) = 0$, for all $i > d$ by [6, Lemma 2.1].

(ii) If M is a minimax R-module, then there exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

of R-modules and R-homomorphisms, where N is a finitely generated module and A is an Artinian module. This induces the exact sequence

$$0 \to \Gamma_{\Phi}(N) \to \Gamma_{\Phi}(M) \to \Gamma_{\Phi}(A) \to H^1_{\Phi}(N) \to H^1_{\Phi}(M) \to 0,$$

and $H^i_{\Phi}(M) \cong H^i_{\Phi}(N)$ for all $i \geq 2$.

(iii) If M is an $\mathcal{AF}$-module, then there exists a short exact sequence

$$0 \to A \to M \to N \to 0,$$

of R-modules and R-homomorphisms, where A is an Artinian module and N is a finitely generated module. So, we get the exact sequence

$$0 \to \Gamma_{\Phi}(A) \to \Gamma_{\Phi}(M) \to \Gamma_{\Phi}(N) \to 0,$$

and $H^i_{\Phi}(M) \cong H^i_{\Phi}(N)$ for all $i \geq 1$.

As the first result of this paper, we give the following proposition which is a generalization of [3, Proposition 2.1] and also it will be useful to prove next results.

Proposition 2.2. *Let M be an R-module of finite dimension, Φ a system of ideals of R, and $\mathcal{S}$ a Serre subcategory of* $\mathrm{Mod}(R)$*. Assume that $n \in \mathbb{N}$ is such that $H^i_{\Phi}(M) \in \mathcal{S}$ for all $i > n$. Then $\frac{H^i_{\Phi}(M)}{x^j H^i_{\Phi}(M)} \in \mathcal{S}$ for all $x \notin \mathrm{Zdv}_R(M)$, all $i \geq n$, and all $j \geq 0$. Consequently, $H^n_{\Phi}(M) \in \mathcal{S}$ if and only if there exist $x \notin \mathrm{Zdv}_R(M)$ and $m \geq 0$ such that $x^m H^n_{\Phi}(M) \in \mathcal{S}$.*

Proof. Let x be an arbitrary non-zerodivisor on M. Note that it is enough to prove the desired result only for $i = n$ and $j = 1$. To do this, we use induction on $d := \dim M$. When $d = 0$, the result follows by Remark 2.1 (i). So, suppose that $d > 0$ and the result has been proved for all R-modules with dimension of less than d. Considering the exact sequence

$$H^i_\Phi(M) \overset{.x}{\to} H^i_\Phi(M) \to H^i_\Phi(M/xM) \to H^{i+1}_\Phi(M),$$

for all $i \geq 0$, and the assumption, we have $H^i_\Phi(M/xM) \in \mathcal{S}$ for all $i > n$. Thus, by the inductive hypothesis, we get

$$\frac{H^n_\Phi(M/xM)}{xH^n_\Phi(M/xM)} \in \mathcal{S}.$$

Now, the exact sequence

$$H^n_\Phi(M) \overset{.x}{\to} H^n_\Phi(M) \overset{\alpha}{\to} H^n_\Phi(M/xM) \overset{\beta}{\to} H^{n+1}_\Phi(M),$$

induces the following exact sequences:

$$H^n_\Phi(M) \overset{.x}{\to} H^n_\Phi(M) \to N := \mathrm{Im}\ \alpha \to 0$$

and

$$0 \to N \to H^n_\Phi(M/xM) \to K := \mathrm{Im}\ \beta \to 0.$$

Thus, the following sequences

$$\frac{H^n_\Phi(M)}{xH^n_\Phi(M)} \overset{.x}{\to} \frac{H^n_\Phi(M)}{xH^n_\Phi(M)} \to \frac{N}{xN} \to 0 \tag{1}$$

and

$$\mathrm{Tor}^R_1(\frac{R}{xR}, K) \to \frac{N}{xN} \to \frac{H^n_\Phi(M/xM)}{xH^n_\Phi(M/xM)} \to \frac{K}{xK} \to 0 \tag{2}$$

are both exact. From the exact sequence (1), we get

$$\frac{N}{xN} \cong \frac{H^n_\Phi(M)}{xH^n_\Phi(M)}.$$

On the other hand, by [3, Proposition 2.1], we have $\mathrm{Tor}^R_1(\frac{R}{xR}, K) \in \mathcal{S}$. Therefore $\frac{N}{xN} \in \mathcal{S}$ by the exact sequence (2) and this completes the proof. For the last part, apply the following short exact sequence:

$$0 \to x^m H^n_\Phi(M) \to H^n_\Phi(M) \to \frac{H^n_\Phi(M)}{x^m H^n_\Phi(M)} \to 0. \qquad \square$$

In [8], the authors introduced the concept of ZD-modules. An R-module M is said to be a ZD-module (zerodivisor-module) if for any submodule N of M, the set of zerodivisors of M/N is a union of finitely many prime ideals in $\mathrm{Ass}_R(M/N)$. For the properties of this modules, see [8]. As a consequence of Proposition 2.2, we obtain the following.

Corollary 2.3. *Let M be an ZD-module of finite dimension and Φ a system of ideals of R. Let $n \in \mathbb{N}$ be such that $H^i_\Phi(M) \in \mathcal{S}$ for all $i > n$. Then $\frac{H^i_\Phi(M)}{\mathfrak{a}^j H^i_\Phi(M)} \in \mathcal{S}$ for any $\mathfrak{a} \in \Phi$, all $i \geq n$ and all $j \geq 0$. Consequently, $H^n_\Phi(M) \in \mathcal{S}$ if and only if there exist $\mathfrak{a} \in \Phi$ and $m \geq 0$ such that $\mathfrak{a}^m H^n_\Phi(M) \in \mathcal{S}$.*

Proof. Let $\mathfrak{a} \in \Phi$. Since $H^i_\Phi(M) \cong H^i_\Phi(M/\Gamma_\Phi(M))$ for all $i > 0$, and $M/\Gamma_\Phi(M)$ is a Φ-torsion-free R-module, we may assume that $\Gamma_\Phi(M) = 0$ and so $\Gamma_\mathfrak{a}(M) = 0$. Since M is a ZD-module, the Prime Avoidance Theorem, follows that there exists $x \in \mathfrak{a}$ which $x \notin \mathrm{Zdv}_R(M)$ and so $\frac{H^i_\Phi(M)}{x^j H^i_\Phi(M)} \in \mathcal{S}$ for all $i \geq n$ and all $j \geq 0$, by Proposition 2.2. Now, the assertion follows immediately from the epimorphism $\dfrac{R}{x^j R} \to \dfrac{R}{\mathfrak{a}^j} \to 0$. $\square$

To achieve further results, we need Proposition 2.4, which is a generalization of [22, Proposition 3.1].

