

Quasi-Newton Methods for Solving Non-Smooth Multiobjective Optimization

N. Hoseini Monjezi

Univetsity of Isfahan

Abstract. In this paper, a quasi-Newton type algorithm for non-smooth multiobjective optimization is presented. In this algorithm, in every iteration a quadratic subproblem solves until a critical point is reached. Moreover, the global convergence of the algorithm is established under suitable assumptions.

AMS Subject Classification: 90C30; 90C46; 49J52

Keywords and Phrases: Multiobjective optimization, non-smooth analysis, quasi-newton, critical point, global convergence

1. Introduction

In many areas in engineering, economics and science new developments are only possible by the application of modern optimization methods. The optimization problems raising nowadays in applications are mostly multiobjective i.e. many competing objectives are aspired all at once [10, 11]. A single solution would not generally minimize every objective function simultaneously. A concept of optimality which is useful in the multiobjective framework is that of Pareto optimality.

There are two solution approaches for multiobjective optimization problems. One is the scalarization approach, i.e. by some scalarization procedures one or several equivalent parameterized single objective problems are solved and the corresponding number of Pareto optimal points are obtained [2, 5]. Another approach is the non-scalarization methods whereby from the current non-critical solution, a Pareto optimal point

that dominates the solution is generated. Fliege and Svaiter in [4] have presented a steepest descent method for solving multiobjective optimization problems. Vieira et al [12] generalise those ideas in which Armijo rule is replaced with a multiobjective golden section line search.

In [8] a trust region algorithm is presented for solving both smooth and non-smooth multiobjective optimization. Our aim in this work is to extend the result obtained in [8] for nonsmooth multiobjective problems. Moreover, we reduce the assumptions that have been introduced in [8]. In this paper, we present a quasi-Newton method for finding the critical solutions of non-smooth multiobjective optimization problems (the Pareto optimal solutions)[9].

Indeed, we do not require the differentiability of the functions involved. In this work, we use the quasi-Newton method to approximate second order information of each objective function, though the second derivative may not exist. Our algorithm replaces each objective function with a quadratic model for each objective function until the critical point is reached. This model improves the performance by constraining the descent direction norms and a small positive scalar to control the descent direction. For the new algorithm we propose global convergence result under suitable assumptions.

The organization of this paper is as follows. In Section 2, we present some preliminaries and introduce some notations. We studied the relation between critically and descent direction in the non-smooth case. The new algorithm and its properties are described in Section 3. In Section 4, the global convergence of the algorithm is obtained under some suitable assumptions.

2. Preliminaries

We begin this section by introducing some notations. Let \mathbb{R} be the set of real numbers; \mathbb{R}_+ be the set of non-negative real numbers and \mathbb{R}_{++} be the set of strictly positive real numbers.

For any $u, v \in \mathbb{R}^m$, denote

$$\begin{aligned} u \leq v &\iff v - u \in \mathbb{R}_+^m \iff v_j - u_j \geq 0 & j = 1, \dots, m, \\ u < v &\iff v - u \in \mathbb{R}_{++}^m \iff v_j - u_j > 0 & j = 1, \dots, m. \end{aligned}$$

Given any open set $X \subset \mathbb{R}^n$. A function $h : X \rightarrow \mathbb{R}$ is said to be Lipschitz of rank $\mu > 0$ near $x \in X$ if for some $\epsilon > 0$

$$|h(y) - h(z)| \leq \mu \|y - z\| \quad \forall y, z \in B(x, \epsilon).$$

By $h \in C^{0,1}(\mathbb{R}^n, \mathbb{R})$, we indicate that h is locally Lipschitz. For any $h \in C^{0,1}(\mathbb{R}^n, \mathbb{R})$, the classic directional derivative of h at x in direction d is defined as

$$h'(x; d) = \lim_{\alpha \rightarrow 0} \frac{h(x + \alpha d) - h(x)}{\alpha}.$$

Let us recall some basic concepts and tools from nonsmooth analysis. Most of the material included here can be found in [1].

