Approximation of Discrete Data by Discrete Weighted Transform

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Abstract. The aim of this paper is to introduce the notion of discrete and inverse discrete weighted transform. We show that discrete input data can be converted to a continuous approximation through the inverse discrete weighted transform.

1. Introduction

In many problems we prefer to transform the original model into a special space where a computation is simpler. Various kinds of transforms such as, Fourier, Laplace and Wavelet are used in methods for construction of approximation models in a space without complexity of computation. Among them, the fuzzy transform which introduced by Perfilieva [4], provides a relation between $\mathbb{R}^n$ and the space of continuous functions defined on the real line. The investigation of fuzzy transform has attracted attention in many real world fields, such as, solution of ordinary and partial differential equations, image compression and reconstruction, signal processing, and so on [3, 4, 5, 6]. Constructing a fuzzy transform

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for approximating a continuous function on $\mathbb{R}$ or a function which is known only at some points in the universe has become an active topic in all these applications properties.

First of all we briefly recall the basic facts of a fuzzy transform of continuous function $f : [a, b] \to \mathbb{R}$ which has been shown in [4].

Let $a = t_0 < ... < t_n = b$, $n \geqslant 3$, be fixed points of $[a, b]$. A fuzzy partition is a set of continuous functions $\mathbf{A} = \{A_1, ..., A_n\}$ such that for each $i = 1, ..., n$;

(i) $A_i : [a, b] \to [0, 1], A_i(t_i) = 1$;

(ii) $supp A_i = (t_{i-1}, t_{i+1})$;

(iii) $A_i(t)$ strictly increases for $i = 2, ..., n$, and decreases on $[t_i, t_{i+1}]$ for $i = 1, ..., n - 1$;

(iv) For all $t \in [a, b], \sum_{i=1}^{n} A_i(t) = 1$.

$A_1, ..., A_n$ are called basic functions. The fuzzy partition $\mathbf{A} = \{A_1, ..., A_n\}$ is uniform if the points $t_0, ..., t_n$, $n \geqslant 3$, are equidistant with two requirement additional properties:

(v) $A_i(t_i - x) = A_i(t_i + x)$ for all $x \in [0, h], i = 2, ..., n - 1$;

(vi) $A_i(x) = A_{i-1}(x - h)$ for all $i = 2, ..., n - 1$ and $x \in [t_i, t_{i+1}]$ and $A_{i+1}(x) = A_i(x - h)$ for all $i = 2, ..., n - 1$ and $x \in [t_i, t_{i+1}]$, where $h = \frac{b-a}{n-1}$.

According to [4], the continuous fuzzy transform is given by

$$F_i = \frac{\int_a^b f(x)A_i(x)dx}{\int_a^b A_i(x)dx}, \quad i = 1, ..., n.$$

The discrete fuzzy transform is given by

$$F_i = \frac{\sum_{j=1}^{m} f(x_j)A_i(x_j)}{\sum_{j=1}^{m} A_i(x_j)}$$

where f is given at nodes $x_j \in [a, b]$, $j = 1, ..., m$ and $\mathbf{A} = \{A_1, ..., A_n\}$, $n < m$, is the fuzzy partition of $[a, b]$, for which there exists at least one

node in each subinterval (t_{i-1}, t_{i+1}), i.e., $A_i(x_j) > 0$ for all $i = 1, ..., n$ and some $j \in \{1, ..., m\}$.

For reconstruction of the function f, the continuous inverse fuzzy transform $f_{F,n}$, has been defined by

$$f_{F,n}(t) = \sum_{i=1}^{n} F_i A_i(t)$$

Although $f_{F,n}$ is not the inverse of fuzzy transform but $f_{F,n}$, approximate f up to an arbitrary precision [3, 4]. In [3], Patane investigated the relation between the least square approximation technique and the fuzzy transform. He showed that the discrete fuzzy transform is invariant with respect to interpolating and least square approximation of the given data set.

In this paper, in accordance to [1, 2, 6], we specialize the fuzzy transform to the case of weighted membership functions and show the relation between this case and the previous work . In particular, we show that the result of [3] is valid for discrete weighted transform. In this way, we compute the discrete weighted transform of functions defined on a subset $\mathcal{D}$ of $\mathbb{R}^m$ and prove the convergence property.

2.   Main Results

Let $\mathcal{D}$ be a bounded subset of $\mathbb{R}^m$, which contains its infimum and supremum. Let $\omega : \mathcal{D} \to (0, 1]$ be a continuous function. At first, we give the concept of the weighted partition.

