LIE SYMMETRIES OF SCHRÖDINGER EQUATION ON A SPHERE

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Abstract. We are going to discuss an object with a mass attached to a spring and vibrating on the surface of a sphere (see Figure 1). To do this, we first introduce a new equation on a sphere using the Schrödinger equation. In fact, the paper considers the question of a quantum system obeying the Schrodinger equation on a Sphere. After a brief introduction we set up the Hamiltonian of the system and the corresponding Schrodinger equation, consider infinitesimal generators and their Lie algebra and its Adjoint presentation. The paper also contains a section on symmetry reduction using similarity variables which are used in our study of the 3D quantum harmonic oscillator on a sphere as a special case of the new equation, and possible solutions are proposed.

1. Introduction

Studies in this area are in progress since such equations depict the states and properties of nonlinear phenomena, broaden vision in terms of physical aspects, and then become more practical in engineering and other sciences, so the search for accurate solutions is important in Nonlinear is in several ways like plasma laser radiation [1, 9, 10]. To obtain

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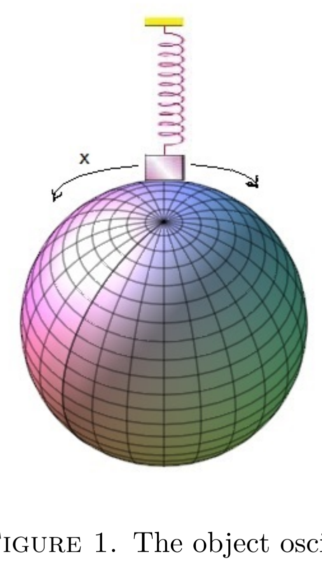


Figure 1. The object oscillates on a sphere.

an equation that can describe the oscillation of an object attached to a spring on a sphere, in general, we first consider Schrödinger equation in three-dimensional space:

$$-\frac{h^2}{2m}\Delta\Psi + v(x,y,z)\Psi = ih\Psi_t. \tag{1}$$

This equation reflects the wave nature of our quantum solutions $\Psi(x, y, z)$. A significant part of quantum mechanics is devoted to the study of solutions to the Schrödinger equation. Equation (1) governs the time dependence of the wave-function of an object moving inside a given potential, $v(x, y, z)$. A unique role is played by solutions to (1) that have the simple form: $\Psi = \psi(x,y,z)exp(\dfrac{-iEt}{h})$ where the function ψ satisfies the Schrödinger equation of the Schrödinger eigenvalue equation:

$$\begin{cases} -\dfrac{h^2}{2m}\Delta\psi + v(x,y,z)\psi = E\psi, \\ \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \end{cases} \tag{2}$$

which is time-independent. In both of these equations, h and m represent real constants. E is a constant that emerges during the separation of variables procedure. Sometimes quantum problems arise on a sphere. Therefore, it is necessary to examine Equation (2) on a sphere. It is well known, the metric on $S^2 \times R$ is:

$$ds^2 = dz^2 - dx^2 - sin^2 x dy^2 \qquad f \in C^\infty(G). \tag{3}$$

Adjusting the metric (3) on $S^2 \times R$ and rewriting Equation (2), the differential equation on the sphere would be:

$$u_{zz} = u_{xx} + (cotx)u_x + (csc^2 x)u_{yy} + (2m/h^2)(E - v)u \tag{4}$$

where u is the function ψ defined on $S^2 \times R$.
The Schrodinger equation for a harmonic oscillator may be obtained by using the classical spring potential

$$\begin{array}{c} v(x,y,z) = \frac{1}{2}q_x x^2 + \frac{1}{2}q_y y^2 + \frac{1}{2}q_z z^2, \\ \omega_x^2 = \frac{q_x}{m}, \qquad \omega_y^2 = \frac{q_y}{m}, \qquad \omega_z^2 = \frac{q_z}{m}, \end{array} \tag{5}$$

where ω is angular frequency and q_x, q_y and q_z are bond force constants. Adjusting the metric (3) on $S^2 \times R$ and rewriting Equation (5), the spring potential on the sphere would be:

$$v(x,y,z) = \tfrac{1}{2}q_x x^2 + \tfrac{1}{2}q_y sin^4(x) y^2 + \tfrac{1}{2}q_z z^2, \tag{6}$$

The Schrodinger equation (4) with this form of potential is

$$\begin{array}{c} u_{zz} = u_{xx} + (cotx)u_x + (csc^2 x)u_{yy} \\ +(2m/h^2)(E - \frac{1}{2}q_x x^2 - \frac{1}{2}q_y sin^4(x) y^2 - \frac{1}{2}q_z z^2)u, \end{array} \tag{7}$$

where the particle oscillates on a sphere.

Equation (4) is the general state of the Equation (7), which we try to solve by the method of Lie symmetry groups. Equation (7) describes an oscillating object on the surface of a sphere. This equation is reported in the last section, and possible solutions are presented. Besides, the symmetry group approach or Lie's approach itself, which is a computational method to find invariant solutions, is crucially utilized in revealing the answer of PDEs and ODEs. Performing the Lie symmetry group procedure; the problem of symmetry classification for different equations is widely considered in various spaces [2, 5–8]. Utilizing this plan of action, one finds appropriate solutions via studied ones, investigates the invariant solutions, and decreases the order of ODEs [3, 4, 11]. In this article, utilizing Lie's procedure, we get symmetries of the Schrödinger differential equation on the sphere. Next, utilizing Ibragimov's method an optimal sub-algebras structure related to the symmetry Lie algebra is presented. The article is collected as follows:

- We describe the symmetry algebra infinitesimal generators of Equation (4), and gain several outcomes.
- We build the optimal systems of sub-algebras.
- We give the Lie invariants, and some other concepts related to the infinitesimal symmetries of Equation (4).
- Possible solutions of the 3D Harmonic Oscillator on the sphere are discussed.

2. Infinitesimal generators of Equation (4)

Commonly,

$$\begin{cases} \Delta_\hbar(x, u^{(j)}) = 0, \quad \hbar = 1, ..., n, \\ x = (x^1, ..., x^p), \\ u = (u^1, ..., u^q), \end{cases} \tag{8}$$

define a PDE structure of order jth, where u is dependent on x, and $u^{(i)}$ means $\partial_i u/(\partial x)^i$. Local infinitesimal generators of the above structure that as a Lie group acts on the manifold $X \times U$, is:

$$\tilde{x}^i = x^i + \delta\varsigma^i(x, u) + \emptyset(\delta^2), \qquad i = 1, ..., p, \tag{9}$$

$$\tilde{u}^j = u^j + \delta\phi_j(x, u) + \emptyset(\delta^2), \qquad j = 1, ..., q, \tag{10}$$

where ς^i and ϕ^j represent the infinitesimal transformations for $\{x^1, ..., x^p\}$ and $\{u^1, ..., u^q\}$, respectively. A given local infinitesimal generators related to the all transformations (9) as a group, is

$$\mathfrak{X} = \sum_{i=1}^{p} \varsigma^i(x, u)\partial_{x^i} + \sum_{j=1}^{q} \phi_j(x, u)\partial_{u^j} \tag{11}$$

Now to utilize the mentioned technique for Equation (4), infinitesimal transformations with one parameter as a Lie group is assumed: (x^1, x^2 and x^3 are substituted by x, y and z respectively to not use index,)

$$\begin{cases} \tilde{x} &= x + \delta\varsigma^1(x, y, z, u, v) + \emptyset(\delta^2), \\ \tilde{y} &= y + \delta\varsigma^2(x, y, z, u, v) + \emptyset(\delta^2), \\ \tilde{z} &= z + \delta\varsigma^3(x, y, z, u, v) + \emptyset(\delta^2), \\ \tilde{u} &= u + \delta\phi_1(x, y, z, u, v) + \emptyset(\delta^2), \\ \tilde{v} &= v + \delta\phi_2(x, y, z, u, v) + \emptyset(\delta^2). \end{cases}$$