Let $h \in C^{0,1}(\mathbb{R}^n, \mathbb{R})$, the Clarke directional derivative of h at x in the direction d , denoted $h^\circ(x, d)$ is defined as follows

$$h^\circ(x; d) = \limsup_{\substack{y \rightarrow x \\ \alpha \downarrow 0}} \frac{h(y + \alpha d) - h(y)}{\alpha}.$$

The generalized gradient of h at x , denoted by $\partial_c h(x)$, is defined by

$$\partial_c h(x) := \{\xi \in \mathbb{R}^n : \langle \xi, d \rangle \leq h^\circ(x; d) \quad \forall d \in \mathbb{R}^n\}.$$

It is well-known that $\partial_c h(x)$ is a nonempty convex compact set in \mathbb{R}^n , and the set-valued mapping $x \rightarrow \partial_c h(x)$ is upper semicontinuous. It is worth mentioning that for a convex function h , $\partial_c h(x)$ coincides with the subdifferential $\partial h(x)$, defined as follows:

$$\partial_c \phi(x) := \{\xi \in \mathbb{R}^n \mid \phi(y) \geq \phi(x) + \langle \xi, y - x \rangle \quad \forall y \in \mathbb{R}^n\}.$$

We recall the following theorems from [1].

Theorem 2.1. *Let φ and ψ are locally Lipschitz from \mathbb{R}^n to \mathbb{R} , and $\hat{x} \in \text{dom}(\varphi) \cap \text{dom}(\psi)$. Then, the following properties hold:*

$$(a) \quad \varphi^0(\hat{x}; d) = \max\{\langle \xi, d \rangle \mid \xi \in \partial_c \varphi(\hat{x})\}, \quad \forall d \in \mathbb{R}^n.$$

$$(b) \quad \partial_c(\lambda\varphi(\hat{x})) = \lambda\partial_c\varphi(\hat{x}) \quad \forall \lambda \in \mathbb{R}.$$

$$(c) \quad \partial_c(\varphi + \psi)(\hat{x}) \subseteq \partial_c\varphi(\hat{x}) + \partial_c\psi(\hat{x}).$$

We present a numerical algorithm for the following non-smooth multi-objective problems:

$$\min_{x \in \mathbb{R}^n} F(x),$$

where $F = (F_1, \dots, F_m) : X \rightarrow \mathbb{R}^m$ is locally Lipschitz and $X \subseteq \mathbb{R}^n$ is the domain of F which is assumed to be open and define the index set $I = \{1, 2, \dots, m\}$.

Since no unique solution which minimize simultaneously all objective functions exists, so the concept of optimality has to be replaced by the concept of Pareto optimality or efficiency, as explained below:

Definition 2.2. Given a point $x^* \in X$, (i). x^* is said to be a globally efficient point or Pareto optimum of F if and only if there does not exist $x \in X$ such that

$$F(x) \leq F(x^*) \quad \text{and} \quad F(x) \neq F(x^*);$$

(ii). x^* is said to be a weakly efficient point or weak Pareto optimum of F if and only if there does not exist $x \in X$ such that

$$F(x) < F(x^*);$$

(iii). The point x^* is said to be a locally efficient point or locally weakly efficient for F if and only if there exists a neighbourhood $V \subset X$ such that the point x^* is efficient or weakly efficient for F restricted to V .

It is easy to see that any globally Pareto optimal solution is locally Pareto optimal. The converse is true if X is convex and F is \mathbb{R}^m -convex (i.e. if F is componentwise convex.). But under non-convex situation the equivalence is invalid.

Let us recall some definitions from [8].

Definition 2.3. (i). A point $x^* \in S$ is said to be critical (or stationary) for F , if

$$\Re(\partial_c F(x^*)) \cap (-R_{++}^m) = \emptyset,$$

where $\Re(\partial_c h(x))$ denotes the range or image space of the generalized Jacobian of h at x . (ii). A direction $d \in \mathbb{R}^n$ is said to be a descent direction for F at x if for any $j \in I$, the directional derivative of the corresponding componentwise function F_j in direction d satisfies the following condition:

$$F'_j(x; d) < 0, \quad (1)$$

where the directional derivative at x in the direction d is defined as

$$F'_j(x; d) = \lim_{\alpha \rightarrow 0} \frac{F_j(x + \alpha d) - F_j(x)}{\alpha}. \quad (2)$$

Let $F \in C^1(X, \mathbb{R}^m)$, then it follow from (2.) that $F'_j(x; d) = \nabla F_j(x)^t d$. The descent direction d will reduce every objective function value if it is used to update the design x . It is easy to see by (8) and (2.), that d is a descent direction for F at x if and only if

$$\nabla F_j(x)^t d < 0 \quad \forall j \in I.$$

Definition 2.3 means that if $x^* \in X$ is critical for F , then there does not exist a descent direction at x^* .