Proposition 2.4. ([20, Theorem 3.1]) *Let M be a finite dimensional R-module, Φ be a system of ideals of R, and $t \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$H^i_\Phi(M) = 0$, for all $i \geq t$;*

(ii) *$H^i_\Phi(M)$ is finitely generated for all $i \geq t$;*

(iii) *There exists $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \sqrt{(0 :_R H^i_\Phi(M))}$ for all $i \geq t$ (or equivalently, there exists $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} H^i_\Phi(M) = 0$ for all $i \geq t$).*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are clear.
For (iii) $\Rightarrow$ (i), assume that there exists $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \sqrt{(0 :_R H^i_\Phi(M))}$

for all $i \geq t$. So there exists a non-negative integer $m \in \mathbb{N}$ such that $\mathfrak{a}^m H^i_\Phi(M) = 0$ for all $i \geq t$. To prove the assertion, it is sufficient for us to prove that $H^t_\Phi(M) = 0$. On the other hand, as any R-module is direct limit of its finitely generated submodules, so we may assume that M is finitely generated R-module, by [5, Proposition 2.4]. We use induction on $d := \dim M$. When $d = 0$, it is clear that $H^i_\mathfrak{a}(M) = 0$ for all $\mathfrak{a} \in \Phi$ and all $i \geq t$. Now, suppose inductively that $d > 0$ and the result has been proved for all finitely generated R-modules of dimension smaller than d. Since, $H^i_\Phi(M) \cong H^i_\Phi(\frac{M}{\Gamma_\Phi(M)})$ for all $i \geq 1$, we may assume that $\Gamma_\Phi(M) = 0$ and therefore $\Gamma_\mathfrak{a}(M) = 0$. Hence there exists $x \in \mathfrak{a}$ which is a non-zerodivisor on M. Now, consider the following long exact sequence

$$\cdots \to H^t_\Phi(M) \overset{.x^m}{\to} H^t_\Phi(M) \to H^t_\Phi(\frac{M}{x^m M}) \to H^{t+1}_\Phi(M) \to \cdots . \quad (3)$$

Thus $\mathfrak{a} \subseteq \sqrt{(0 :_R H^t_\Phi(\frac{M}{x^m M}))}$ for all $i \geq t$, by [7, Lemma 9.1.1]. Since $\dim \frac{M}{x^m M} < d$, by induction hypothesis, we have $H^t_\Phi(\frac{M}{x^m M}) = 0$. Now, the long exact sequence (3) implies that $H^t_\Phi(M) = x^m H^t_\Phi(M)$. Therefore $H^t_\Phi(M) = 0$, as required. $\square$

Remark 2.5. By the proof of Proposition 2.4 (iii) $\Rightarrow$ (i), we can replace the condition (iii) with $Rx \subseteq \sqrt{(0 :_R H^i_\Phi(M))}$ for some $x \notin \mathrm{Zdv}_R(M)$ and for all $i \geq t$.

The following corollary is a generalization of [2, Proposition 2.3] for a system of ideals of R.

Corollary 2.6. *Let M be a finite dimensional R-module, Φ be a system of ideals of R and $t \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *$H^i_\Phi(M)$ is Artinian for all $i \geq t$;*

(ii) *$H^i_\Phi(M)$ is minimax for all $i \geq t$;*

(iii) *$x^m H^i_\Phi(M)$ is minimax for some non-zerodivisor x on M, some $n \in \mathbb{N}$, and all $i \geq t$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) are obvious.
(iii) $\Rightarrow$ (ii) This follows easily from Proposition 2.2.

(ii) $\Rightarrow$ (i) First, we show that $\mathrm{Supp}_R(H^i_\Phi(M)) \subseteq \mathrm{Max}(R)$ for all $i \geq t$. For this purpose, let $\mathfrak{p} \in \mathrm{Spec}(R) \setminus \mathrm{Max}(R)$ and $i \geq t$. By assumption, there exists a short exact sequence

$$0 \longrightarrow N \longrightarrow H^i_\Phi(M) \longrightarrow A \longrightarrow 0, \tag{4}$$

in which N is a Noetherian module and A is an Artinian module. It is easy to see that $(H^i_\Phi(M))_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module for all $i \geq t$. Then by Proposition 2.4, $(H^i_\Phi(M))_\mathfrak{p} = 0$ for all $i \geq t$. Hence $\mathrm{Supp}_R(H^i_\Phi(M)) \subseteq \mathrm{Max}(R)$ for all $i \geq t$. Considering the exact sequence (4) and since N is Noetherian, there exists a finite set $\{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\} \subseteq \mathrm{Max}(R)$ such that

$$\mathrm{Var}(0 \underset{R}{:} N) = \mathrm{Supp}_R(N) = \mathrm{Ass}_R(N) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_n\}.$$

This deduces that N is Artinian and so $H^i_\Phi(M)$ is Artinian for all $i \geq t$. □

Now, we are going to establish some results on top general local cohomology modules.