The related symmetry generator will be:

$$\begin{aligned}\mathfrak{X} = \varsigma^1(x,y,z,u,v)\partial_x + \varsigma^2(x,y,z,u,v)\partial_y + \varsigma^3(x,y,z,u,v)\partial_t + \\ \phi_1(x,y,z,u,v)\partial_u + \phi_2(x,y,z,u,v)\partial_v.\end{aligned} \tag{12}$$

The status of existence of invariance is equivalent to the following explanation:

$$\begin{aligned}&Pr^{(2)}\mathfrak{X}[u_{xx} + (cotx)u_x + (csc^2x)u_{yy} - u_{zz} + (2m/h^2)(E - v(x,y,z))u] = 0,\\ &\text{whenever :}\\ &u_{xx} + (cotx)u_x + (csc^2x)u_{yy} - u_{zz} + (2m/h^2)(E - v(x,y,z))u = 0.\end{aligned}$$

Since, ς^1, ς^2, ς^3, ϕ_1 and ϕ_2 are functions with variables x, y, z, u and v, vanishing the sole coefficients, we earn the following specific equations:

$$\begin{aligned}&h^2 sin(x)\varsigma^2_v = 0, \quad h^2 sin(x)\varsigma^2_{uu} = 0, \quad h^2 sin(x)\varsigma^1_v = 0,\\ &h^2 sin(x)\varsigma^1_{uu} = 0, \quad h^2 sin(x)\varsigma^2_{vv} = 0, \quad h^2 sin(x)\varsigma^3_v = 0,\\ &\ldots.\end{aligned}$$

The number of these equations is 89. Examining these PDEs, we have a statement as:

Theorem 2.1. *The point symmetry group of Equation* (4) *as a Lie group owns a Lie sub-algebra consists of* (12) *which* ξs *and* ϕs *are the infinitesimals as follows:*

$$\begin{aligned}
\varsigma^1 =\ & ((c_6 sin(y) + c_7 cos(y))cos(z) + sin(z)(c_3 sin(y) + c_4 cos(y)))cos(x)\\
& + c_1 sin(y) + c_2 cos(y) + c_5 sin(z) sin(x) + c_8 cos(z) sin(x),\\
\varsigma^2 =\ & \frac{(c_6 cos(y) - c_7 sin(y))cos(z) + sin(z)(c_3 cos(y) + c_4 sin(y))}{sin(x)}\\
& + \frac{c_1 cos(y) - c_2 sin(y)}{tan(x)} + c_{10},\\
\varsigma^3 =\ & (-(c_6 sin(y) + c_7 cos(y))sin(z) + cos(z)(c_3 sin(y) + c_4 cos(y)))sin(x)\\
& - c_5 cos(z)cos(x) + c_8 cos(x) sin(z) + c_9,\\
\phi_1 =\ & \frac{1}{2}((c_6 sin(y) + c_7 cos(y))cos(z) + sin(z)(c_3 sin(y) + c_4 cos(y)))sin(x)u,\\
& \frac{1}{2}c_5 sin(z)cos(x)u + \frac{1}{2}c_8 cos(z)cos(x)u + c_{11}u + \alpha(u),\\
\phi_2 =\ & \frac{1}{4sin^2(x)mu}((((c_6 sin(y) + c_7 cos(y))cos(z) + sin(z)(c_3 sin(y) + c_4 cos(y)))\\
& sin^2(x) - (c_5 sin(z) + c_8 cos(z))cos(x) sin(x))(\frac{h^2}{8} + mE + mv)\\
& (-8) sin(x)u + 2h^2 sin^2(x)(\alpha_{xx} - \alpha_{zz}) + 2h^2\alpha_{yy}\\
& + (\frac{1}{4}h^2 cos(x)u_x + \frac{1}{2}sin(x)m(E - v))8 sin(x)).
\end{aligned}$$

where c_i, $i = 1, ..., 11$ *are real constant.*

Corollary 2.2. *Every Lie group consists of symmetries with one-parameter of* (4) *has eleven-dimensional Lie subalgebra obtained from the following generators:*

$$
\begin{aligned}
\mathfrak{X}_1 =\ & \partial_y,\\
\mathfrak{X}_2 =\ & \partial_z,\\
\mathfrak{X}_3 =\ & sin(y)\partial_x + cos(y)cot(x)\partial_y,\\
\mathfrak{X}_4 =\ & cos(y)\partial_x - sin(y)cot(x)\partial_y,\\
\mathfrak{X}_5 =\ & sin(x)sin(z)\partial_x - cos(x)cos(z)\partial_z - \frac{1}{2}ucos(x)sin(z)\partial_u\\
& +\Omega cos(x)sin(z)\partial_v,\\
\mathfrak{X}_6 =\ & sin(x)cos(z)\partial_x + sin(z)cos(x)\partial_z - \frac{1}{2}ucos(x)cos(z)\partial_u\\
& +\Omega cos(x)cos(z)\partial_v,\\
\mathfrak{X}_7 =\ & cos(x)sin(z)sin(y)\partial_x + \frac{cos(y)sin(z)}{sin(x)}\partial_y + sin(x)sin(y)cos(z)\partial_z\\
& +\frac{1}{2}usin(x)sin(y)sin(z)\partial_u - \Omega sin(x)sin(y)sin(z)\partial_v,\\
\mathfrak{X}_8 =\ & cos(x)sin(z)cos(y)\partial_x - \frac{sin(y)sin(z)}{sin(x)}\partial_y + sin(x)cos(y)cos(z)\partial_z\\
& +\frac{1}{2}usin(x)cos(y)sin(z)\partial_u - \Omega sin(x)cos(y)sin(z)\partial_v,\\
\mathfrak{X}_9 =\ & cos(x)cos(z)sin(y)\partial_x + \frac{cos(y)cos(z)}{sin(x)}\partial_y - sin(x)sin(y)sin(z)\partial_z\\
& +\frac{1}{2}usin(x)sin(y)cos(z)\partial_u - \Omega sin(x)sin(y)cos(z)\partial_v,\\
\mathfrak{X}_{10} =\ & cos(x)cos(z)cos(y)\partial_x - \frac{sin(y)cos(z)}{sin(x)}\partial_y - sin(x)cos(y)sin(z)\partial_z\\
& +\frac{1}{2}usin(x)cos(y)cos(z)\partial_u - \Omega sin(x)cos(y)cos(z)\partial_v,\\
\mathfrak{X}_{11} =\ & u\partial_u,\\
\mathfrak{X}_\alpha =\ & \alpha\partial_u + \frac{(E-v)\alpha\partial_v}{u},
\end{aligned}
$$

where $\Omega = \dfrac{8mE - 8mv + h^2}{4m}$ *(and* $\partial_x \equiv \dfrac{\partial}{\partial x}, \cdots$*).*

We deliver Lie bracket for Eq.(4) by Table (1). The phrase $[\mathfrak{X}_i, \mathfrak{X}_j] = \mathfrak{X}_i\mathfrak{X}_j - \mathfrak{X}_j\mathfrak{X}_i$ characterizes the Values in row i^{th} and column j^{th}, $i, j = 1, ..., 11$.

For instance, the flow of $\mathfrak{X}_2$ in Corollary 2.2 is expressed by

$$\Phi_\epsilon = (x, y, z + \epsilon).$$

3. 1D subalgebras of Equation (4)

Utilizing the symmetry technique, one can specify the one parameter optimal structure of Equation (4). Providing special subgroups that offer different sorts of solutions is essential. Thus, we want to look for an invariant solution that is not identical to a transformation from the whole symmetry group. Such an issue causes to express the sense of an optimal structure of sub-algebra. In the study of 1D sub-algebras, the classification

TABLE 1. Lie algebra for Eq.(4).

