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On the Capability of Finite Abelian Pairs of Groups

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Abstract. A group G is called capable if it is isomorphic to the group of inner automorphisms of some group H. The notion of capable groups was extended to capable pairs by G. Ellis, in 1996. Recently, a classification of capable pairs of finite abelian groups was given by A. Pourmirzaei, A. Hokmabadi and S. Kayvanfar. In this paper, we give a different characterization of capable pairs of finite abelian groups in terms of a condition on the lattice of subgroups.

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1. Introduction

A group G is called capable if it is isomorphic to the group of inner automorphisms of another group, or equivalently $G \cong E/Z(E)$ for some group E. As P. Hall [3] remarked, characterization of capable groups are important in classifying groups of prime power order. The study of

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capable groups was started by R. Baer [1], who determined all capable groups which are direct sums of cyclic groups. Consequently, all capable finite abelian groups are characterized by R. Baer's result. Recently, Z. Sunic [5] provided a different characterization of capable finite abelian groups by considering a condition on the lattice of subgroups. He proved that a finite abelian group G is capable if and only if there exists a family $\{H_i\}_{i\in I}$ of subgroups of G with trivial intersection such that the union generates G, and all quotients G/H_i $(i \in I)$ have the same exponent.
The theory of the capability of groups was extended in an interesting way to a theory for pairs of groups, by G. Ellis [2]. By a pair of groups we mean a group G with a normal subgroup N, and it is denoted by (G, N). Ellis [2] defined a capable pair of groups in terms of a relative central extension. In 2011, A. Pourmirzaei, A. Hokmabadi and S. Kayvanfar [4], determined all capable pairs of finitely generated abelian groups as follows.

Theorem 1.1. ([4]) *Let G be a finitely generated abelian group as follows:*

$$G = \langle x_1\rangle \oplus ... \oplus \langle x_m\rangle \oplus \langle y_1\rangle \oplus ... \oplus \langle y_r\rangle,$$

where $\langle x_i\rangle \cong \mathbb{Z}$, for $1 \leqslant i \leqslant m$ and $o(y_i) = d_i$ for $1 \leqslant i \leqslant r$, such that $d_{i+1} \mid d_i$. If $N \leqslant G$ such that $N = \langle x_1^{\alpha_1}\rangle \oplus ... \oplus \langle x_m^{\alpha_m}\rangle \oplus \langle y_1^{\beta_1}\rangle \oplus ... \oplus \langle y_r^{\beta_r}\rangle$, then (G, N) is capable if and only if

(i) $m \geqslant 2$, or

(ii) $m = 0$, $r \geqslant 2$ and $d_1 \mid [d_2, \beta_1]$,

in which $[d_2, \beta_1]$ means the least common multiple of d_2 and β_1.

In this paper, we give a condition in terms of proper subgroups which is equivalent to the capability of the pair (G, N), where $G = \langle y_1\rangle \oplus \ldots \oplus \langle y_k\rangle$ and $N = \langle y_1^{\beta_1}\rangle \oplus \ldots \oplus \langle y_{k-1}^{\beta_{k-1}}\rangle \oplus \langle y_k^{\beta_k}\rangle$, such that $o(y_i^{\beta_i}) \mid o(y_{k-1}^{\beta_k})$ for $1 \leqslant i \leqslant k-2$.

2. Main Result

The following lemmas are essential to prove the main result.

Lemma 2.1. [5, Lemma 2] *Let $A = \langle a \rangle \cong \mathbb{Z}_n$ and $B = \langle b \rangle \cong \mathbb{Z}_n$, where $n > 1$ and let $C = AB$ be the direct product of A and B (written internally). Let X be the set of elements of C of order n.*

(a) $a^i b^j \in X$ if and only if the greatest common divisor of i, j and n is 1.

(b) For every $x \in X$ there exists $y \in X$ such that, $C = \langle x \rangle \langle y \rangle$ as an internal direct product.

(c) $C = \cup_{x \in X} \langle x \rangle$.

Lemma 2.2. *Let $G = \langle y_1 \rangle \oplus \ldots \oplus \langle y_k \rangle$ be a finite non trivial abelian group such that $o(y_i) = n_i$ $(1 \leqslant i \leqslant k)$ and $n_{i-1} | n_i$. Consider*

$$N^* = \langle y_1^{\beta_1} \rangle \oplus \ldots \oplus \langle y_{k-2}^{\beta_{k-2}} \rangle \oplus \langle y_{k-1}^{\beta_k} \rangle \oplus \langle y_k^{\beta_k} \rangle$$

as a subgroup of G such that $o(y_i^{\beta_i}) | o(y_{k-1}^{\beta_k})$ $(1 \leqslant i \leqslant k-2)$. Suppose that $m = [n_k, \beta_k]/[n_{k-1}, \beta_k]$ and let p^t be the highest power of prime number p such that $p^t \mid m$ and A_p be the unique subgroup of $N_k = \langle y_k^{\beta_k} \rangle$ of order p^t. Then for any subgroup H of N^*, $A_p \subseteq H$ if and only if N^*/H has exponent dividing $[n_k, \beta_k]/\beta_k p^t$.

