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Original Research Paper

On $(p, q)-$Centralizers of Certain Banach Algebras

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Abstract. Let A be an algebra with a right identity. In this paper, we study $(p, q)-$centralizers of A and show that every $(p, q)-$centralizer of A is a two-sided centralizer. In the case where, A is normed algebra, we also prove that $(p, q)-$centralizers of A are bounded. Then, we apply the results for some group algebras and verify that $L^1(G)$ has a nonzero weakly compact $(p, q)-$centralizer if and only if G is compact and the center of $L^1(G)$ is non-zero. Finally, we investigate $(p, q)-$Jordan centralizers of A and determine them.

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1 Introduction

Let T be a map on an algebra A, and let p and q be distinct non-negative integers. Then T is called a $(p, q)-$*centralizer* if for every $a, b \in A$

$$(p + q)T(ab) = pT(a)b + qaT(b)$$

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for all $a \in A$. We denote by $C_{p,q}(A)$ the space of all $(p,q)-$centralizers of A. Every $(1,0)-$centralizer is called a *left centralizer* and every $(0,1)-$centralizer is called a *right centralizer*. Moreover, T is said to be a *two-sided centralizer* if T is both a left centralizer and a right centralizer. Let $C_{ts}(A)$ be the set of all two-sided centralizers of A. It is easy to see that if $a \in A$, then the linear mapping $\rho_a : A \to A$ defined by

$$\rho_a(b) = ba$$

is a right centralizer. Also, the linear mapping $\lambda_a : A \to A$ defined by $\lambda_a(b) = ab$ is a left centralizer. Note that if $a \in Z(A)$, the center of A, then

$$\rho_a = \lambda_a \in C_{ts}(A).$$

An additive mapping T on an algebra A is a $(p,q)-$*Jordan centralizer* if

$$(p+q)T(a^2) = pT(a)a + qaT(a)$$

The set of all $(p,q)-$Jordan centralizers of A is denoted by $C^J_{p,q}(A)$. Clearly,

$$C_{ts}(A) \subseteq C_{p,q}(A) \subseteq C^J_{p,q}(A).$$

Vukman [16] introduced the notion of $(p,q)-$Jordan centralizers. For a prime ring R with a suitable characteristic and positive integers p and q, he proved that

$$C^J_{p,q}(R) = C_{ts}(R),$$

when $Z(R)$ is non-zero. Kosi-Ulbl and Vukman [6] proved this result for semiprime rings; see [2, 4] for generalized $(p,q)-$Jordan centralizers.

In this paper, we investigate $(p,q)-$centralizers and $(p,q)-$Jordan centralizers of algebras. In Section 2, we show that $(p,q)-$centralizers of an algebra A with a right identity are two-sided centralizers. In the case where, A is a normed algebra, every $(p,q)-$centralizer of A is bounded. We prove that every bounded $(p,q)-$centralizer of an algebra with a bounded approximate identity is a two-sided centralizer. In Section 3, we give some results on the range of $(p,q)-$centralizers. In Section 4, we investigate $(p,q)-$centralizers of some group algebras and determine the set of all their $(p,q)-$centralizers. For a locally compact group G, we show that $L^1(G)$ has a non-zero weakly compact $(p,q)-$centralizer if and only if G is compact and the center of $L^1(G)$ is non-zero. In Section 5, we give a characterization of $(p,q)-$Jordan centralizers of an algebra with a right identity. We also show that every $(p,q)-$Jordan centralizer of a commutative algebra is always a two-sided centralizer.

Convention: In this paper, p and q are distinct positive integers.

2 $(p,q)-$Centralizers on Algebras

First, we characterize $(p,q)-$centralizers of algebras.

Theorem 2.1. *Let A be an algebra with a right identity u. If $T : A \to A$ is a map, then the following assertions are equivalent.*

(a) $T \in C_{p,q}(A)$.

(b) $T = \rho_{T(u)} \in C_{ts}(A)$.

(c) *For every $a, b \in A$, $aT(b) = T(a)b$.*

In this case, T is linear.

Proof. Let $T \in C_{p,q}(A)$. Then for every $a \in A$, we have

$$\begin{aligned}(p+q)T(a) &= (p+q)T(au)\\ &= pT(a) + qaT(u).\end{aligned}$$

Thus

$$T(a) = aT(u). \tag{1}$$

It follows that

$$\begin{aligned}(p+q)abT(u) &= (p+q)T(ab)\\ &= pT(a)b + qaT(b)\\ &= paT(u)b + qabT(u).\end{aligned}$$

Hence $abT(u) = aT(u)b$. From this and (1) we see that $aT(b) = T(a)b$. Thus for every $a, b \in A$,

$$\begin{aligned}(p+q)T(ab) &= pT(a)b + qaT(b)\\ &= (p+q)T(a)b.\end{aligned}$$

So, $T(ab) = T(a)b = aT(b)$. That is, $T \in C_{ts}(A)$. Therefore, (a)⇒(b). The implications (b)⇒(c)⇒(a) are clear. □

For a normed algebra A, let $C^b_{p,q}(A)$ be the space of all bounded $(p, q)-$centralizers of A.