Corollary 2.7. *Let M be an R-module of dimension $d \geq 1$ and Φ be a system of ideals of R. Then the following statements hold:*

(i) *$H^d_\Phi(M) = x^j H^d_\Phi(M)$ for any non-zerodivisor x on M and all $j \geq 0$.*

(ii) *$H^d_\Phi(M) = 0$ if and only if there exists a non-zerodivisor x on M and $m \in \mathbb{N}$ such that $x^m H^d_\Phi(M)$ is a finitely generated R-module.*

(iii) *$H^d_\Phi(M)$ is an Artinian R-module if and only if there exist a non-zerodivisor x on M and $m \in \mathbb{N}$ such that $x^m H^d_\Phi(M)$ is a minimax R-module.*

Proof. (i) Use Remark 2.1 (i) and Proposition 2.2 for the class of zero modules.
(ii) It follows from part (i) and Proposition 2.4 for the class of finitely generated modules.
(iii) Apply Corollary 2.6 and Proposition 2.2 for the class of Artinian modules. □

Corollary 2.8. *Let $(R, \mathfrak{m})$ be a local ring, M be a minimax R-module of dimension d, and Φ be a system of ideals of R. Then*

(i) *$H^d_\Phi(M)$ is an Artinian R-module.*

(ii) *If $d > 1$, then $\dfrac{H^{d-1}_\Phi(M)}{x^j H^{d-1}_\Phi(M)}$ is an Artinian R-module for all $x \notin \mathrm{Zdv}_R(M)$ and all $j \geq 0$. In particular, $H^{d-1}_\Phi(M)$ is Artinian if and only if $x^m H^{d-1}_\Phi(M)$ is minimax for some $x \notin \mathrm{Zdv}_R(M)$ and some $m \in \mathbb{N}$.*

(iii) *Suppose that $d > 2$. If there exists $x \notin \mathrm{Zdv}_R(M)$ and $m \in \mathbb{N}$ such that $x^m H^{d-1}_\Phi(M)$ is minimax, then $\dfrac{H^{d-2}_\Phi(M)}{x^j H^{d-2}_\Phi(M)}$ is Artinian for all $j \geq 0$.*

Proof. (i) It follows easily from Remark 2.1 (ii) and [9, Theorem 2.6].
(ii) The result follows by part (i), Corollary 2.6, and Proposition 2.2.
(iii) This immediately follows from part (ii), Corollary 2.6, and Proposition 2.2. □

Following, we present some applications of the previous results to the ordinary local cohomology modules.

Proposition 2.9. *Let M be a minimax R-module of dimension d and $\mathfrak{a}$ be an ideal of R. Then*

(i) *$H^d_\mathfrak{a}(M)$ is an $\mathfrak{a}$-cofinite Artinian module.*

(ii) *If $d > 1$, then $\dfrac{H^{d-1}_\mathfrak{a}(M)}{\mathfrak{a}^j H^{d-1}_\mathfrak{a}(M)}$ has finite length for all $j \geq 0$.*

(iii) *If $d > 2$ and $x^m H^{d-1}_\mathfrak{a}(M)$ is minimax module, for some $x \notin \mathrm{Zdv}_R(M)$ and some $m \in \mathbb{N}$, then $\dfrac{H^{d-2}_\mathfrak{a}(M)}{x^j H^{d-2}_\mathfrak{a}(M)}$ is an Artinian module for all $j \geq 0$. Consequently, $\dfrac{H^{d-2}_\mathfrak{a}(M)}{\mathfrak{a}^j H^{d-2}_\mathfrak{a}(M)}$ is Artinian for all $j \geq 0$.*

Proof. (i) It follows easily using Remark 2.1 (ii) for $\Phi = \{\mathfrak{a}^i | i \geq 0\}$ and [17, Proposition 5.1].
(ii) Set $X := \dfrac{H^{d-1}_\mathfrak{a}(M)}{\mathfrak{a}^j H^{d-1}_\mathfrak{a}(M)}$. In view of part (i), $H^i_\mathfrak{a}(M)$ is Artinian $\mathfrak{a}$-cofinite R-module for all $i > d-1$. Hence, using Corollary 2.3 for $\Phi = \{\mathfrak{a}^i | i \geq 0\}$ and the class of $\mathfrak{a}$-cofinite minimax modules (which is a

Serre subcategory by [17, Corollary 4.4]), we get that X is an $\mathfrak{a}$-cofinite for all $j \geq 0$. So, $(0 :_X \mathfrak{a}) = X$ is a finitely generated R-module. On the other hand, using Corollary 2.3 for the class of Artinian R-modules, X is Artinian and so has finite length.
(iii) By assumption, part (i) and Corollary 2.6, we deduce that $H^i_{\mathfrak{a}}(M)$ is an Artinian R-module for all $i > d-2$. Now, the assertion follows from Proposition 2.2 and Corollary 2.3. □

Proposition 2.10. *Let $\mathcal{S}$ be a Serre subcategory of* $\mathrm{Mod}(R)$*. Then the following hold:*

(i) *$\mathcal{S} \neq 0$ if and only if $R/\mathfrak{m} \in \mathcal{S}$ for some $\mathfrak{m} \in \mathrm{Max}(R)$.*

(ii) *Let $\mathcal{FL}$ be the class of finite length R-modules. Then $\mathcal{FL} \subseteq \mathcal{S}$ if and only if $R/\mathfrak{m} \in \mathcal{S}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$.*

(iii) *If $(R, \mathfrak{m})$ is a local ring and $\mathcal{S} \neq 0$, then $\mathcal{FL} \subseteq \mathcal{S}$.*

(iv) *If $R/\mathfrak{m} \in \mathcal{S}$, for any $\mathfrak{m} \in \mathrm{Max}(R)$ and M is a finitely generated or an Artinian R-module, then $\mathrm{Ext}^j_R(R/\mathfrak{m}, M) \in \mathcal{S}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$ and all $j \geq 0$.*

(v) *If $R/\mathfrak{m} \in \mathcal{S}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$, and M is a minimax R-module, then $\mathrm{Ext}^j_R(R/\mathfrak{m}, M) \in \mathcal{S}$ for any $\mathfrak{m} \in \mathrm{Max}(R)$ and all $j \geq 0$.*