Proof. Assume that H is a subgroup of N^* that contains A_p and suppose that $g \in N^*$. Since $o(y_i^{\beta_i})$ divides $o(y_{k-1}^{\beta_k}) = [n_{k-1}, \beta_k]/\beta_k$, for all $1 \leqslant i \leqslant k-2$, we have $g^{[n_{k-1}, \beta_k]/\beta_k} \in N_k$.
So $(g^{[n_{k-1}, \beta_k]/\beta_k})^{m/p^t} = g^{[n_k, \beta_k]/\beta_k p^t} \in N_k$, and $o(g^{[n_k, \beta_k]/\beta_k p^t}) \mid p^t$. Therefore $g^{[n_k, \beta_k]/\beta_k p^t}$ belongs to A_p and $(gH)^{[n_k, \beta_k]/\beta_k p^t} = 1_{N^*/H}$. This implies that the exponent of N^*/H divides $[n_k, \beta_k]/\beta_k p^t$.
For the converse, let H be a subgroup of N^* and $\exp(N^*/H)$ divide $[n_k, \beta_k]/\beta_k p^t$. Put $g = y_k^{\beta_k}$. By the assumption we have $g^{[n_k, \beta_k]/\beta_k p^t} \in H$. On the other hand, one can see that $g^{[n_k, \beta_k]/\beta_k p^t}$ is an element of N_k of order p^t. This implies that $A_p \subseteq H$ and the assertion follows. $\square$

Now we are ready to state and prove the main result.

Theorem 2.3. *Let $G = \langle y_1 \rangle \oplus \ldots \oplus \langle y_k \rangle$ be a finite non trivial abelian group, where $o(y_i) = n_i$ $(1 \leqslant i \leqslant k)$, such that $n_{i-1} | n_i$ for $i = 2, \cdots, k$ and $N = \langle y_1^{\beta_1} \rangle \oplus \ldots \oplus \langle y_{k-1}^{\beta_{k-1}} \rangle \oplus \langle y_k^{\beta_k} \rangle$ such that $o(y_i^{\beta_i}) | o(y_{k-1}^{\beta_k})$, for*

$1 \leqslant i \leqslant k-2$. *Put*

$$N^* = \langle y_1^{\beta_1} \rangle \oplus \ldots \oplus \langle y_{k-2}^{\beta_{k-2}} \rangle \oplus \langle y_{k-1}^{\beta_k} \rangle \oplus \langle y_k^{\beta_k} \rangle.$$

Then the following conditions are equivalent.

(a) *The pair (G, N) is capable.*

(b) *There exists a family of proper subgroups $\{H_i\}_{i \in I}$ of N^* such that*
(i) $\cap_{i \in i} H_i = 1$,
(ii) $\cup_{i \in I} H_i = N^*$,
(iii) $(N^*/H_i) \cong (N^*/H_j)$, *for every* $i, j \in I$,
(iv) $H_i \cong H_j$, *for every* $i, j \in I$.

(c) *There exists a family of subgroups $\{H_i\}_{i \in I}$ of N^* such that*
(i) $\cap_{i \in i} H_i = 1$,
(ii) $\langle \cup_{i \in I} H_i \rangle = N^*$,
(iii) $\exp(N^*/H_i) = \exp(N^*/H_j)$, *for every* $i, j \in I$.

(d) *There exists a family of subgroups $\{H_i\}_{i \in I}$ of N^* such that*
(i) $\cap_{i \in i} H_i = 1$,
(ii) $\langle \cup_{i \in I} H_i \rangle = N^*$,
(iii) $\exp(H_i) = \exp(H_j)$, *for every* $i, j \in I$.