Definition 2.1. *Let* $\{\mathcal{D}_i\}_{i=1}^{n}$ *be a partition of* $\mathcal{D} \subseteq \mathbb{R}^m$ *and* $\{x_1, ..., x_n\}$ *a set of points in* $\mathcal{D}$ *such that* $x_i \in \mathcal{D}_i$, $i = 1, ..., n$. *The family* $\mathcal{P} = \{P_1, ..., P_n\}$ *is a weighted partition of* $\mathcal{D}$ *if the following hold for each* $i = 1, ..., n;$

(i) $P_i : \mathcal{D} \to [0, 1]$ *is continuous;* $P_i(x) \neq 0$, $x \in \mathcal{D}_i;$
(ii) $P_i(x_i) = \omega(x_i);$
(iii) $P_i(x) = 0$, $x \in \mathcal{D} \setminus \mathcal{D}_i;$

(iv) $\sum_{i=1}^{n} P_i(x) = \omega(x)$, $x \in \mathcal{D}$.

Note that any fuzzy partition can be considered as a weighted partition with the constant weight function $\omega \equiv 1$ on $\mathcal{D}$.

Following example shows that a weighted partition need not necessarily to be a fuzzy partition.

Example 2.2. Let $\mathcal{D} = [a, b]$, $\omega(x) = e^{-x}$ and $a = x_1 < ... < x_n = b$. Define

$$P_1(x) = \begin{cases} e^{-x} \cdot \left(0.5 \left(\cos\frac{\pi}{h}(x - x_1) + 1 \right) \right) & x \in [x_1, x_2] \\ \\ 0 & otherwise. \end{cases}$$

$$P_k(x) = \begin{cases} e^{-x} \cdot \left(0.5 \left(\cos\frac{\pi}{h}(x - x_k) + 1 \right) \right) & x \in [x_{k-1}, x_{k+1}] \\ \\ 0 & otherwise, \end{cases}$$

for $k = 2, ..., n - 1$, and

$$P_n(x) = \begin{cases} e^{-x} \cdot \left(0.5 \left(\cos\frac{\pi}{h}(x - x_n) + 1 \right) \right) & x \in [x_{n-1}, x_n] \\ \\ 0 & otherwise, \end{cases}$$

where $h = \dfrac{b - a}{n - 1}$. The weighted partition P_k, $1 \leqslant k \leqslant n$, with $n = 7$, $a = 0$ and $b = 3$ is shown in Figure 1.
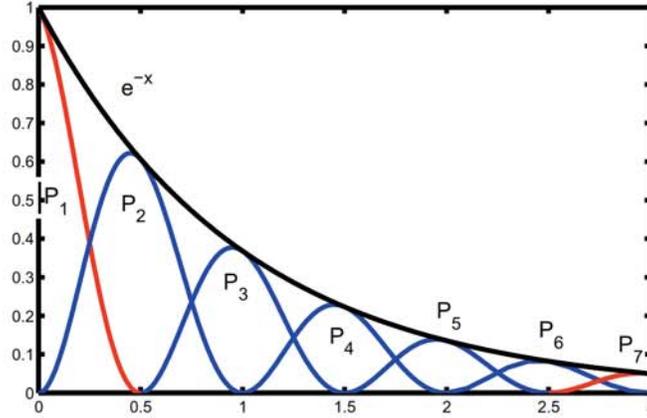
Figure 1. A weighted partition on $[0, 3]$

Suppose that $f : \mathcal{D} \to \mathbb{R}$ is given at some nodes $\{x_1, ..., x_l\}$ of $\mathcal{D} \subseteq \mathbb{R}^m$. Let $\mathcal{P} = \{P_1, ..., P_n\}$, $n < l$, be the weighted partition such that for each $1 \leqslant i \leqslant n$, there exists at least one node in $\mathcal{D}_i$, i.e., $P_i(x_j) > 0$ for some $j = 1, ..., l$.

Now with respect to f and the weighted partition $\mathcal{P}$ we define the discrete weighted transform as follows.

Definition 2.3. *Let $\omega : \mathcal{D} \to (0, 1]$ be a continuous function and $\mathcal{P} = \{P_1, ..., P_n\}$ be a weighted partition of $\mathcal{D}$. The discrete weighted transform of $f : \mathcal{D} \to \mathbb{R}$ with respect to $\mathcal{P}$ is $\mathcal{F}_p = [F_1, ..., F_n]$ if*

$$F_i = \frac{\sum_{j=1}^{l} f(x_j) P_i(x_j)}{\sum_{j=1}^{l} P_i(x_j)}; \qquad i = 1, ..., n.$$

where f is known at discrete data $\{x_1, ..., x_n\}$ in $\mathcal{D}$.

An interesting property of discrete weighted transform is the following optimal property.