(vi) *If $(R, \mathfrak{m})$ is a local ring, $\mathcal{S} \neq 0$, and M is a minimax R-module, then $\mathrm{Ext}^j_R(R/\mathfrak{m}, M) \in \mathcal{S}$ for all $j \geq 0$.*

Proof. (i) ($\Rightarrow$) Since $\mathcal{S} \neq 0$, there exists a non-zero R-module L in $\mathcal{S}$. Let $0 \neq x \in L$. Then $(0 :_R x) \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{Max}(R)$. Now, since $Rx \in \mathcal{S}$, the assertion follows from the natural epimorphism $Rx \cong R/(0 :_R x) \to R/\mathfrak{m}$.
($\Leftarrow$) It is clear.
(ii) Let $\mathcal{FL} \subseteq \mathcal{S}$ and $\mathfrak{m} \in \mathrm{Max}(R)$. Since $\ell_R(R/\mathfrak{m}) < \infty$, $R/\mathfrak{m} \in \mathcal{S}$. Conversely, let $N \in \mathcal{FL}$ and set $l := \ell_R(N)$. Hence, there is a chain of R-submodules of N as follows:

$$0 = N_0 \subseteq N_1 \subseteq ... \subseteq N_l = N,$$

in which $N_j/N_{j-1} \cong R/\mathfrak{m}$ for all $1 \leq j \leq l$ and some $\mathfrak{m} \in \mathrm{Max}(R)$. Now, the assertion is followed by induction on l.
(iii) The result follows from parts (i) and (ii).

(iv) Let $\mathfrak{m} \in \operatorname{Max}(R)$ and $j \geq 0$. Let M be a finitely generated or an Artinian R-module. Since $\operatorname{Ext}_R^j(R/\mathfrak{m}, M)$ is annihilated by $\mathfrak{m}$, it has finite length. Therefore it belongs to $\mathcal{S}$, by part (ii).
(v) Since M is minimax, there exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is a finitely generated R-module and A is Artinian R-module. This induces the long exact sequence

$$\cdots \to \operatorname{Ext}_R^j(R/\mathfrak{m}, N) \to \operatorname{Ext}_R^j(R/\mathfrak{m}, M) \to \operatorname{Ext}_R^j(R/\mathfrak{m}, A) \to \cdots .$$

Now, apply part (iv).
(vi) The result is a consequence of parts (i) and (v). □

Corollary 2.11. *Let $\mathcal{S}$ be a Serre subcategory of* $\operatorname{Mod}(R)$*, M be a minimax R-module of dimension $d \neq 1$, and $\mathfrak{a}$ be an ideal of R. Then the following hold:*

(i) *If $R/\mathfrak{m} \in \mathcal{S}$ for any $\mathfrak{m} \in \operatorname{Max}(R)$, then $\dfrac{H_{\mathfrak{a}}^{d-1}(M)}{\mathfrak{a}^j H_{\mathfrak{a}}^{d-1}(M)} \in \mathcal{S}$ for all $j \geq 0$.*

(ii) *If $(R, \mathfrak{m})$ be a local ring and $\mathcal{S} \neq \{0\}$, then $\dfrac{H_{\mathfrak{a}}^{d-1}(M)}{\mathfrak{a}^j H_{\mathfrak{a}}^{d-1}(M)} \in \mathcal{S}$ for all $j \geq 0$.*

Proof. Each both parts immediately follow from Propositions 2.9 and 2.10. □

Corollary 2.12. *Let M be a minimax R-module of dimension $d \neq 1$. Then $H_{\mathfrak{a}}^{d-1}(M)$ is an Artinian R-module if and only if there exists a non-zerodivisor x on M and $n \in \mathbb{N}$ such that $x^n H_{\mathfrak{a}}^{d-1}(M)$ is minimax.*

Proof. Use Corollary 2.6 for $t = d - 1$ and $\Phi = \{\mathfrak{a}^i | i \geq 0\}$.
□

Remark and Examples 2.13.

(i) The assertions of Corollary 2.12, may not be hold when $\dim M = 1$. For example, let M be a finitely generated $\mathfrak{a}$-torsion R-module of dimension d=1. Then, it is clear that $H_{\mathfrak{a}}^{d-1}(M) = M$ is not Artinian.

(ii) In Proposition 2.9 (iii), $\dfrac{H^{d-2}_{\mathfrak{a}}(M)}{\mathfrak{a}^j H^{d-2}_{\mathfrak{a}}(M)}$ does not have necessary finite length. To see this, let k be a field and $R = k[[X_1, X_2, X_3, X_4]], I = \langle X_1, X_2\rangle, J = \langle X_3, X_4\rangle$, and $\mathfrak{a} = I \cap J$. By the Mayer-Vietoris sequence, we get $H^2_{\mathfrak{a}}(R) \cong H^2_I(R) \oplus H^2_J(R)$ and so,

$$\frac{H^2_{\mathfrak{a}}(R)}{\mathfrak{a}H^2_{\mathfrak{a}}(R)} \cong \frac{H^2_I(R)}{\mathfrak{a}H^2_I(R)} \oplus \frac{H^2_J(R)}{\mathfrak{a}H^2_J(R)} \cong H^2_I(R/\mathfrak{a}) \oplus H^2_J(R/\mathfrak{a}).$$

Since $H^2_I(R/\mathfrak{a}) \neq 0$, by the Lichtenbaum-Hartshorne Vanishing Theorem [7, Theorem 8.2.1], we have $\operatorname{cd}(I, R/\mathfrak{a}) = 2$. On the other hand, $H^2_I(R/\mathfrak{a})$ is not a finitely generated R-module by [12, Remark 2.5]. Thus, $\dfrac{H^2_{\mathfrak{a}}(R)}{\mathfrak{a}H^2_{\mathfrak{a}}(R)}$ is not, too. However, note that it is an Artinian R-module.