[,]	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$\mathfrak{X}_4$	$\mathfrak{X}_5$	$\mathfrak{X}_6$	$\mathfrak{X}_7$	$\mathfrak{X}_8$	$\mathfrak{X}_9$	$\mathfrak{X}_{10}$	$\mathfrak{X}_{11}$
$\mathfrak{X}_1$	0	0	$\mathfrak{X}_4$	$-\mathfrak{X}_3$	0	0	$\mathfrak{X}_8$	$-\mathfrak{X}_7$	$\mathfrak{X}_{10}$	$-\mathfrak{X}_9$	0
$\mathfrak{X}_2$	$*$	0	0	0	$\mathfrak{X}_6$	$-\mathfrak{X}_5$	$\mathfrak{X}_9$	$\mathfrak{X}_{10}$	$-\mathfrak{X}_7$	$\mathfrak{X}_8$	0
$\mathfrak{X}_3$	$*$	$*$	0	$\mathfrak{X}_1$	$\mathfrak{X}_7$	$\mathfrak{X}_9$	$-\mathfrak{X}_5$	0	$-\mathfrak{X}_6$	0	0
$\mathfrak{X}_4$	$*$	$*$	$*$	0	$\mathfrak{X}_8$	$\mathfrak{X}_{10}$	0	$-\mathfrak{X}_5$	0	$-\mathfrak{X}_6$	0
$\mathfrak{X}_5$	$*$	$*$	$*$	$*$	0	$-\mathfrak{X}_2$	$-\mathfrak{X}_3$	$-\mathfrak{X}_4$	0	0	0
$\mathfrak{X}_6$	$*$	$*$	$*$	$*$	$*$	0	0	0	$-\mathfrak{X}_3$	$-\mathfrak{X}_4$	0
$\mathfrak{X}_7$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-\mathfrak{X}_1$	$-\mathfrak{X}_2$	0	0
$\mathfrak{X}_8$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0	$-\mathfrak{X}_2$	0
$\mathfrak{X}_9$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	$-\mathfrak{X}_1$	0
$\mathfrak{X}_{10}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0	0
$\mathfrak{X}_{11}$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	0

question turns into arranging the adjoint representation orbits. An optimal structure of the sub-algebras question is replied by presuming a candidate of any set of related sub-algebras [13] and [12]. Adjoint candidate of every X_i, for $i = 1, ..., 11$ is characterized as:

$$\mathrm{Ad}(e^{(\mathfrak{s}.\mathfrak{X}_i)}.\mathfrak{X}_j) = \mathfrak{X}_j - \mathfrak{s}.[\mathfrak{X}_i, \mathfrak{X}_j] + \frac{\mathfrak{s}^2}{2}.[\mathfrak{X}_i, [\mathfrak{X}_i, \mathfrak{X}_j]] - \cdots , \tag{13}$$

where $\mathfrak{s}$ is a parameter and $[\mathfrak{X}_i, \mathfrak{X}_j]$ has characterized in Table (1) for $1 \le i, j \le 11$ ([12],p (199)). We show the Lie algebra of (13) by $\mathfrak{g}$, and we collect the adjoint action in Table (2). An optimal system of one-dimensional subalgebras is constructed by utilizing Ibragimov's method.

Theorem 3.1. *A 1D optimal structure of Eq.*(4) *is presented as:*

1) $\mathfrak{X}_1 \pm X_{11}$,
2) $\mathfrak{X}_2 \pm X_{11}$,
3) $\mathfrak{X}_3 \pm X_{11}$,
4) $\mathfrak{X}_4 \pm X_{11}$,
5) $\mathfrak{X}_5 \pm X_{11}$,
6) $\mathfrak{X}_6 \pm X_{11}$,
7) $\mathfrak{X}_7 \pm X_{11}$,
8) $\mathfrak{X}_8 \pm X_{11}$,
9) $\mathfrak{X}_9 \pm X_{11}$,
10) $\mathfrak{X}_{10} \pm X_{11}$,
11) $\mathfrak{X}_{11}$,
12) $\mathfrak{X}_1 \pm \mathfrak{X}_2 \pm X_{11}$,
13) $\mathfrak{X}_1 \pm \mathfrak{X}_5 \pm X_{11}$,
14) $\mathfrak{X}_1 \pm \mathfrak{X}_6 \pm X_{11}$,
15) $\mathfrak{X}_2 \pm \mathfrak{X}_3 \pm X_{11}$,
16) $\mathfrak{X}_2 \pm \mathfrak{X}_4 \pm X_{11}$,
17) $\mathfrak{X}_3 \pm \mathfrak{X}_8 \pm X_{11}$,
18) $\mathfrak{X}_3 \pm \mathfrak{X}_{10} \pm X_{11}$,
19) $\mathfrak{X}_4 \pm \mathfrak{X}_7 \pm X_{11}$,
20) $\mathfrak{X}_4 \pm \mathfrak{X}_9 \pm X_{11}$,
21) $\mathfrak{X}_5 \pm \mathfrak{X}_9 \pm X_{11}$,
22) $\mathfrak{X}_5 \pm \mathfrak{X}_{10} \pm X_{11}$,
23) $\mathfrak{X}_6 \pm \mathfrak{X}_7 \pm X_{11}$,
24) $\mathfrak{X}_6 \pm \mathfrak{X}_8 \pm X_{11}$,
25) $\mathfrak{X}_7 \pm \mathfrak{X}_{10} \pm X_{11}$,
26) $\mathfrak{X}_8 \pm \mathfrak{X}_9 \pm X_{11}$,

where $c_i \in \mathbb{R}$ are real numeral coefficients for $i = 1, \cdots, 4$.

TABLE 2. Adjoint presentation

Ad	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$\mathfrak{X}_4$
$\mathfrak{X}_1$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$cos(\mathfrak{s})\mathfrak{X}_3 - sin(\mathfrak{s})\mathfrak{X}_4$	$cos(\mathfrak{s})\mathfrak{X}_4 + sin(\mathfrak{s})\mathfrak{X}_3$
$\mathfrak{X}_2$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$\mathfrak{X}_4$
$\mathfrak{X}_3$	$cos(\mathfrak{s})\mathfrak{X}_1 + sin(\mathfrak{s})\mathfrak{X}_4$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$cos(\mathfrak{s})\mathfrak{X}_4 - sin(\mathfrak{s})\mathfrak{X}_1$
$\mathfrak{X}_4$	$cos(\mathfrak{s})\mathfrak{X}_1 - sin(\mathfrak{s})\mathfrak{X}_3$	$\mathfrak{X}_2$	$cos(\mathfrak{s})\mathfrak{X}_3 + sin(\mathfrak{s})\mathfrak{X}_1$	$\mathfrak{X}_4$
$\mathfrak{X}_5$	$\mathfrak{X}_1$	$cosh(\mathfrak{s})\mathfrak{X}_2 - sinh(\mathfrak{s})\mathfrak{X}_6$	$cosh(\mathfrak{s})\mathfrak{X}_3 - sinh(\mathfrak{s})\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_4 - sinh(\mathfrak{s})\mathfrak{X}_8$
$\mathfrak{X}_6$	$\mathfrak{X}_1$	$cosh(\mathfrak{s})\mathfrak{X}_2 + sinh(\mathfrak{s})\mathfrak{X}_5$	$cosh(\mathfrak{s})\mathfrak{X}_3 - sinh(\mathfrak{s})\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_4 - sinh(\mathfrak{s})\mathfrak{X}_{10}$
$\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_1 - sinh(\mathfrak{s})\mathfrak{X}_8$	$cosh(\mathfrak{s})\mathfrak{X}_2 - sinh(\mathfrak{s})\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_3 + sinh(\mathfrak{s})\mathfrak{X}_5$	$\mathfrak{X}_4$
$\mathfrak{X}_8$	$cosh(\mathfrak{s})\mathfrak{X}_1 + sinh(\mathfrak{s})\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_2 - sinh(\mathfrak{s})\mathfrak{X}_{10}$	$\mathfrak{X}_3$	$cosh(\mathfrak{s})\mathfrak{X}_4 + sinh(\mathfrak{s})\mathfrak{X}_5$
$\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_1 - sinh(\mathfrak{s})\mathfrak{X}_{10}$	$cosh(\mathfrak{s})\mathfrak{X}_2 + sinh(\mathfrak{s})\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_3 + sinh(\mathfrak{s})\mathfrak{X}_6$	$\mathfrak{X}_4$
$\mathfrak{X}_{10}$	$cosh(\mathfrak{s})\mathfrak{X}_1 + sinh(\mathfrak{s})\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_2 + sinh(\mathfrak{s})\mathfrak{X}_8$	$\mathfrak{X}_3$	$cosh(\mathfrak{s})\mathfrak{X}_4 + sinh(\mathfrak{s})\mathfrak{X}_6$
$\mathfrak{X}_{11}$	$\mathfrak{X}_1$	$\mathfrak{X}_2$	$\mathfrak{X}_3$	$\mathfrak{X}_4$