Proof. Let $N', N_1, \ldots, N_k$ be subgroups of N^* such that $N' = \langle y_{k-1}^{\beta_k} \rangle \cong \mathbb{Z}_{m'_{k-1}}$ and $N_i = \langle y_i^{\beta_i} \rangle \cong \mathbb{Z}_{m_i}$ $(1 \leqslant i \leqslant k)$. Then $N^* = N_1 \ldots N_{k-2} N' N_k$. First we show that (a) implies (b). Let (G, N) be capable. Then in view of Theorem 1.1, we have $k \geqslant 2$ and $n_k | [n_{k-1}, \beta_k]$. Let for $i = 1, \ldots, k-2$,

$$H_i = \langle y_1^{\beta_1}, \ldots, y_{i-1}^{\beta_{i-1}}, y_i^{\beta_i} y_k^{\beta_k m_k / m_i}, y_{i+1}^{\beta_{i+1}}, \ldots, y_{k-1}^{\beta_k} \rangle,$$

$$H_{k-1} = \langle y_1^{\beta_1}, \ldots, y_{k-2}^{\beta_{k-2}}, y_{k-1}^{\beta_k} y_k^{\beta_k m_k / m'_{k-1}} \rangle,$$

and

$$H_k = \langle y_1^{\beta_1}, \ldots, y_{k-2}^{\beta_{k-2}}, y_{k-1}^{\beta_k} \rangle.$$

Step 1. $H_i \cong N_1 \ldots N_{k-2} N'$, for $1 \leqslant i \leqslant k$.

It is clear that, $H_k \cong N_1 \ldots N_{k-2} N'$. Also, since $o(y_{k-1}^{\beta_k} y_k^{\beta_k m_k / m'_{k-1}}) = m'_{k-1}$ and $\langle y_{k-1}^{\beta_k} y_k^{\beta_k m_k / m'_{k-1}} \rangle \cap \langle y_1^{\beta_1}, \ldots, y_{k-2}^{\beta_{k-2}} \rangle = 1$, we have $H_{k-1} \cong$

$N_1 \dots N_{k-2}N'$.
Similarly, one can show that $H_i \cong N_1 \dots N_{k-2}N'$, for $1 \leqslant i \leqslant k-2$.

Step 2. $N^*/H_i \cong N_k$, for $i = 1, \dots, k-2$.
It is clear that $N^*/H_k \cong N_k$. Using Dedekind's Modular Law, we have

$$\frac{N^*}{H_{k-1}} \cong \frac{N'N_k}{\langle y_{k-1}^{\beta_k} y_k^{\beta_k m_k/m'_{k-1}} \rangle}.$$

On the other hand, since $N'N_k = \langle y_{k-1}^{\beta_k} y_k^{\beta_k m_k/m'_{k-1}} \rangle N_k$, we have

$$N^*/H_{k-1} \cong N_k/\langle y_{k-1}^{\beta_k} y_k^{\beta_k m_k/m'_{k-1}} \rangle \cap N_k.$$

Now it is enough to show that

$$\langle y_{k-1}^{\beta_k} y_k^{\beta_k m_k/m'_{k-1}} \rangle \cap N_k = 1.$$

Assume that $g = (y_{k-1}^{\beta_k} y_k^{\beta_k m_k/m'_{k-1}})^s \in N_k$, for some integer s. Then there exists an integer ℓ such that

$$g = (y_{k-1}^{\beta_k})^s (y_k^{\beta_k m_k/m'_{k-1}})^s = y_k^{\beta_k \ell}.$$

Thus $(y_{k-1}^{\beta_k})^s \in N' \cap N_k = 1$ and so $(y_{k-1}^{\beta_k})^s = 1$. Then m'_{k-1} must divide s. It follows that $g = 1$. Therefore $N^*/H_{k-1} \cong N_k$.
Similarly one can show that $N^*/H_i \cong N_k$, for $i = 1, \dots, k-2$.

Step 3. $\cap_{i=1}^k H_i = 1$.
Let $g = (y_1^{\beta_1})^{s_1} \dots (y_{k-1}^{\beta_k})^{s_{k-1}} (y_k^{\beta_k})^{s_k}$ be an arbitrary element in $\cap_{i=1}^k H_i$, such that $0 \leqslant s_i < m_i$, for $i = 1, \dots, k$. Since $g \in H_k$, we have $s_k = 0$. Also for $1 \leqslant i \leqslant k-2$, $g \in H_i$ implies that

$$g = (y_1^{\beta_1})^{s_1} \dots (y_{k-1}^{\beta_k})^{s_{k-1}} = (y_1^{\beta_1})^{r_1} \dots (y_i^{\beta_i})^{r_i} (y_k^{\beta_k m_k/m_i})^{r_i} \dots (y_{k-1}^{\beta_k})^{r_{k-1}},$$

for some integers $r_1, \dots, r_{k-1}$. Then we have $s_j = r_j$, for all $1 \leqslant j \leqslant k-1$, and $(y_k^{\beta_k})^{m_k r_i/m_i} = 1$. Therefore m_k divides $m_k r_i/m_i = m_k s_i/m_i$ and so $m_i \mid s_i$, for $i = 1, \dots, k-1$. This implies that $g = 1$.