Corollary 2.14. *Let M be a minimax R-module of dimension $d \neq 1$ and $\mathfrak{a} \subseteq Jac(R)$ be an ideal of R. Suppose that for some non-negative integer m, the R-module $\mathfrak{a}^m H^{d-1}_{\mathfrak{a}}(M)$ is finitely generated. Then the following hold:*

(i) *$H^{d-1}_{\mathfrak{a}}(M)$ has finite length and so there exists a non-negative integer n, such that $\mathfrak{a}^n H^{d-1}_{\mathfrak{a}}(M) = 0$.*

(ii) *If $\mathcal{S}$ is a Serre subcategory of $\operatorname{Mod}(R)$ such that $R/\mathfrak{m} \in \mathcal{S}$ for all $\mathfrak{m} \in \operatorname{Max}(R)$, then $H^{d-1}_{\mathfrak{a}}(M) \in \mathcal{S}$.*

(iii) *If $(R, \mathfrak{m})$ is a local ring and $\mathcal{S} \neq \{0\}$, then $H^{d-1}_{\mathfrak{a}}(M) \in \mathcal{S}$.*

Proof. (i) Let $\mathfrak{a}^m H^{d-1}_{\mathfrak{a}}(M)$ be a finitely generated R-module for some non-negative integer m. Consider the exact sequence

$$0 \to \mathfrak{a}^m H^{d-1}_{\mathfrak{a}}(M) \to H^{d-1}_{\mathfrak{a}}(M) \to \frac{H^{d-1}_{\mathfrak{a}}(M)}{\mathfrak{a}^m H^{d-1}_{\mathfrak{a}}(M)} \to 0.$$

By Proposition 2.9 (ii) and Corollary 2.12, the R-module $H^{d-1}_{\mathfrak{a}}(M)$ has finite length.
(ii) It follows from part (i) and Proposition 2.10 (ii) .
(iii) This follows immediately by part (i) and Proposition 2.10 (iii). □

3 Finiteness of Support and Associated Prime Ideals

As it is mentioned in the Introduction, for an ideal $\mathfrak{a}$ of R, let $\overline{A}^*(\mathfrak{a})$ denote $\bigcup\limits_{n\geqslant 0} \mathrm{Ass}_R(R/\ \overline{\mathfrak{a}^n}\ )$, where $\overline{\mathfrak{a}^n}$ denotes the integral closure of $\mathfrak{a}^n$. Ratliff in [18] showed that, if R is Noetherian, then $\overline{A}^*(\mathfrak{a})$ is a finite set for all ideal $\mathfrak{a}$. Using this idea, Marley in [15, Corollary 2.7] showed that if $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module of dimension at most three, then the set of $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M))$ is finite for each ideal $\mathfrak{a}$ of R and all $i \geq 0$.

In this section, the first result will play a crucial role. Consequently, we obtain a generalization of [15, Corollary 2.7], whenever R is a semi-local ring and M is a minimax R-module. For an R-module M and $j \geq 0$, we set $\mathrm{Supp}^j_R(M) := \{\mathfrak{p} \in \mathrm{Supp}_R(M) |\ \mathrm{ht}\mathfrak{p} = j\}$.

Proposition 3.1. *Let* $\dim R = n$. *Then*

$$\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a}) \bigcup (\bigcup_{i=1}^{n-j} \mathrm{Supp}^{i+j}_R(H^j_{\mathfrak{a}}(M))),$$

for all R*-modules* M*, all ideals* $\mathfrak{a}$ *of* R*, and all* $j \geq 0$*.*

Proof. Let $j \geq 0$ and $\mathfrak{p} \in \mathrm{Supp}_R(H^j_{\mathfrak{a}}(M))$. Hence $\mathrm{ht}\mathfrak{p} \geq j$, by [7, Lemma 6.3.1]. Now, if $\mathrm{ht}\mathfrak{p} > j$, then $\mathfrak{p} \in \bigcup_{i=1}^{n-j} \mathrm{Supp}^{j+i}_R(H^j_{\mathfrak{a}}(M))$ and when $\mathrm{ht}\mathfrak{p} = j$, we get $\mathfrak{p} \in \mathrm{Supp}^j_R(H^j_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a})$, by [15, Proposition 2.3]. □

Remark 3.2. For an R-module M and $j \geq 0$, set

$$(\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)))_{<j} := \{\mathfrak{p} \in \mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) |\ \mathrm{ht}\mathfrak{p} < j\}$$

and

$$(\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)))_{\geq j} := \{\mathfrak{p} \in \mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) |\ \mathrm{ht}\mathfrak{p} \geq j\}.$$

In view of the proof of Proposition 3.1, we have $\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)) = (\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)))_{\geq j}$. Therefore $\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M))$ is a finite set if and only if $(\mathrm{Supp}_R(H^j_{\mathfrak{a}}(M)))_{>j}$ is a finite set.

An immediate consequence of Proposition 3.1 is the following, which is a generalization of [15, Corollaries 2.4 and 2.5].

Corollary 3.3. *Let R be a semi-local ring of dimension n. Then for any ideal $\mathfrak{a}$ of R and any R-module M, $\mathrm{Supp}_R(H^n_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a})$ and $\mathrm{Supp}_R(H^{n-1}_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a}) \cup \mathrm{Max}(R)$. Consequently, $\mathrm{Ass}_R(H^n_{\mathfrak{a}}(M))$ and $\mathrm{Ass}_R(H^{n-1}_{\mathfrak{a}}(M))$ are finite sets. In particular, the assertion holds in any local ring R.*

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a local ring of dimension n, $\mathfrak{a}$ be an ideal of R, and M be a minimax R-module such that $H^n_{\mathfrak{a}}(M) \neq 0$. Then $\mathrm{Supp}_R(H^{n-1}_{\mathfrak{a}}(M)) \subseteq \overline{A}^*(\mathfrak{a})$.*

Proof. According to Corollary 2.7 (i), $H^n_{\mathfrak{a}}(M)$ is an Artinian R-module and so $\mathrm{Supp}_R(H^n_{\mathfrak{a}}(M)) = \{\mathfrak{m}\}$. Now, the assertion follows from Corollary 3.3. □

Corollary 3.5. *Let R be a semi-local ring of dimension n and M be a ZD-module. Then for all $j \geq 0$, $\mathrm{Supp}_R(\frac{H^{n-2}_{\mathfrak{a}}(M)}{\mathfrak{a}^j H^{n-2}_{\mathfrak{a}}(M)})$ is a finite set.*

Proof. By Corollary 3.3, $H^i_{\mathfrak{a}}(M)$ has finite support for all $i > n-2$. Now, the assertion is deduced from Corollary 2.3 for the class of R-modules with finite support. □

The following, as the last result of this section is a generalization of [15, Corollary 2.7], in the case that R is a semi-local ring and M is a minimax module.