Ad	$\mathfrak{X}_5$	$\mathfrak{X}_6$	$\mathfrak{X}_7$	$\mathfrak{X}_8$
$\mathfrak{X}_1$	$\mathfrak{X}_5$	$\mathfrak{X}_6$	$cos(\mathfrak{s})\mathfrak{X}_7 - sin(\mathfrak{s})\mathfrak{X}_8$	$cos(\mathfrak{s})\mathfrak{X}_8 + sin(\mathfrak{s})\mathfrak{X}_7$
$\mathfrak{X}_2$	$cos(\mathfrak{s})\mathfrak{X}_5 - sin(\mathfrak{s})\mathfrak{X}_6$	$cos(\mathfrak{s})\mathfrak{X}_6 + sin(\mathfrak{s})\mathfrak{X}_5$	$cos(\mathfrak{s})\mathfrak{X}_7 - sin(\mathfrak{s})\mathfrak{X}_9$	$cos(\mathfrak{s})\mathfrak{X}_8 - sin(\mathfrak{s})\mathfrak{X}_{10}$
$\mathfrak{X}_3$	$cos(\mathfrak{s})\mathfrak{X}_5 - sin(\mathfrak{s})\mathfrak{X}_7$	$cos(\mathfrak{s})\mathfrak{X}_6 - sin(\mathfrak{s})\mathfrak{X}_9$	$cos(\mathfrak{s})\mathfrak{X}_7 + sin(\mathfrak{s})\mathfrak{X}_5$	$\mathfrak{X}_8$
$\mathfrak{X}_4$	$cos(\mathfrak{s})\mathfrak{X}_5 - sin(\mathfrak{s})\mathfrak{X}_8$	$cos(\mathfrak{s})\mathfrak{X}_6 - sin(\mathfrak{s})\mathfrak{X}_{10}$	$\mathfrak{X}_7$	$cos(\mathfrak{s})\mathfrak{X}_8 + sin(\mathfrak{s})\mathfrak{X}_5$
$\mathfrak{X}_5$	$\mathfrak{X}_5$	$cosh(\mathfrak{s})\mathfrak{X}_6 - sinh(\mathfrak{s})\mathfrak{X}_2$	$cosh(\mathfrak{s})\mathfrak{X}_7 - sinh(\mathfrak{s})\mathfrak{X}_3$	$cosh(\mathfrak{s})\mathfrak{X}_8 - sinh(\mathfrak{s})\mathfrak{X}_4$
$\mathfrak{X}_6$	$cosh(\mathfrak{s})\mathfrak{X}_7 + sinh(\mathfrak{s})\mathfrak{X}_2$	$\mathfrak{X}_6$	$\mathfrak{X}_7$	$\mathfrak{X}_8$
$\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_7 + sinh(\mathfrak{s})\mathfrak{X}_3$	$\mathfrak{X}_6$	$\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_8 - sinh(\mathfrak{s})\mathfrak{X}_1$
$\mathfrak{X}_8$	$cosh(\mathfrak{s})\mathfrak{X}_7 + sinh(\mathfrak{s})\mathfrak{X}_4$	$\mathfrak{X}_6$	$cosh(\mathfrak{s})\mathfrak{X}_7 + sinh(\mathfrak{s})\mathfrak{X}_1$	$\mathfrak{X}_8$
$\mathfrak{X}_9$	$\mathfrak{X}_5$	$cosh(\mathfrak{s})\mathfrak{X}_6 + sinh(\mathfrak{s})\mathfrak{X}_3$	$cosh(\mathfrak{s})\mathfrak{X}_7 + sinh(\mathfrak{s})\mathfrak{X}_2$	$\mathfrak{X}_8$
$\mathfrak{X}_{10}$	$\mathfrak{X}_5$	$cosh(\mathfrak{s})\mathfrak{X}_6 + sinh(\mathfrak{s})\mathfrak{X}_4$	$\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_8 + sinh(\mathfrak{s})\mathfrak{X}_2$
$\mathfrak{X}_{11}$	$\mathfrak{X}_5$	$\mathfrak{X}_6$	$\mathfrak{X}_7$	$\mathfrak{X}_8$

Ad	$\mathfrak{X}_9$	$\mathfrak{X}_{10}$	$\mathfrak{X}_{11}$
$\mathfrak{X}_1$	$cos(\mathfrak{s})\mathfrak{X}_9 - sin(\mathfrak{s})\mathfrak{X}_{10}$	$cos(\mathfrak{s})\mathfrak{X}_{10} + sin(\mathfrak{s})\mathfrak{X}_9$	$\mathfrak{X}_1$
$\mathfrak{X}_2$	$cos(\mathfrak{s})\mathfrak{X}_9 + sin(\mathfrak{s})\mathfrak{X}_7$	$cos(\mathfrak{s})\mathfrak{X}_{10} + sin(\mathfrak{s})\mathfrak{X}_8$	$\mathfrak{X}_2$
$\mathfrak{X}_3$	$cos(\mathfrak{s})\mathfrak{X}_9 + sin(\mathfrak{s})\mathfrak{X}_6$	$\mathfrak{X}_{10}$	$\mathfrak{X}_3$
$\mathfrak{X}_4$	$\mathfrak{X}_9$	$cos(\mathfrak{s})\mathfrak{X}_{10} + sin(\mathfrak{s})\mathfrak{X}_6$	$\mathfrak{X}_4$
$\mathfrak{X}_5$	$\mathfrak{X}_9$	$\mathfrak{X}_{10}$	$\mathfrak{X}_5$
$\mathfrak{X}_6$	$cosh(\mathfrak{s})\mathfrak{X}_9 - sinh(\mathfrak{s})\mathfrak{X}_3$	$cosh(\mathfrak{s})\mathfrak{X}_{10} - sinh(\mathfrak{s})\mathfrak{X}_4$	$\mathfrak{X}_6$
$\mathfrak{X}_7$	$cosh(\mathfrak{s})\mathfrak{X}_9 - sinh(\mathfrak{s})\mathfrak{X}_2$	$\mathfrak{X}_{10}$	$\mathfrak{X}_7$
$\mathfrak{X}_8$	$\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_{10} - sinh(\mathfrak{s})\mathfrak{X}_2$	$\mathfrak{X}_8$
$\mathfrak{X}_9$	$\mathfrak{X}_9$	$cosh(\mathfrak{s})\mathfrak{X}_{10} - sinh(\mathfrak{s})\mathfrak{X}_1$	$\mathfrak{X}_9$
$\mathfrak{X}_{10}$	$cosh(\mathfrak{s})\mathfrak{X}_9 + sinh(\mathfrak{s})\mathfrak{X}_1$	$\mathfrak{X}_{10}$	$\mathfrak{X}_{10}$
$\mathfrak{X}_{11}$	$\mathfrak{X}_9$	$\mathfrak{X}_{10}$	$\mathfrak{X}_{11}$

Proof. Here, we use Ibragimov's method. Due to the Table (1), $\langle \mathfrak{X}_{11} \rangle$ is the center of $\mathfrak{g}$, so we need to specify the sub-algebras of

$$\langle \mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4, \mathfrak{X}_5, \mathfrak{X}_6, \mathfrak{X}_7, \mathfrak{X}_8, \mathfrak{X}_9, \mathfrak{X}_{10} \rangle.$$