We now make a basic assumption.

(A1) For any $x \in X$ and $d \in \mathbb{R}^n$ the function F is regular at x , this means that the directional derivatives $F'(x, d)$ at x for all d exists and

$$F^\circ(x, d) = F'(x, d).$$

Under this assumption, we obtain the following conclusion for the critical points.

Lemma 2.4. Suppose that assumption (A1) holds, then $x^* \in S$ is critical if and only if either one of the following two conditions is satisfied

- (a). *There does not exist a descent direction at x^* for F .*
- (b). *In the special case, there exists at least one $j_0 \in I$ such that $0 \in \partial_c F_{j_0}(x^*)$.*

Proof. (a). It follows from (A1) that directional derivative $F'_j(x, d)$ exists and is equal to $F_j^\circ(x, d)$ i.e.

$$F^\circ(x; d) = F'(x; d).$$

Now if $x^* \in X$ is a critical point for F ; we have

$$\Re(\partial_c F(x^*)) \cap (-R_{++}^m) = \emptyset,$$

Therefore for every $d \in \mathbb{R}^n$, there exists at least one $j_0 = j_0(d) \in I$ such that $F_{j_0}^\circ(x, d) \geq 0$. Therefore, (a) hold. Obviously (b) is a especial case of (a).

On the other hand, if either one of (a) and (b) is satisfied. Then

$$\exists j_0 = j_0(d) \in I, \quad F_{j_0}^\circ(x^*; d) \geq 0.$$

By using (A1) we have $F'_{j_0}(x^*; d) \geq 0$, then from Definition 2.3. it follows that x^* is critical for F . \square

It is worth to mention that, in general efficiency is not equivalent to critically and they are related as follows.

Theorem 2.5. *Suppose assumption (A1) holds.*

- (a). *If $x^* \in X$ is a locally weak Pareto optimal, then x^* is a critical point for F .*
- (b). *If X is convex, F is \mathbb{R}^m – convex and $x^* \in X$ is critical for F , then x^* is weak Pareto optimal.*
- (c). *If X is convex, F is strictly \mathbb{R}^m – convex and $x^* \in X$ is critical for F , then x^* is Pareto optimal.*

Proof. (a). Suppose that $x^* \in X$ is locally weak Pareto optimal. Therefore, it is sufficient to prove $x^* \in X$ is a critical point. This can be proved

by contradiction i.e. suppose that $x^* \in X$ is not critical, therefore

$$\Re(\partial_c F(x^*)) \cap (-R_{++}^m) \neq \emptyset.$$

Then,

$$\exists d \in \mathbb{R}^n \quad \text{s.t.} \quad F_j^\circ(x^*; d) < 0 \quad \forall j \in I.$$

By using (A1), we get

$$F_j'(x^*; d) = F_j^\circ(x^*; d) < 0.$$

Hence,

$$\exists \bar{\alpha} > 0 \quad \text{s.t.} \quad \alpha \in (0, \bar{\alpha}] \quad F_j(x^* + \alpha d) < F_j(x^*),$$

This contradicts that $x^* \in X$ is locally weak Pareto optimal.

To prove part (b), take any $x \in X$. We can set $d = x - x^*$. Since $x^* \in X$ is critical, it follows that there exists $j_0 = j_0(d) \in I$ such that $F_{j_0}^\circ(x; d) = F_{j_0}'(x; d)$ exists and non-negative.