Proposition 3.6. *Let R be a semi-local ring and M a minimax R-module of dimension at most three. Then $\mathrm{Ass}_R(H^i_{\mathfrak{a}}(M))$ is a finite set for all $i \geq 0$.*

Proof. We first consider M as a finitely generated R-module. Therefore, by replacing R with $R/(0 :_R M)$, we can assume that $\dim R \leq 3$. For $i = 2, 3$, the assertion follows from Corollary 3.3. For $i = 0, 1$, the result is also clear from [15, Proposition 1.1]. Now, suppose M is a minimax R-module. Then, for $i = 0, 2, 3$, the result follows easily from Remark 2.1 (ii). Also, for $i = 1$, the assertion follows from the exact sequence $\Gamma_{\mathfrak{a}}(A) \xrightarrow{g} H^1_{\mathfrak{a}}(N) \xrightarrow{f} H^1_{\mathfrak{a}}(M) \to 0$, which is mentioned in Remark 2.1 (ii), and the fact that $\mathrm{Ass}_R(H^1_{\mathfrak{a}}(M)) \subseteq \mathrm{Ass}_R(H^1_{\mathfrak{a}}(N)) \cup \mathrm{Supp}_R(\mathrm{Ker} f)$. □

4 Finiteness of Bass Numbers and Betti Numbers

In this section, we show that Hartshorne's question is true in some special cases. First of all, in Proposition 4.2, we show that these discussions are established for minimax modules of dimension at most 2, which is a summary of main results in [14]. To this end, we begin by the following lemma.

Lemma 4.1. *Let M be a finitely generated R-module such that* $\dim M \leq 2$. *Then the R-module $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.*

Proof. This is clear for $i = 0$. Moreover, for $i \geq 2$, the assertion follows from [17, Proposition 5.1] and Grothendieck's Vanishing Theorem. Finally, for $i = 1$, we apply [17, Proposition 3.11]. □

Proposition 4.2. *Let M be a minimax R-module and suppose that one of the following cases holds:*

(a) $\dim M \leqslant 2$,

(b) $\dim R \leqslant 2$,

(c) $\dim(R/\mathfrak{a}) = 1$,

(d) $\mathrm{cd}(\mathfrak{a}, M) \leqslant 1$,

(e) $\mathrm{cd}(\mathfrak{a}, R) \leqslant 1$.

Then

(i) *$H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 1$ and it is $\mathfrak{a}$-cominimax for all $i \geq 0$. Thus, $\mathrm{Ext}^j_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is a minimax R- module for all $i, j \geq 0$.*

(ii) *$\mathrm{Tor}^R_j(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is a finitely generated R-module for all $j \geq 0$ and all $i \geq 1$. Thus, $\mathrm{Tor}^R_j(R/\mathfrak{a}, H^i_{\mathfrak{a}}(M))$ is a minimax R- module for all $i, j \geq 0$.*

(iii) *The Bass numbers and the Betti numbers of $H^i_{\mathfrak{a}}(M)$ are finite for all $i \geq 0$.*

Moreover, if $\mathrm{Hom}_R(R/\mathfrak{a}, M)$ is finitely generated, then all of the statements hold for all $i, j \geq 0$.

Proof. (i) Let the situation be as in (a). In view of the notation of Remark 2.1 (ii), we have the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(N) \to \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(A) \to H^1_{\mathfrak{a}}(N) \xrightarrow{f} H^1_{\mathfrak{a}}(M) \to 0,$$

where N is a finitely generated module and $H^i_{\mathfrak{a}}(M) \cong H^i_{\mathfrak{a}}(N)$ for all $i \geq 2$. Hence, by lemma 4.1, $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 2$. Now, it remains to prove the result for $i = 1$. To do this, set $L := \mathrm{Ker} f$. Again, by Lemma 4.1, $H^1_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite. Thus $(0 :_L \mathfrak{a})$ has finite length. Now, in view of [17, Proposition 4.1], we deduce that L is $\mathfrak{a}$-cofinite. On the other hand, the long exact sequence

$$\cdots \to \mathrm{Ext}^i_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(N)) \to \mathrm{Ext}^i_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(M)) \to \mathrm{Ext}^{i+1}_R(R/\mathfrak{a}, L) \to \cdots,$$

yields that $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite. By the fact that M is minimax, we conclude that $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cominimax for all $i \geq 0$.
In the case (b), the assertion clearly holds, because $\dim M \leq \dim R \leq 1$.
In the case (c), since $\dim R/\mathfrak{a} = 1$, the claim concludes by [14, Theorem 2.4].
In the case (d), if $\mathrm{cd}(\mathfrak{a}, M) = 0$, then there is nothing to prove. If $\mathrm{cd}(\mathfrak{a}, M) = 1$, then, $H^i_{\mathfrak{a}}(M) = 0$ for all $i \geq 2$. So, it is sufficient to verify the assertion for $H^1_{\mathfrak{a}}(M)$. To do this, note that $\mathrm{cd}(\mathfrak{a}, N) \leq \mathrm{cd}(\mathfrak{a}, M) \leq 1$, by Remark 2.1 (ii). If $\mathrm{cd}(\mathfrak{a}, N) = 0$, then $H^1_{\mathfrak{a}}(M) = 0$. Let $\mathrm{cd}(\mathfrak{a}, N) = 1$. Similar to the method of the proof of case (a), it is easy to see that $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite.
Finally, in the case (e), the assertion follows from the fact that $\mathrm{cd}(\mathfrak{a}, M) \leq \mathrm{cd}(\mathfrak{a}, R)$.
(ii) This follows from part (i) and [17, Theorem 2.5].
(iii) This follows from parts (i), (ii), and the fact that the Bass numbers and the Betti numbers of the minimax modules are finite. □

Recall that an R-module M is said locally minimax, if $M_{\mathfrak{m}}$ is minimax for any $\mathfrak{m} \in \mathrm{Max}(R)$. It is clear that the class of finitely generated, Artinian, minimax and $\mathcal{AF}$ modules, whole are locally minimax.