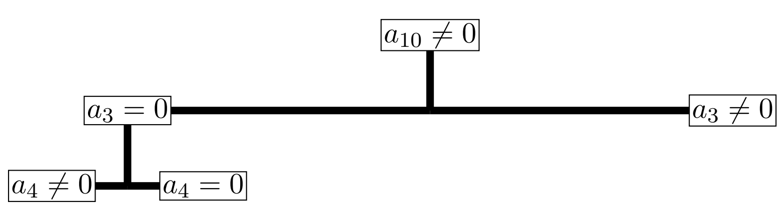
$F_i^{\mathfrak{s}} : \mathfrak{g} \to \mathfrak{g}$ characterized as the linear map $\mathfrak{X} \mapsto \mathrm{Ad}(\exp(\mathfrak{s}\mathfrak{X}_i).\mathfrak{X})$, where $1 \leq i, j \leq 11$. Some matrices of $F_i^{\mathfrak{s}}$, $1 \leq i, j \leq 11$, namely $M_1^{\mathfrak{s}}$ and $M_5^{\mathfrak{s}}$, according to basis $\{\mathfrak{X}_1, \cdots, \mathfrak{X}_{11}\}$

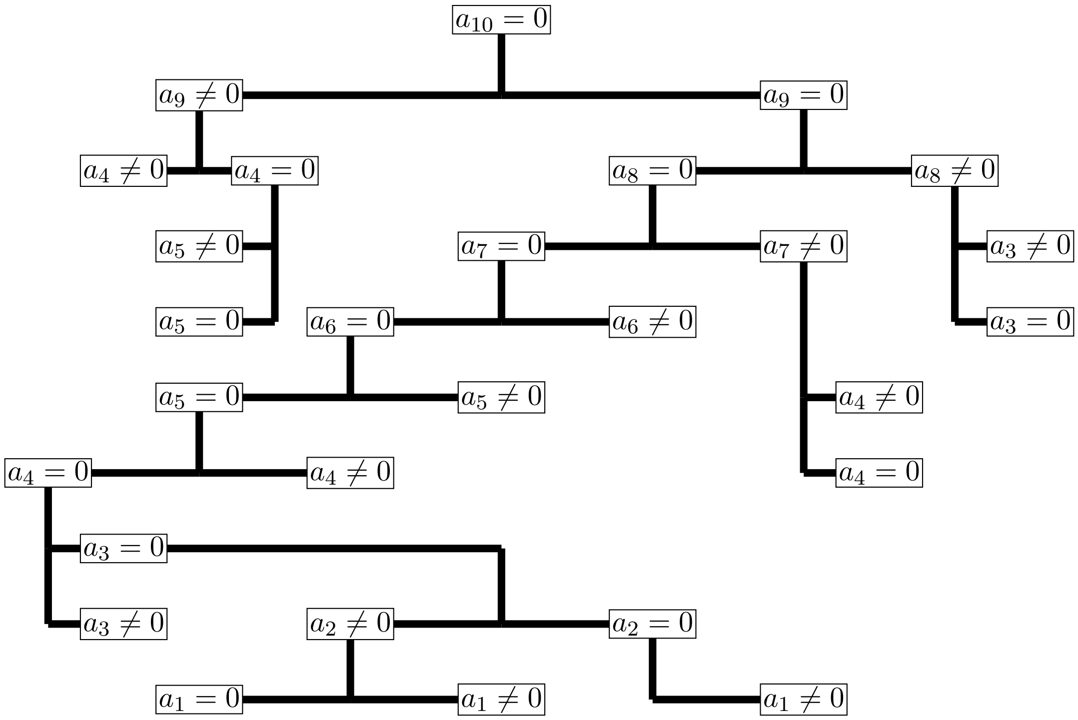
are reported as:

$$M_1^{\mathfrak{s}}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & cos(\mathfrak{s}) & -sin(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sin(\mathfrak{s}) & cos(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & cos(\mathfrak{s}) & -sin(\mathfrak{s}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & sin(\mathfrak{s}) & cos(\mathfrak{s}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & cos(\mathfrak{s}) & -sin(\mathfrak{s}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & sin(\mathfrak{s}) & cos(\mathfrak{s}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_5^{\mathfrak{s}}=\begin{bmatrix} cosh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & -sinh(\mathfrak{s}) & 0 & 0 & 0 \\ 0 & cosh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & -sinh(\mathfrak{s}) & 0 & 0 \\ 0 & 0 & cosh(\mathfrak{s}) & 0 & sinh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & sinh(\mathfrak{s}) & 0 & cosh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -sinh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & cosh(\mathfrak{s}) & 0 & 0 & 0 \\ 0 & -sinh(\mathfrak{s}) & 0 & 0 & 0 & 0 & 0 & 0 & cosh(\mathfrak{s}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Acting the eleven matrices mentioned above on a generator $\mathfrak{X}=\sum_{i=1}^{11} a_i\mathfrak{X}_i$ periodically we specify $\mathfrak{X}$. In order to clarify the proof, the following two diagrams are given.





Case I. Let $a_{10} \neq 0$. Consider a vector

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}) \quad a_{10} \neq 0. \tag{14}$$

the coefficients of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_4, \mathfrak{X}_6, \mathfrak{X}_8$ and $\mathfrak{X}_9$ can be disappeared by setting $\mathfrak{s}_9 = tan^{-1}(a_1/a_{10})$, $\mathfrak{s}_8 = tanh^{-1}(a_2/a_{10})$, $\mathfrak{s}_6 = tanh^{-1}(a_4/a_{10})$, $\mathfrak{s}_4 = tan^{-1}(a_6/a_{10})$, $\mathfrak{s}_2 = tan^{-1}(a_8/a_{10})$ and $\mathfrak{s}_1 = tan^{-1}(a_9/a_{10})$ respectively. Thus, (14) is reduced to

$$(0, 0, a_3, 0, a_5, 0, a_7, 0, 0, a_{10}). \tag{15}$$

- Let $a_{10} = a_3 \neq 0$, for vector (15), the coefficients of $\mathfrak{X}_5$ and $\mathfrak{X}_7$ would be disappeared by setting $s_7 = -tanh^{-1}(a_5/a_3)$, and $\mathfrak{s}_5 = tanh^{-1}(a_7/a_3)$ respectively. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_3 = 1$ and $a_{10} = \pm 1$.Thus, X gives rise to case (18).
- Let $a_{10} \neq 0$, $a_3 = 0$ and $a_5 \neq 0$, for vector (15), the coefficient of $\mathfrak{X}_7$ would be disappeared by setting $\mathfrak{s}_3 = -tan^{-1}(a_7/a_5)$. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_5 = 1$ and $a_{10} = \pm 1$. Thus, X gives rise to case (22).
- Let $a_{10} \neq 0$ and $a_3 = a_5 = 0$. In order to simplify the phrase, for vector (15) by scaling $\mathfrak{X}$, assume that $a_7 = 1$ and $a_{10} = \pm 1$. Thus, X gives rise to cases (10) and (25).