On the other hand, by the convexity of F_{j_0} we obtain

$$\begin{aligned} F_{j_0}(x) - F_{j_0}(x^* + \alpha d) &\geq \xi^t(x - x^* - \alpha d) \quad \forall \xi \in \partial_c F_{j_0}(x^* + \alpha d) \\ &= \xi^t\left(x - x^* - \alpha(x - x^*)\right) \\ &= (1 - \alpha)\xi^t(x - x^*). \end{aligned}$$

Hence,

$$F_{j_0}(x) - F_{j_0}(x^* + \alpha d) \geq (1 - \alpha)\xi^t(x - x^*) \quad \forall \xi \in \partial_c F_{j_0}(x^* + \alpha d).$$

Let $\alpha \rightarrow 0$, then according to the continuity of F_{j_0} we have

$$F_{j_0}(x) - F_{j_0}(x^*) \geq F_{j_0}^\circ(x; x - x^*) \geq 0,$$

This mean that x^* is weak Pareto optimal.

To prove part (c), take any $x \in X$. From the strict convexity of F , the inequality holds strictly for any $x \neq x^*$. Then

$$F_{j_0}(x) - F_{j_0}(x^*) > F_{j_0}^\circ(x; x - x^*) \geq 0.$$

Then the conclusion immediately follows. \square

3. Algorithm

In this section, we state an algorithm for obtaining a critical point of non-smooth multiobjective optimization. For computing the search direction, in each iteration, our algorithm replaces each objective function with a quadratic model for it until the critical point is found. This model improves the performance by constraining the descent direction norms and a small positive scalar to control the descent direction.

For any $x \in X$, given sufficient small $\epsilon > 0$, we define $d(x)$ as the optimal solution of

$$NSP_\epsilon(x) \begin{cases} \min & t \\ \text{s.t.} & \\ & F_j^\circ(x; d) + \frac{1}{2} d^t B_j(x) d \leq t \quad j \in I, \\ & \|d\| \leq 1, \quad t \leq -\epsilon, \quad d \in \mathbb{R}^n \end{cases}$$

where $F_j^\circ(x; d)$ is the Clarke generalized directional derivative of F_j at x in direction d and $\frac{1}{2}d^t B_j(x) d$ is the approximation Hessian obtained by quasi-Newton methods such that $F_j^\circ(x; d)$ and $\frac{1}{2}d^t B_j(x) d$ carry certain first-order and second-order information of $F_j(x)$ respectively, although the first and second derivatives of $F_j(x)$ may not exist in general. The constraint direction norms $\|d\| \leq 1$ is used to improve performance as $\|d\| \leq 1$ and it eliminates the possible case $\|d\| \rightarrow \infty$.

By using Lemma 2.4, for every critical point conditions (a) or (b) should be satisfied. In this paper we propose a solution approach satisfying item (a) of Lemma 2.4. Because the item (b) is not general (It is a special case).

Lemma 3.1. *Given $x \in X$. Suppose that assumption (A1) holds. Let $B_j(x) \quad j \in I$ be positive semidefinite. For sufficient small positive scalar ϵ , if the feasible set of $NSP_\epsilon(x)$ is nonempty. Then x is noncritical and any feasible point $d_\epsilon(x)$ is a descent direction for F ; Otherwise, x is a good approximate of the critical point for F .*

Proof. Suppose that the feasible set of $NSP_\epsilon(x)$ is nonempty. Then there exists $d_\epsilon(x)$, such that it is a feasible point to $NSP_\epsilon(x)$ as $\epsilon > 0$,

and

$$F_j^\circ(x; d_\epsilon(x)) + \frac{1}{2} d_\epsilon(x)^t B_j(x) d_\epsilon(x) \leq t \leq -\epsilon < 0 \quad \forall j \in I,$$

From the positivity of $B_j(x)$, we have

$$F_j^\circ(x; d_\epsilon(x)) < -\frac{1}{2} d_\epsilon(x)^t B_j(x) d_\epsilon(x) \leq 0 \quad \forall j \in I.$$

Hence,

$$F_j'(x; d) = F_j^\circ(x; d) < 0 \quad \forall j \in I.$$

This means $d_\epsilon(x)$ is a descent direction for F and therefore x is noncritical.