Proposition 4.3. *Let M be an $\mathcal{AF}$ module and $\mathfrak{a}$ be an ideal of R. Let x be a non-zerodivisor on M such that $x^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$. Then $H^1_{\mathfrak{a}}(M)$ is minimax and so the Bass numbers and* *the Betti numbers of $H^i_{\mathfrak{a}}(M)$ are finite for all $i \leq 1$. In particular, the results hold when $\mathfrak{a}^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$.*

Proof. Let $x \notin \mathrm{Zdv}_R(M)$ and $m \in \mathbb{N}_0$ be such that $x^m H^1_{\mathfrak{a}}(M)$ is locally minimax. By Remark 2.1 (iii), we may assume that M is a finitely generated R-module and $\mathrm{grade}(\mathfrak{a}, M) \geq 1$. Let E be the injective envelope of M and put $T := E/M$. Then $\Gamma_{\mathfrak{a}}(E) = 0$. Now, by the exact sequence $0 \to M \to E \to T \to 0$, we have $\mathrm{Hom}_R(R/\mathfrak{a}, T) \cong \mathrm{Ext}^1_R(R/\mathfrak{a}, M)$ and $\Gamma_{\mathfrak{a}}(T) \cong H^1_{\mathfrak{a}}(M)$. Since $(0 :_T \mathfrak{a}) = (0 :_{\Gamma_{\mathfrak{a}}(T)} \mathfrak{a}) \cong (0 :_{H^1_{\mathfrak{a}}(M)} \mathfrak{a})$ is finitely generated, $(0 :_{x^m H^1_{\mathfrak{a}}(M)} (M)$ is finitely generated, too. Thus $x^m H^1_{\mathfrak{a}}(M)$ is minimax, by [2, Theorem 2.6]. Now, from the long exact sequence

$$\cdots \to \Gamma_{\mathfrak{a}}(M/xM) \to H^1_{\mathfrak{a}}(M) \to x^m H^1_{\mathfrak{a}}(M) \to \cdots,$$

we conclude that $H^i_{\mathfrak{a}}(M)$ is minimax and so the Bass numbers and the Betti numbers of $H^i_{\mathfrak{a}}(M)$ are finite for all $i \leq 1$, by Proposition 2.10 (v) and [17, Theorem 2.1]. □

Melkersson in [16, Theorem 2.2] showed that Grothendieck's conjecture and Hartshorne's question is not true in general for the rings of Krull dimensions 3, even if the ring is local. The next result is about $\mathfrak{a}$-cofiniteness of local cohomology modules over Noetherian rings (not necessarily local) and finitely generated modules of Krull dimensions at most 3.

Proposition 4.4. *Let R be a Noetherian ring and M be a finitely generated R-module with $\dim M \leq 3$. Let x be a non-zerodivisor on M such that $x^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$. Then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite and so the Bass numbers and the Betti numbers of $H^i_{\mathfrak{a}}(M)$ are finite for all $i \geq 0$. In particular, the results hold when $\mathfrak{a}^m H^1_{\mathfrak{a}}(M)$ is locally minimax for some $m \in \mathbb{N}_0$.*

Proof. By Proposition 4.2, we may assume that $\dim M = 3$. By the same method of the proof of Proposition 4.3, we get $(0 :_{H^1_{\mathfrak{a}}(M)} \mathfrak{a})$ is finitely generated. Hence, $H^1_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite minimax, by [17, Proposition 4.3] and Proposition 4.3. Consequently, $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for $i \neq 2$, by [17, Proposition 5.1] and Grothendieck's Vanishing Theorem.

Now, the assertion follows from [17, Proposition 3.11]. □

The next corollary shows that, the conditions of Proposition 4.4 are available.

Corollary 4.5. *Let M be a finitely generated R-module with* $\dim M \leq 3$. *Then $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$ if one of the following conditions holds:*

(i) $\mathrm{Supp}_R(\mathrm{Ext}^i_R(R/\mathfrak{a}, M)) \subseteq \mathrm{Max}(R)$ *for* $i = 0, 1$;

(ii) $\mathrm{Supp}_R(\mathrm{Ext}^1_R(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M))) \subseteq \mathrm{Max}(R)$;

(iii) $\mathrm{Supp}_R(\mathrm{Ext}^1_R(R/\mathfrak{a}, M))$ *and* $\mathrm{Supp}(\mathrm{Ext}^2_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)))$ *are contained in* $\mathrm{Max}(R)$.

Proof. In view of Proposition 4.4, it is enough to show that $H^1_{\mathfrak{a}}(M)$ is an Artinian module.