Case II. Let $a_{10} = 0$. Consider a vector

$$(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, 0). \tag{16}$$

- Let $a_{10} = 0$ and $a_9 \neq 0$, for vector (16), the coefficients of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_6$ and $\mathfrak{X}_7$ can be disappeared by setting $\mathfrak{s}_{10} = -tanh^{-1}(a_1/a_9)$, $\mathfrak{s}_7 = tanh^{-1}(a_2/a_9)$,

$\mathfrak{s}_6 = tanh^{-1}(a_3/a_9)$, $\mathfrak{s}_3 = tan^{-1}(a_6/a_9)$ and $\mathfrak{s}_2 = tan^{-1}(a_7/a_9)$ respectively. Thus, (16) is reduced to

$$(0, 0, 0, a_4, a_5, 0, 0, a_8, a_9, 0). \tag{17}$$

- Let $a_{10} = 0$ and $a_9 = a_4 \neq 0$, for vector (17), the coefficient of $\mathfrak{X}_5$ can be disappeared by setting $\mathfrak{s}_8 = -tanh^{-1}(a_5/a_4)$. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_4 = 1$ and $a_9 = \pm 1$. Thus, X gives rise to case (20).
- Let $a_{10} = a_4 = 0$ and $a_9 = a_5 \neq 0$, for vector (17), the coefficient of $\mathfrak{X}_8$ can be disappeared by setting $\mathfrak{s}_4 = -tan^{-1}(a_8/a_5)$. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_5 = 1$ and $a_9 = \pm 1$. Thus, X gives rise to case (21).
- Let $a_{10} = a_4 = a_5 = 0$ and $a_9 \neq 0$, in vector (17). In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_8 = 1$ and $a_9 = \pm 1$. Thus, X gives rise to cases (9) and (26).
- Let $a_{10} = a_9 = 0$ and $a_8 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_4, \mathfrak{X}_5$ and $\mathfrak{X}_7$ can be disappeared by setting $\mathfrak{s}_7 = tanh^{-1}(a_1/a_8)$, $\mathfrak{s}_{10} = -tanh^{-1}(a_2/a_8)$, $\mathfrak{s}_5 = tanh^{-1}(a_4/a_8)$, $\mathfrak{s}_4 = tan^{-1}(a_5/a_8)$ and $\mathfrak{s}_1 = tan^{-1}(a_7/a_8)$ respectively. Thus, (16) is reduced to

$$(0, 0, a_3, 0, 0, a_6, 0, a_8, 0, 0). \tag{18}$$

- Let $a_{10} = a_9 = 0$ and $a_8 = a_3 \neq 0$, for vector (18), the coefficient of $\mathfrak{X}_6$ can be disappeared by setting $\mathfrak{s}_9 = -tanh^{-1}(a_6/a_3)$. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_3 = 1$ and $a_8 = \pm 1$. Thus, X gives rise to case (17).
- Let $a_{10} = a_9 = a_3 = 0$ and $a_8 \neq 0$, in vector (18). In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_6 = 1$ and $a_8 = \pm 1$. Thus, X gives rise to cases (8) and (24).
- Let $a_{10} = a_9 = a_8 = 0$ and $a_7 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_1, \mathfrak{X}_2, \mathfrak{X}_3$ and $\mathfrak{X}_5$ can be disappeared by setting $\mathfrak{s}_8 = -tanh^{-1}(a_1/a_7)$, $\mathfrak{s}_9 = -tanh^{-1}(a_2/a_7)$, $\mathfrak{s}_5 = tanh^{-1}(a_3/a_7)$, and $\mathfrak{s}_3 = tan^{-1}(a_5/a_7)$ respectively. Thus, (16) is reduced to

$$(0, 0, 0, a_4, 0, a_6, a_7, 0, 0, 0). \tag{19}$$

- Let $a_{10} = a_9 = a_8 = 0$ and $a_7 = a_4 \neq 0$, for vector (19), the coefficient of $\mathfrak{X}_6$ can be disappeared by setting $\mathfrak{s}_{10} = -tanh^{-1}(a_6/a_4)$. In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_4 = 1$ and $a_7 = \pm 1$. Thus, X gives rise to case (19).
- Let $a_{10} = a_9 = a_8 = a_4 = 0$ and $a_7 \neq 0$, in vector (19). In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_6 = 1$ and $a_7 = \pm 1$. Thus, X gives rise to cases (7) and (23).
- Let $a_{10} = a_9 = a_8 = a_7 = 0$ and $a_6 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_2, \mathfrak{X}_3, \mathfrak{X}_4$ and $\mathfrak{X}_5$ can be disappeared by setting $\mathfrak{s}_5 = tanh^{-1}(a_2/a_6)$, $\mathfrak{s}_9 = -tanh^{-1}(a_3/a_6)$, $\mathfrak{s}_{10} = -tanh^{-1}(a_4/a_6)$, and $\mathfrak{s}_2 = tan^{-1}(a_5/a_6)$ respectively. Thus, (16) is reduced to

$$(a_1, 0, 0, 0, 0, a_6, 0, 0, 0, 0). \tag{20}$$

In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_1 = 1$ and $a_6 = \pm 1$. Thus, X gives rise to cases (6) and (14).

- Let $a_{10} = a_9 = a_8 = a_7 = a_6 = 0$ and $a_5 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_2, \mathfrak{X}_3$ and $\mathfrak{X}_4$ can be disappeared by setting $\mathfrak{s}_6 = -tanh^{-1}(a_2/a_5)$, $\mathfrak{s}_7 = -tanh^{-1}(a_3/a_5)$ and $\mathfrak{s}_8 = -tanh^{-1}(a_4/a_5)$ respectively. Thus, (16) is reduced to

$$(a_1, 0, 0, 0, a_5, 0, 0, 0, 0, 0). \tag{21}$$

In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_1 = 1$ and $a_5 = \pm 1$. Thus, X gives rise to cases (5) and (13).

- Let $a_{10} = a_9 = a_8 = a_7 = a_6 = a_5 = 0$ and $a_4 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_1$ and $\mathfrak{X}_3$ can be disappeared by setting $\mathfrak{s}_3 = -tan^{-1}(a_1/a_4)$, and $\mathfrak{s}_1 = tan^{-1}(a_3/a_4)$ respectively. Thus, (16) is reduced to

$$(0, a_2, 0, a_4, 0, 0, 0, 0, 0, 0). \tag{22}$$

In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_2 = 1$ and $a_4 = \pm 1$. Thus, X gives rise to cases (4) and (16).

- Let $a_{10} = a_9 = a_8 = a_7 = a_6 = a_5 = a_4 = 0$ and $a_3 \neq 0$, for vector (16), the coefficient of $\mathfrak{X}_1$ can be disappeared by setting $\mathfrak{s}_4 = tan^{-1}(a_1/a_3)$. Thus, (16) is reduced to

$$(0, a_2, a_3, 0, 0, 0, 0, 0, 0, 0). \tag{23}$$

In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_2 = 0, 1$ and $a_3 = \pm 1$. Thus, X gives rise to cases (3) and (15).

- Let $a_{10} = a_9 = a_8 = a_7 = a_6 = a_5 = a_4 = a_3 = 0$. Thus, (16) is reduced to

$$(a_1, a_2, 0, 0, 0, 0, 0, 0, 0, 0). \tag{24}$$

In order to simplify the phrase, by scaling $\mathfrak{X}$, assume that $a_1 = 0, 1$ and $a_2 = 0, \pm 1$. Thus, X gives rise to cases (1), (2) and (12).

□

4. Some reduced equations of Eq.(4)

Now, we are going to offer a classified symmetry reduction of Eq.(4) regarding subalgebras of Theorem 3.1. For this purpose, we have to look for a new shape of Eq.(4) in particular coordinates to reduce it. A coordinate like this would be built by realizing independent invariant ς, η, k, h corresponds to the infinitesimal solution. Therefore, representing the problem in other coordinates, utilizing the derivative will reduce the order of PDE. Every 1D sub-algebras in 3.1, the similarity variables ς_i, η_i, k_i, and h_i are brought in Table 3, where, in cases (16) and (18), one puts $\alpha = 1$. Every similarity variable is utilized to reduce Eq.(4) to a new PDE which, we bring in Table 4.