Now suppose that the feasible set of $NSP_\epsilon(x)$ is empty. Then we should show that x is a good approximate of critical point. This can be proved by contradiction i.e. suppose that there exists a descent direction $\bar{d} \in \mathbb{R}^n$ such that

$$F_j'(x; \bar{d}) < 0 \quad \forall j \in I,$$

By Assumption (A1), we have $F_j'(x; d) = F_j^\circ(x; d) < 0$. It follows that there exists a positive scalar $\bar{\alpha} > 0$, such that for any $\alpha \in (0, \bar{\alpha}]$

$$F_j^\circ(x; \alpha \bar{d}) + \frac{1}{2} \alpha^2 \bar{d}^t B_j(x) \bar{d} < 0 \quad j \in I.$$

If for any $\alpha \in (0, \bar{\alpha}]$, we define $-\epsilon = F_j^\circ(x; \alpha \bar{d}) + \frac{1}{2} \alpha^2 \bar{d}^t B_j(x) \bar{d}$, then $\alpha \bar{d}$ is a feasible point to $NSP_\epsilon(x)$. This contradicts that the feasible set of $NSP_\epsilon(x)$ is empty. Therefore x is a good approximate critical point for F . \square

Based on the above lemma, we now state the following algorithm.

Algorithm 3.2. Quasi-Newton Algorithm

Step 0: Let $x_0 \in X$ be the initial point. Given a sufficient small positive scalar $\epsilon > 0$ and the positive initial matrix $B_j(x_0)$ for $j \in I$. Set $k := 0$.

Step 1: Generate $NSP_\epsilon(x)$;

If the feasible set is empty stop and x_k is a good approximate critical point. Else solve subproblem $NSP_\epsilon(x_k)$ and compute $d(x_k)$ that is the optimal solution of $NSP_\epsilon(x_k)$.

Step 2: Choose step size α_k by Armijo-like rule such that $x_{k+1} = x_k + \alpha_k d(x_k) \in X$.

Step 3: Update positive definite matrix $B_j(x_{k+1})$ for $j \in I$. Set $k := k + 1$. Go to step 1.

In step 3, we use *BFGS* update criterion to update $B_j(x_{k+1})$ similar to the update in [7].

4. Global Convergence

First we make some basic assumptions. Then we prove the global convergence of Algorithm 1.

(A2). Assume that the level set $L_0 = \{x \in \mathbb{R}^n : F(x) \leq F(x_0)\}$ is bounded.

(A3). Assume that the step-length $\alpha_k = 1$ is accepted for any sufficient large K .

Now define

$$\pi(x) := \sup_{\|d\| \leq 1} \min_{j \in I} \{-F_j^\circ(x; d)\}.$$

• If $x \in X$ is noncritical. we have

$$\Re(\partial_c F(x^*)) \cap (-R_{++}^m) \neq \emptyset.$$

Therefore,

$$\pi(x) = \sup_{\|d\| \leq 1} \min_{j \in I} \{-F_j^\circ(x; d)\} > 0$$

• If $x \in X$ is critical. we have

$$\Re(\partial_c F(x^*)) \cap (-R_{++}^m) = \emptyset.$$

Thus,

$$\exists j_0 \in I, \quad F_{j_0}^\circ(x; d) \geq 0$$

Therefore

$$\begin{aligned} \min_{j \in I} \{-F_j^\circ(x; d)\} &\leq 0 \\ \sup_{\|d\| \leq 1} \min_{j \in I} \{-F_j^\circ(x; d)\} &= 0 \\ \pi(x) &= 0. \end{aligned}$$

So for every $x \in X$, $\pi(x) \geq 0$ and x is critical for F if and only if $\pi(x) = 0$.

Theorem 4.1. *Suppose assumptions (A1), (A2) and (A3) hold, then every accumulation point of the sequence $\{x_k\}$ is critical point for F .*

Proof. Based on the above discussions, it suffices to show that any accumulation point of $\{x_k\}$ is the solution of $\pi(\cdot) = 0$. Let x^* be an accumulation point of $\{x_k\}$. Without any loss of generality we may assume that the subsequence $\{x_k\}_{k \in \kappa}$ converges to x^* .