Proposition 2.4. *Let f be a function which is known only at a given set of points $X = \{x_1, ..., x_l\} \subseteq \mathcal{D}$ and $\mathcal{P} = \{P_1, ..., P_n\}$ be a weighted partition of $\mathcal{D}$. Then for $i = 1, ..., n$ the ith element of discrete weighted*

transform $\mathcal{F}_p$ minimizes the expression

$$\sum_{j=1}^{l} \Big(f(x_j) - y\Big)^2 P_i(x_j).$$

Proof. Define

$$H(y) = \sum_{j=1}^{l} \Big(f(x_j) - y\Big)^2 P_i(x_j).$$

So H is continuously differentiable with respect to y in the range of f. Deriving the function H with respect to y. If $H'(y) = 0$ then $y = \dfrac{\sum_{j=1}^{l} f(x_j) P_i(x_j)}{\sum_{j=1}^{l} P_i(x_j)} = F_i$. So the proof is complete. $\square$

Definition 2.5. *Let $f : \mathcal{D} \to \mathbb{R}$ be given at some points of $X = \{x_1, ..., x_l\}$ and $\omega : \mathcal{D} \to (0, 1]$ be a continuous function. Suppose that $\mathcal{F}_p = [F_1, ..., F_n]$ is the discrete weighted transform of f with respect to $\mathcal{P} = \{P_1, ..., P_n\}$. The function $T_{P,f}(x) = \sum_{k=1}^{n} \dfrac{F_k P_k(x)}{\omega(x)}$ is called the inverse discrete weighted transform of f.*

In order to show that the inverse discrete weighted transform can approximate the original function f at common nodes with an arbitrary precision, we will use the method employed in [1, 6].

Theorem 2.6. *Let ω be a continuous function from $\mathcal{D}$ into $(0, 1]$ and f be given at points $x_1, ..., x_l$ of $\mathcal{D}$. Then for any positive ε, there exists a weighted partition $\{P_1, ..., P_{n_\varepsilon}\}$ such that for each $1 \leqslant i \leqslant n_\varepsilon$,*

$$||f - T_{P,f}||_\infty < \varepsilon.$$

Proof. Suppose that $X = \{x_1, ..., x_l\}$ is ordered lexicographically, that is, $x_i = (x_i^1, ..., x_i^m) \prec x_j = (x_j^1, ..., x_j^n)$ if either $x_i^r = x_j^r$ for all $r = 1, ..., m$ or $x_i^n < x_j^n$ at the first place n where they differ. We choose the points $\{x_1, ..., x_{n_\varepsilon}\}$ from X as follows and construct the weighted partition $\{P_1, ..., P_{n_\varepsilon}\}$ from these nodes.

1. $k := 0$;

2. $k := k + 1$. Put $x_k := \min X$, $X = X \setminus x_k$. If $X = \emptyset$ then stop;

3. If the set $X' = \left\{ x \in X : x_k \prec x \text{ and } \left| f(x) - f(x_k) \right| \leqslant \varepsilon/2 \right\}$ is empty go to the step 2.

To choice the members of X' we check each x which is greater than x_k from less to great whenever $\left| f(x) - f(x_k) \right| \leqslant \varepsilon/2$, otherwise we cease. If $X' \neq \emptyset$, Then put $k := k + 1$, $x_k := \max X'$ and $X = X \setminus X'$. If $X \neq \emptyset$ then go to step 2, otherwise, put $n_\varepsilon := k$ and stop.

Now in order to $\{x_1, ..., x_{n_\varepsilon}\}$ the weighted partition $\mathcal{P} = \{P_1, ..., P_{n_\varepsilon}\}$ will be defined corresponding to Definition 1. For $1 \leqslant k \leqslant n - 1$, and $x \in \mathcal{D} \bigcap \mathcal{D}_k$, we have

$$
\begin{aligned}
\left| f(x) - F_k \right| &= \left| f(x) - \frac{\sum_{j=1}^{l} f(x_j) P_k(x_j)}{\sum_{j=1}^{l} P_k(x_j)} \right| \\
&\leqslant \frac{\sum_{j=1}^{l} \left| f(x) - f(x_j) \right| P_k(x_j)}{\sum_{j=1}^{l} P_k(x_j)} \\
&= \frac{\sum_{x_j \in \mathcal{D} \bigcap \mathcal{D}_j} \left| f(x) - f(x_j) \right| P_k(x_j)}{\sum_{x_j \in \mathcal{D} \bigcap \mathcal{D}_j} P_k(x_j)} \\
&< \varepsilon.
\end{aligned}
$$

Similarly $\left| f(x) - F_n \right| < \varepsilon$. This completes the proof. $\square$

We apply the matrix formulation [3] to simplify the computation and show the validity of some results of [1, 3] for discrete and inverse weighted transform.