Step 4. $N^* = \langle \cup_{i \in I} H_i \rangle$.
By hypothesis, $n_k | [n_{k-1}, \beta_k]$. It implies that $m'_{k-1} = m_k$ and so $y_{k-1}^{\beta_k} y_k^{\beta_k} \in$

H_{k-1}. On the other hand, $y_1^{\beta_1}, \ldots, y_{k-2}^{\beta_{k-2}}, y_{k-1}^{\beta_k} \in \cup_{i=1}^k H_i$. Therefore we have $N^* \subseteq \langle \cup_{i\in I} H_i \rangle$.
Now the family $H = \{H_i\}_{i=1}^k$ satisfies the conditions (ii), (iii), (iv) of (b), but this family is not enough for condition (i). Therefore we need to extend this family to a larger family of subgroups such that all conditions hold.
Since $m'_{k-1} = m_k$, we can consider $n_{k-1}/(n_{k-1}, \beta_k) = n_k/(n_k, \beta_k) = m$. Put $X = \{x \in N'N_k | o(x) = m\}$. Assume that $H_x = N_1 \ldots N_{k-2}\langle x \rangle$, for all $x \in X$. It is clear that, $H_x \cong N_1 \ldots N_{k-2}N'$. Also by part (b) of Lemma 2.1, $N'N_k = \langle x \rangle \langle y \rangle$, for some $y \in X$. Then we have

$$\frac{N^*}{H_x} \cong \frac{N_1 \ldots N_{k-2}N'N_k}{N_1 \ldots N_{k-2}\langle x \rangle} \cong \frac{N'N_k}{\langle x \rangle} \cong \frac{\langle x \rangle \langle y \rangle}{\langle x \rangle} \cong \langle y \rangle \cong N_k.$$

Also in view of part (c) of Lemma 2.1,

$$\begin{aligned} \cup_{x\in X} H_x &= \cup_{x\in X} N_1 \ldots N_{k-2}\langle x \rangle \\ &= N_1 \ldots N_{k-2} \cup_{x\in X} \langle x \rangle \\ &= N_1 \ldots N_{k-2}N'N_k \\ &= N^*. \end{aligned}$$

Now the union of families $\{H_i\}_{1\leqslant i\leqslant k}$ and $\{H_x\}_{x\in X}$ satisfies the condition (i), (ii), (iii) and (iv) and the result holds.
Clearly (b) implies (c) and (d).
Now, we show that (c) implies (a). By Theorem 1.1, it is enough to show that $k \geqslant 2$ and $n_k | [n_{k-1}, \beta_k]$.

Step 1. $k \geqslant 2$.
By the method of reductio ad absurdum, suppose that $k = 1$ and G is a cyclic group. Then for every subgroup H_i $(i \in I)$ of N^*, we have $\exp(N^*/H_i) = |N^*|/|H_i|$. Then condition (iii) implies that $|N^*|/|H_i| = |N^*|/|H_j|$ and thus $|H_i| = |H_j| = d$, for all $i, j \in I$. Since N^* is cyclic, it has a unique subgroup of order d. Hence the family $\{H_i\}_{i\in I}$ contains only one subgroup which is a contradiction. Therefore $k \geqslant 2$.

Step 2. $n_k | [n_{k-1}, \beta_k]$.
By the method of reductio ad absurdum, suppose that n_k does not divide $[n_{k-1}, \beta_k]$. Then $m = [n_k, \beta_k]/[n_{k-1}, \beta_k] > 1$. Assume that p is a

prime number which divides m and A_p is the unique subgroup of N_k of order p^t, where p^t is the highest power of p such that $p^t \mid m$. Also, let p^T be the highest power of p such that $p^T|[n_k, \beta_k]/\beta_k$. On the other hand, $\exp(N^*) = [n_k, \beta_k]/\beta_k$ and also $N^* = \langle \cup_{i\in I} H_i \rangle$ by condition (ii) of (c). Hence there exists at least one subgroup in the family $\{H_i\}_{i\in I}$ whose exponent is divisible by p^T. Let H_j be such a subgroup with $p^T|\exp(H_j)$. Since H_j is abelian, so it contains an element h of order p^T. Then $h^{p^{T-t}}$ is an element of N_k of order p^t, and so $h^{p^{T-t}}$ generates A_p. This implies that $A_p \subseteq H_j$. Then using Lemma 2.2, one can see that $\exp(N^*/H_j)|[n_k, \beta_k]/\beta_k p^t$. Hence by condition (iii), we have $\exp(N^*/H_i)|[n_k, \beta_k]/p^t$, for all $i \in I$. Therefore in view of Lemma 2.2, every subgroup H_i in the family of $\{H_i\}_{i\in I}$ should contain A_p and so $A_p \subseteq \cap_{i\in I} H_i$ which is a contradiction.
Using a similar method one can see that (d) implies (a) and hence the proof is completed.

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