Suppose that d_k is the optimal solution of $NSP_\epsilon(x_k)$. By Theorem 2.5 in [1] we have

$$F_j(x_k) - F_j(x_{k+1}) \geq \langle \xi, x_k - x_{k+1} \rangle - o(\|\alpha_k d_k\|)^2,$$

so

$$F_j(x_k) - F_j(x_{k+1}) \geq \langle \xi, -\alpha_k d_k \rangle - o(\|x_k - x_{k+1}\|)^2.$$

Therefore, from Assumption (A3)

$$F_j(x_k) - F_j(x_{k+1}) \geq -\langle \xi, d_k \rangle.$$

By using Theorem 2.1 for all $j \in I$, we have

$$F_j^\circ(x_k; d_k) \geq \langle \xi, d_k \rangle \quad \forall \xi \in \partial_c F_j(x_k).$$

Therefore, we have

$$-F_j^\circ(x_k; d_k) \leq -\langle \xi, d_k \rangle \leq F_j(x_k) - F_j(x_{k+1})$$

Thus,

$$\begin{aligned} -F_j^\circ(x_k; d_k) &\leq F_j(x_k) - F_j(x_{k+1}) \implies \\ \min_{j \in I} \{-F_j^\circ(x_k; d_k)\} &\leq \min_{j \in I} \{F_j(x_k) - F_j(x_{k+1})\}. \end{aligned}$$

Therefore

$$\pi(x_k) \leq \min_{j \in I} \{F_j(x_k) - F_j(x_{k+1})\}.$$

By using Assumption (A2)

$$\begin{aligned} \sum_{\substack{k \rightarrow \infty \\ k \in \kappa}} \pi(x_k) &\leq \sum_{\substack{k \rightarrow \infty \\ k \in \kappa}} \min_{j \in I} \{F_j(x_k) - F_j(x_{k+1})\} \\ &\leq \min_{j \in I} \{F_j(x_0) - F_j(x^*)\} \\ &\leq M. \end{aligned} \tag{3}$$

Next we prove $\pi(x^*) = 0$ by contradiction i.e. assume $\pi(x^*) > 0$, which implies there are $\beta > 0$ and $\epsilon_0 > 0$ such that

$$\forall 0 < \epsilon < \epsilon_0, \quad \|x_k - x^*\| \leq \epsilon, \quad \pi(x_k) \geq \beta > 0.$$

This means that

$$\sum_{\substack{k \rightarrow \infty \\ k \in \kappa}} \pi(x_k) \geq \sum_{\substack{k \rightarrow \infty \\ k \in \kappa}} \beta = \infty.$$

This contradicts (3). Therefore we have $\pi(x^*) = 0$ and x^* is critical point for F . \square

References

- [1] F. H. Clarke, Yu. S. Ledyaev, R. J. Stern, and P. R. Wolensk, *Nonsmooth Analysis and Control Theory*, Springer, New York, 1998.
- [2] G. Eichfelder, *Adaptive Scalarization Methods in Multiobjective Optimization*, Springer Verlag, Berlin, Heidelberg, 2008.

- [3] J. Fliege and B. F. Svaiter, Steepest descent method for multicriteria optimization, *Operations Research*, 51 (2000), 479-494.
- [4] A. M. Geoffrion, Proper efficiency and the theory of vector maximization, *J. Optim. Theory Appl.*, 22 (1968), 618-630.
- [5] S. Qu, M. Goh, and F. T. S. Chan, Quasi-Newton methods for solving multiobjective optimization, *Operations Research Letters*, 39 (2011), 397-399.
- [6] S. Qu, M. Goh, and B. Liang, Trust region methods for solving multiobjective optimization, *Optimization Methods and Softwares*, (2012), 1-16.
- [7] Y. Sawaragi, H. Nakayama, and T. Tanino, *Theory of Multiobjective Optimization*, Vol. 176 of Mathematics in Science and Engineering, Academic Press, Orlando, San Diego, New York, London, 1985.
- [8] S. Shan and G. G. Wang, An efficient pareto set identification approach for multiobjective optimization on black-box functions, *J. Mechanical Design*, 127 (2005), 866-874.
- [9] W. Stadtler and J. Dauer, *Multicriteria Optimization in Engineering: A Tutorial and Survey*, American Institute of Aeronautics and Astronautics, 1992.
- [10] D. A. G. Vieira, R. H. C. Takahashi, and R. R. Saldanha, Multicriteria optimization with a multiobjective golden section line search, *Math. Programming*, Ser. A, 131 (2012), 131-161.

Najmeh Hoseini Monjezi
Department of Mathematics
Ph.D Student of Mathematics
Univetsity of Isfahan
Isfahan, Iran
E-mail: hoseini_najmeh@yahoo.com