Let us to consider $V = \left[f(x_1), ..., f(x_n) \right]^t$, $Q = diag \left[\sum_{i=1}^{n} P_k(x_i) \right]$, $1 \leqslant k \leqslant n$ and the coefficient matrix $P = \left[P_k(x_i) \right]$, $1 \leqslant k \leqslant n$, $1 \leqslant i \leqslant n$. Hence the discrete weighted transform can be written in the matrix form $\mathcal{F}_P = \left[F_1, ..., F_n \right] = Q^{-1} P V$.

Theorem 2.7. *Let the function f be given at nodes $X = \{x_1, ..., x_n\} \subseteq \mathcal{D}$ and f^* be the least square approximation with respect to the values of f.*

Then the discrete weighted transform of f and f^* are the same.

Proof. By Definition 3, the kth component of discrete weighted of f^* is

$$
\begin{aligned}
F_k^* &= \frac{\sum_{i=1}^n f^*(x_i) P_k(x_i)}{\sum_{i=1}^n P_k(x_i)} \\
&= \frac{\sum_{i=1}^n \left(\sum_{j=1}^n f_j^*(x_i) P_j(x_i) \right) P_k(x_i)}{\sum_{i=1}^n P_k(x_i)}, \qquad 1 \leqslant k \leqslant n.
\end{aligned}
$$

So by using matrix formulation we have

$$
\mathcal{F}_{\mathcal{P}}^* = \left[F_1^*, ..., F_n^* \right] = Q^{-1} P M^{-1} \left(M V \right) = Q^{-1} P V = \mathcal{F}_P,
$$

where $M = diag\left(\omega(x_i) \right), 1 \leqslant i \leqslant n$. $\quad\square$

Proposition 2.8. *Let f be a function given at points $\{x_1, ..., x_l\} \subseteq \mathcal{D}$ and $\mathcal{F} = [F_1, ..., F_n]$, $n \leqslant l$, be the weighted transform of f with respect to $\mathcal{P} = \{P_1, ..., P_n\}$. Then $\| \mathcal{F}_P \|_\infty \leqslant \| V \|_\infty$ and $\| \mathcal{F}_P \|_2 \leqslant \sqrt{n} \| V \|_\infty$, where $V = \left[f(x_1), ..., f(x_l) \right]^t$.*

Proof. Note that, $\| \mathcal{F}_p \|_\infty = \max \left| F_k \right|_{1 \leqslant k \leqslant n}$. So by Definition 3,

$$
\begin{aligned}
\left| F_k \right| &= \frac{\left| \sum_{i=1}^l f(x_i) P_k(x_i) \right|}{\sum_{i=1}^l P_k(x_i)} \\
&\leqslant \frac{\sum_{i=1}^l \left| f(x_i) \right| P_k(x_i)}{\sum_{i=1}^l P_k(x_i)} \\
&\leqslant \| V \|_\infty . \qquad\qquad (1)
\end{aligned}
$$

Hence $\| \mathcal{F}_p \|_\infty \leqslant \| V \|_\infty$.
Now by relation (1) and Holder's inequality we have

$$
\left(\sum_{k=1}^n \left| F_k \right|^2 \right)^{1/2} \leqslant \left(\sum_{k=1}^n \| f \|_\infty^2 \right)^{1/2} = \sqrt{n} \| f \|_\infty,
$$

and so $\| \mathcal{F}_P \|_2 \leqslant \sqrt{n} \| f \|_\infty$. $\quad\square$

In the following we find a bound for ℓ^2- norm of the discrete weighted transform.

Theorem 2.9.   *Let ω be a continuous function from $\mathcal{D}$ into $(0,1]$. Suppose that function f is known only at $\{x_1, ..., x_n\} \subseteq \mathcal{D}$ and $\mathcal{F}_P = [F_1, ..., F_n]$ is the discrete weighted transform of f with respect to $\mathcal{P} = \{P_1, ..., P_n\}$. Then the ℓ^2- norm of $\mathcal{F}_p$ is bounded by the the ℓ^2- norm of f and the maximum singular value of the coefficient matrix P.*

Proof. By using matrix formulation $\mathcal{F}_P = Q^{-1}PV$ we have

$$
\begin{aligned}
\parallel \mathcal{F}_P \parallel_2 = \parallel Q^{-1}PV \parallel_2 \quad &\leqslant \quad \parallel Q^{-1} \parallel_2 . \parallel P \parallel_2 . \parallel V \parallel_2 \\
&= \quad \left(\min_{1 \leqslant k \leqslant n} \left(\sum_{i=1}^{n} P_k(x_i) \right) \right)^{-1} . \sigma_m(P). \parallel V \parallel_2
\end{aligned}
$$

where $\sigma_m(P)$ is the maximum singular value of the matrix P. $\square$

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