First, assume that the condition (i) occurs. Since $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is a finitely generated module and its support is contained in $\mathrm{Max}(R)$, implies that $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is an Artinian module for $i = 0, 1$. Now, we show that $H^1_{\mathfrak{a}}(M)$ is Artinian. By [7, Lemma 2.1.1], we can assume that $\Gamma_{\mathfrak{a}}(M) = 0$. Let E be the injective envelope of M and put $T := E/M$. Then $\Gamma_{\mathfrak{a}}(E) = 0$. Therefore the exact sequence $0 \to M \to E \to T \to 0$, implies that $\Gamma_{\mathfrak{a}}(T) \cong H^1_{\mathfrak{a}}(M)$ and $\mathrm{Hom}_R(R/\mathfrak{a}, T) \cong \mathrm{Ext}^1_R(R/\mathfrak{a}, M)$. It follows that $(0 :_T \mathfrak{a})$ and thus $(0 :_{\Gamma_{\mathfrak{a}}(T)} \mathfrak{a})$ are Artinian. Now, the assertion follows from [7, Theorem 7.1.2].

Now, let the condition (ii) happens. Set $X := M/\Gamma_{\mathfrak{a}}(M)$. By [7, Corollary 2.1.7 and Lemma 2.1.1], $t := \mathrm{grade}(\mathfrak{a}, X) \geq 1$ and $H^1_{\mathfrak{a}}(M) \cong H^1_{\mathfrak{a}}(X)$. If $t > 1$, then there is nothing to prove. Let $t = 1$. Hence, similar to the method of the proof of part (i), we get

$$\mathrm{Ext}^1_R(R/\mathfrak{a}, X) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(X)) \cong \mathrm{Hom}_R(R/\mathfrak{a}, H^1_{\mathfrak{a}}(M)).$$

Now, Artinianness of $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ and [7, Theorem 7.1.2], imply that $H^1_{\mathfrak{a}}(M)$ is an Artinian module.

Finally, in the case (iii), we use the exact sequence

$$\mathrm{Ext}^1_R\left(R/\mathfrak{a}, M\right) \to \mathrm{Ext}^1_R\left(R/\mathfrak{a}, M/\Gamma_{\mathfrak{a}}(M)\right) \to \mathrm{Ext}^2_R\left(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(M)\right),$$

and part (ii). □

Theorem 4.7 as the last result of this paper, is a generalization of [1, Theorem 2.12] for minimax modules. To prove it, we need Lemma 4.6, as follows.

Lemma 4.6. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \leq 3$ and M be a finitely generated R-module. Then $\operatorname{Ext}_R^j\big(R/\mathfrak{a}, H_\mathfrak{a}^i(M)\big)$ is minimax for all $i, j \geq 0$.*

Proof. As $\operatorname{Ext}_R^j\big(R/\mathfrak{a}, H_\mathfrak{a}^i(M)\big)$ is $\mathfrak{a}$-torsion module for all $i, j \geq 0$, the assertion follows easily from [1, Theorem 2.12] and [4, Remark 2.2 (ii)]. □

Theorem 4.7. *Let $(R, \mathfrak{m})$ be a regular local ring of dimension $d \leq 3$ and let M be a minimax R-module. Then $\operatorname{Ext}_R^j\big(R/\mathfrak{a}, H_\mathfrak{a}^i(M)\big)$ is minimax for all $i, j \geq 0$.*

Proof. Using the notations of Remark 2.1 (i), we get the exact sequence

$$0 \to \Gamma_\mathfrak{a}(N) \to \Gamma_\mathfrak{a}(M) \xrightarrow{\alpha} \Gamma_\mathfrak{a}(A) \longrightarrow H_\mathfrak{a}^1(N) \xrightarrow{\beta} H_\mathfrak{a}^1(M) \to 0,$$

and $H_\mathfrak{a}^i(M) \cong H_\mathfrak{a}^i(N)$ for all $i \geq 2$. Hence, the assertion follows from Lemma 4.6 for any $i \geq 2$. Now, suppose that $i = 0$. The exact sequence $0 \to \Gamma_\mathfrak{a}(N) \to \Gamma_\mathfrak{a}(M) \to \operatorname{Im}\alpha \to 0$ induces the long exact sequence

$$\cdots \to \operatorname{Ext}_R^j\big(R/\mathfrak{a}, \Gamma_\mathfrak{a}(N)\big) \to \operatorname{Ext}_R^j\big(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)\big) \to \operatorname{Ext}_R^j\big(R/\mathfrak{a}, \operatorname{Im}\alpha\big) \to \cdots,$$

for all $j \geq 0$. On the other hand, $\Gamma_\mathfrak{a}(N)$ and $\Gamma_\mathfrak{a}(A)$ are minimax, hence $\operatorname{Ext}_R^j\big(R/\mathfrak{a}, \Gamma_\mathfrak{a}(M)\big)$ is minimax, for all $j \geq 0$, by [3, Lemma 2.1] for the class of minimax R-modules. Finally, for $i = 1$, we consider the short exact sequence $0 \to \operatorname{Ker}\beta \to H_\mathfrak{a}^1(N) \to H_\mathfrak{a}^1(M) \to 0$. The facts that $\operatorname{Ker}\beta$ is Artinian and $(0 :_{H_{\mathfrak{a}(N)}^1} \mathfrak{a})$ is finitely generated, imply that $(0 :_{\operatorname{Ker}\beta} \mathfrak{a})$ has finite length. Thus $\operatorname{Ker}\beta$ is $\mathfrak{a}$-cofinite, by [17, Proposition 4.1]. Now, the assertion follows from Lemma 4.6 and the long exact sequence

$$\cdots \to \operatorname{Ext}_R^j\big(R/\mathfrak{a}, H_\mathfrak{a}^1(N)\big) \to \operatorname{Ext}_R^j\big(R/\mathfrak{a}, H_\mathfrak{a}^1(M)\big) \to \operatorname{Ext}_R^{j+1}\big(R/\mathfrak{a}, \operatorname{Ker}\beta\big) \to \cdots.$$

□

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MirYousef Sadeghi
Assistant Professor of Mathematics
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
E-mail: my.sadeghi@pnu.ac.ir

Zohreh Habibi
Assistant Professor of Mathematics
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
E-mail: Z_habibi@pnu.ac.ir

Fatemeh Shahid
Phd Student of Mathematics
Department of Mathematics, Payame Noor University (PNU), Tehran, Iran
E-mail: Fatemeh.shahid@student.pnu.ac.ir