TABLE 3. Lie group and similarity variable.

i	H_i	ς_i	η_i	t_i	w_i	u_i	v_i
1	$\mathfrak{X}_1$	x	z	u	v	$k(\varsigma,\eta)$	$f(\varsigma,\eta)$
2	$\mathfrak{X}_2$	x	y	u	v	$k(\varsigma,\eta)$	$f(\varsigma,\eta)$
3	$\mathfrak{X}_1+a\mathfrak{X}_2$	x	$y-\frac{z}{a}$	u	v	$k(\varsigma,\eta)$	$f(\varsigma,\eta)$
4	$\mathfrak{X}_1+a\mathfrak{X}_{11}$	x	z	$Ln(u)-ay$	v	$e^{(ay+k(\varsigma,\eta))}$	$f(\varsigma,\eta)$
5	$\mathfrak{X}_2+a\mathfrak{X}_{11}$	x	y	$Ln(u)-az$	v	$e^{(az+k(\varsigma,\eta))}$	$f(\varsigma,\eta)$
6	$\mathfrak{X}_1+a\mathfrak{X}_2+b\mathfrak{X}_{11}$	x	$y-\frac{z}{a}$	$Ln(u)-\frac{b}{a}z$	v	$e^{(\frac{b}{a}z+k(\varsigma,\eta))}$	$f(\varsigma,\eta)$
7	$\mathfrak{X}_1+a\mathfrak{X}_\alpha, \alpha=u$	x	z	$Ln(u)-ay$	$Ln(E-v)+ay$	$e^{(ay+k(\varsigma,\eta))}$	$E-e^{-ay+f(\varsigma,\eta)}$
8	$\mathfrak{X}_2+a\mathfrak{X}_\alpha, \alpha=u$	x	y	$Ln(u)-az$	$Ln(E-v)+az$	$e^{(az+k(\varsigma,\eta))}$	$E-e^{-az+f(\varsigma,\eta)}$
9	$\mathfrak{X}_1+a\mathfrak{X}_\alpha, \alpha=1$	x	z	$u-ay$	$uLn(E-v)+ay$	$ay+k(\varsigma,\eta)$	$E-e^{(-ay+f(\varsigma,\eta))/u}$
10	$\mathfrak{X}_2+a\mathfrak{X}_\alpha, \alpha=1$	x	y	$u-az$	$uLn(E-v)+az$	$az+k(\varsigma,\eta)$	$E-e^{(-az+f(\varsigma,\eta))/u}$
11	$\mathfrak{X}_1+a\mathfrak{X}_\alpha+bX_{11}, \alpha=u$	x	z	$Ln(u)-(a+b)y$	$Ln(E-v)+ay$	$e^{(a+b)y+k(\varsigma,\eta)}$	$E-e^{-ay+f(\varsigma,\eta)}$
12	$\mathfrak{X}_2+a\mathfrak{X}_\alpha+b\mathfrak{X}_{11}, \alpha=u$	x	y	$Ln(u)-(a+b)z$	$Ln(E-v)+az$	$e^{(a+b)z+k(\varsigma,\eta)}$	$E-e^{-az+f(\varsigma,\eta)}$
13	$\mathfrak{X}_1+a\mathfrak{X}_\alpha+b\mathfrak{X}_{11}, \alpha=1$	x	z	$Ln(bu+a)-by$	$uLn(E-v)+ay$	$(e^{(by+k(\varsigma,\eta))}-a)/b$	$E-e^{(-ay+f(\varsigma,\eta))/u}$
14	$\mathfrak{X}_2+a\mathfrak{X}_\alpha+b\mathfrak{X}_{11}, \alpha=1$	x	y	$Ln(bu+a)-bz$	$uLn(E-v)+az$	$(e^{(bz+k(\varsigma,\eta))}-a)/b$	$E-e^{(-az+f(\varsigma,\eta))/u}$
15	$\mathfrak{X}_1+a\mathfrak{X}_2+b\mathfrak{X}_\alpha+c\mathfrak{X}_{11}$, $\alpha=u$	x	$y-\frac{z}{a}$	$Ln(u)-(b+c)y$	$Ln(E-v)+by$	$e^{(b+c)y+k(\varsigma,\eta)}$	$E-e^{-by+f(\varsigma,\eta)}$
16	$\mathfrak{X}_1+a\mathfrak{X}_2+b\mathfrak{X}_\alpha+c\mathfrak{X}_{11}$, $\alpha=1$	x	$y-\frac{z}{a}$	$Ln(cu+b)-cy$	$uLn(E-v)+by$	$(e^{(cy+k(\varsigma,\eta))}-b)/c$	$E-e^{(-by+f(\varsigma,\eta))/u}$
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

TABLE 4. Reduced equations based on similarity variable.

i	Reduction of equations
1	$k_{\varsigma\varsigma}+cot(\varsigma)k_\varsigma-k_{\eta\eta}+(2m/h^2)(E-f)k=0,$
2	$k_{\varsigma\varsigma}+cot(\varsigma)k_\varsigma+csc^2(\varsigma)k_{\eta\eta}+(2m/h^2)(E-f)k=0,$
3	$k_{\varsigma\varsigma}+cot(\varsigma)k_\varsigma+csc^2(\varsigma)k_{\eta\eta}-\frac{1}{a^2}k_{\eta\eta}+(2m/h^2)(E-f)k=0,$
4	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+a^2csc^2(\varsigma)-k_{\eta\eta}-k_\eta^2+(2m/h^2)(E-f)e^{k+ay}=0,$
5	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-a^2+(2m/h^2)(E-f)e^{k+az}=0,$
6	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-\frac{1}{a^2}k_{\eta\eta}-(\frac{1}{a}k_\eta+b)^2+(2m/h^2)(E-f)e^{k+\frac{b}{a}y}=0,$
7	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+a^2csc^2(\varsigma)-k_{\eta\eta}-k_\eta^2+(2m/h^2)e^{f-ay}e^{k+ay}=0,$
8	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-a^2+(2m/h^2)e^{f-az}e^{k+az}=0,$
9	$k_{\varsigma\varsigma}+cot(\varsigma)k_\varsigma-k_{\eta\eta}+(2m/h^2)e^{(f-ay)/u}(k+ay)=0,$
10	$k_{\varsigma\varsigma}+cot(\varsigma)k_\varsigma+csc^2(\varsigma)k_{\eta\eta}+(2m/h^2)e^{(f-az)/u}(k+az)=0,$
11	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+(a+b)^2csc^2(\varsigma)-k_{\eta\eta}-k_\eta^2+(2m/h^2)e^{f-ay}e^{k+(a+b)y}=0,$
12	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-(a+b)^2+(2m/h^2)e^{f-az}e^{k+(a+b)z}=0,$
13	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+b^2csc^2(\varsigma)-k_{\eta\eta}-k_\eta^2+(2m/h^2)e^{(f-ay)/u}(e^{k+by}-a)=0,$
14	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-b^2+(2m/h^2)e^{(f-az)/u}(e^{k+bz}-a)=0,$
15	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-\frac{1}{a^2}k_{\eta\eta}-(\frac{1}{a}k_\eta+\frac{b+c}{a})^2+(2m/h^2)e^{f-ay}e^{k+(b+c)y}=0,$
16	$k_{\varsigma\varsigma}+k_\varsigma^2+cot(\varsigma)k_\varsigma+csc^2(\varsigma)(k_{\eta\eta}+k_\eta^2)-\frac{1}{a^2}k_{\eta\eta}-(\frac{1}{a}k_\eta+\frac{c}{a})^2+(2m/h^2)e^{(f-ay)/u}(e^{k+cy}-b)/c=0.$

As sample, we calculate the invariants related to $H_9 := \mathfrak{X}_1 + a\mathfrak{X}_\alpha$. We integrate the following characteristic expression, assuming $\alpha(u) = 1$.

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dz}{0} = \frac{du}{1} = u\frac{dv}{E-v}.$$

Thus, the variables are calculated as:

$$\varsigma = x, \qquad \eta = z, \qquad t = u - ay, \qquad w = uLn(E-v) + ay,$$

Putting the obtained variables in Eq.(4), and utilizing derivative yields that, the answer of Eq.(4) is as:

$$u = ay + k(\varsigma, \eta), \qquad v = E - e^{(-ay+f(\varsigma,\eta))/u},$$

where $k(\varsigma, \eta)$ and $h(\varsigma, \eta)$ satisfies the following reduced equation with 2 variables

$$k_{\varsigma\varsigma} + cot(\varsigma)k_\varsigma - k_{\eta\eta} + (2m/h^2)e^{(f-ay)/u}(k + ay) = 0. \tag{25}$$

Subalgebra $X_1 + aX_\alpha$ and the reduced Eq.(25) are brought in Tables 3 and 4, by case (9).

5. 3D quantum harmonic oscillator on a sphere

As we saw in the introduction, the Schrodinger equation for a harmonic oscillator on a sphere was introduced by:

$$\begin{aligned} u_{zz} = u_{xx} + (cotx)u_x + (csc^2x)u_{yy} \\ +(2m/h^2)(E - \tfrac{1}{2}q_x x^2 - \tfrac{1}{2}q_y sin^4(x)y^2 - \tfrac{1}{2}q_z z^2)u, \end{aligned} \tag{26}$$

where the particle oscillates on a sphere.

To interact with this equation, it is better to work on the reduction equation of the general form of the Schrodinger equation from Table 4. For this purpose, We select Equation 1 from Table 4 and try to solve it. Obviously, this equation is in terms of two variables, and solving this equation seems simpler than the original Equation (7). So, consider:

$$k_{\varsigma\varsigma} + cot(\varsigma)k_\varsigma - k_{\eta\eta} + (2m/h^2)(E - f)k = 0,$$

where considering the metric (3) on $S^2 \times R$ and the function f will become

$$f(\varsigma, \eta) = \frac{1}{2}\hat{q}\varsigma^2 + \frac{1}{2}\bar{q}\eta^2,$$

where $\hat{q}$ and $\bar{q}$ are constants. Thus (7) turns into

$$k_{\varsigma\varsigma} + cot(\varsigma)k_\varsigma - k_{\eta\eta} + (2m/h^2)(E - \tfrac{1}{2}\hat{q}\varsigma^2 - \tfrac{1}{2}\bar{q}\eta^2)k = 0. \tag{27}$$

Theorem 5.1. *Assume that $M(\varsigma)$ and $N(\eta)$ are functions. If they satisfy the following two separate ODEs:*

$$\begin{cases} M_{\varsigma\varsigma} + cot(\varsigma)M_\varsigma - (1/h^2)(c_1 + m\hat{q}\varsigma^2)M = 0, \\ N_{\eta\eta} - (1/h^2)(c_1 - m(\bar{q}\eta^2 - 2E))N = 0, \end{cases} \tag{28}$$

then $k(\varsigma, \eta) = M(\varsigma)N(\eta)$ are the solutions of (27).

Proof. It is sufficient to show that $k(\varsigma, \eta) = M(\varsigma)N(\eta)$ satisfies Equation (27). Evaluating the derivative gives:

$$\begin{cases} k_{\varsigma\varsigma} = M_{\varsigma\varsigma}N, \\ cot(\varsigma)k_\varsigma = cot(\varsigma)M_\varsigma N \\ k_{\eta\eta} = N_{\eta\eta}M, \end{cases}$$

so

$$\begin{aligned} 0 &= k_{\varsigma\varsigma} + cot(\varsigma)k_{\varsigma} - k_{\eta\eta} + (2m/h^2)(E - \tfrac{1}{2}\hat{q}\varsigma^2 - \tfrac{1}{2}\bar{q}\eta^2)k \\ &= M_{\varsigma\varsigma}N + cot(\varsigma)M_{\varsigma}N - N_{\eta\eta}M + (2m/h^2)(E - \tfrac{1}{2}\hat{q}\varsigma^2 - \tfrac{1}{2}\bar{q}\eta^2)MN \\ &= M_{\varsigma\varsigma}N + cot(\varsigma)M_{\varsigma}N - N_{\eta\eta}M \\ &+(1/h^2)(2mE - m\hat{q}\varsigma^2 - m\bar{q}\eta^2 + c_1 - c_1)MN \\ &= \Bigg(M_{\varsigma\varsigma} + cot(\varsigma)M_{\varsigma} - (1/h^2)(c_1 + m\hat{q}\varsigma^2)M\Bigg)N \\ &+\Bigg(N_{\eta\eta} - (1/h^2)(c_1 - m(\bar{q}\eta^2 - 2E))N\Bigg)M. \end{aligned}$$

Because M and N are not zero, then the necessary result is obtained.

□

Now to solve Equation (27), we need to consider Equations (28). In this sense, for $c_1 = -2mE$, the solution of the second equation of (28), using Maple, is

$$N = C_1\sqrt{\eta}BesselJ(\frac{1}{4}, \frac{1}{2}\sqrt{\frac{\bar{q}m}{h^2}}\eta^2) + C_2\sqrt{\eta}BesselY(\frac{1}{4}, \frac{1}{2}\sqrt{\frac{\bar{q}m}{h^2}}\eta^2),$$

where $C_1, C_2 \in \mathbf{R}$ and BesselJ and BesselY are the Bessel functions of the first and second kinds, respectively. The first equation of (28) with the assumption $y = \dfrac{M_{\varsigma}}{\varsigma}$ turns into

$$y_{\varsigma} = -y^2 - cot(\varsigma)y + \frac{\hat{q}\varsigma^2 m + c_1}{h^2}.$$

This ODE is called Riccati. Indeed, we started with Equation (7) and finally reached the Riccati equation.

Conclusion

The paper considers the question of a quantum system obeying the Schrodinger equation on a Sphere. After a brief introduction intended to set up the Hamiltonian of the system and the corresponding Schrodinger equation So, we present new Equation (6). In this regard, we tried to solve the new equation. Decreasing the order of the new equation, we present several newer equations with only two variables in Table 4. Using one of the equations in Table 4 for an oscillator object attached to a spring on a sphere, we converted this two-variable PDE into two one-variable ODEs, one of which was solved and the other was the Riccati equation, which can be discussed in exceptional cases. Maybe one will get better results using another of the reduced equations in Table 4.

References

[1] S.C. Anco, M.L. Gandarias, E. Recio, Line-solitons, line-shocks, and conservation laws of a universal KP-like equation in 2+1 dimensions, J. Math. Anal. Appl. 504 (2021), 125319.

[2] Y. AryaNejad, *Symmetry Analysis of Wave Equation on Conformally Flat Spaces*, J. Geom. Phys. 161(2021), 104029.

[3] G.W. Bluman and J.D. Cole, *Similarity Methods for Differential equations,* Appl. Math. Sci., No. 13, Springer-Verlag, New York, 1974.

[4] G.W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Springer, New York, 1989.

[5] M. S. Bruzón, E. Recio, T. M. Garrido, A. P. Márquez, R. de la Rosa *On the similarity solutions and conservation laws of the Cooper-Shepard-Sodano equation*, Mathematical Methods in the Applied Sciences 41 (17), 7325-7332.

[6] M.L. Gandarias, M. Torrisi, A. Valenti, *Symmetry classification and optimal systems of a non-linear wave equation,* Internat. J. Non-Linear Mech. 39 (2004) 389–398.

[7] N.H. Ibragimov (Ed.), *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 3, New Trends in Theoretical Developments and Computational Methods, CRC Press, Boca Raton, 1996.

[8] A. P. Márquez, M. S. Bruzón, *Lie point symmetries, traveling wave solutions and conservation laws of a non-linear viscoelastic wave equation*, Mathematics, 9 (17) (2021) 21–31.

[9] A. P. Márquez, T. M. Garrido, E. Recio, R. de la Rosa, Lie symmetries and exact solutions for a fourth-order nonlinear diffusion equation. Math Meth Appl Sci. 2022;45(17):10614-10627.

[10] M. Nadjafikhah, Y. AryaNejad, N. Zandi, *Conformal Einstein PP-Wave as Quantum Solutions*, Journal of Mathematical Extension. **16**(12)(2022), 1–19.

[11] M. Nadjafikhah, M. Jafary, *Computation of Partially Invariant Solutions for the Einstein Walker Manifolds*, Communications in Nonlinear Science and Numerical Simulation. 18(12) (2011) 3317-3324.

[12] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, 1986.

[13] L.V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

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