Corollary 2.2. *Let A be a normed algebra with a right identity. Then $C_{p,q}(A)$ is a Banach algebra and*

$$C_{p,q}(A) = C^b_{p,q}(A) = \{\rho_a : a \in A\}.$$

Let A and B be two subsets of an algebra $\mathfrak{A}$. We denote by $Z(A, B)$ the set of all $a \in A$ such that $ab = ba$ for all $b \in B$. Note that if $A = B$, then $Z(A, B) = Z(A)$.

Corollary 2.3. *Let A be an algebra with identity 1_A. Then the following statements hold.*

(i) *If $T \in C_{p,q}(A)$, then $T(1_A) \in Z(A)$ and $T = \rho_{T(1_A)} = \lambda_{T(1_A)}$.*

(ii) *If A is also a normed algebra, then $C_{p,q}(A) \cong Z(A)$, where "$\cong$" denotes isometrically isomorphic as Banach algebras.*

Proof. (i) Let $T \in C_{p,q}(A)$. In view of Theorem 2.1, $T \in C_{ts}(A)$. Thus for every $a \in A$, we have

$$T(a) = T(a1_A) = aT(1_A)$$

and

$$T(a) = T(1_Aa) = T(1_A)a.$$

Hence $aT(1_A) = T(1_A)a$ and so $T(1_A) \in Z(A)$.

(ii) Let A be an algebra with the identity 1_A. Then $C_{p,q}(A)$ is a subalgebra of algebra of all bounded linear operators on A. We define the function

$$\Gamma : C_{p,q}(A) \to Z(A)$$

by $\Gamma(T) = T(1_A)$. It is clear that Γ is linear. Let $T \in C_{p,q}(A)$. Then for every $a \in A$,

$$\|T(a)\| = \|aT(1_A)\| \le \|a\|\|T(1_A)\|.$$

This implies that

$$\|T\| = \|T(1_A)\| = \|\Gamma(T)\|.$$

Therefore, Γ is isometric. If $T_1, T_2 \in C_{p,q}(A)$, then

$$\begin{aligned}\Gamma(T_1)\Gamma(T_2) &= T_1(1_A)T_2(1_A)\\ &= T_1(1_A T_2(1_A))\\ &= T_1T_2(1_A)\\ &= \Gamma(T_1T_2).\end{aligned}$$

Hence Γ is an algebra homomorphism. Also, for every $c \in Z(A)$, the linear map ρ_c is an element of $C_{p,q}(A)$. So Γ is isomorphic. $\square$

Let A be a Banach algebra. It is well-known that the second conjugate of A, denoted by A^{**}, with the first Arens product "$\diamond$" defined by

$$\langle F \diamond H, f\rangle = \langle F, Hf\rangle$$

is a Banach algebra, where $\langle Hf, a\rangle = \langle H, fa\rangle$, in which $\langle fa, b\rangle = \langle f, ab\rangle$ for all $F, H \in A^{**}$, $f \in A^*$ and $a, b \in A$; see for example [3].

Theorem 2.4. *Let A be a Banach algebra with a bounded approximate identity. Then $C^b_{p,q}(A)$ is a closed subspace of $C_{ts}(A)$.*

Proof. Let $T \in C^b_{p,q}(A)$. Then

$$(p+q)\langle T^*(f)a, x\rangle = \langle pfT(a), x\rangle + \langle qT^*(fa), x\rangle$$

for all $a, x \in A$. So, if $H \in A^{**}$, then

$$(p+q)\langle HT^*(f), a\rangle = p\langle T^*(Hf), a\rangle + q\langle T^{**}(H)f, a\rangle.$$

Thus for every $F \in A^{**}$ we obtain

$$\langle (p+q)T^{**}(F \diamond H), f\rangle = \langle pT^{**}(F)H + qFT^{**}(H), f\rangle.$$

That is, T^{**} is a $(p,q)-$centralizer. Since A^{**} has a right identity, by Theorem 2.1, $T^{**} \in C_{ts}(A^{**})$. Therefore, $T \in C_{ts}(A)$. $\square$

3 The Range of $(p, q)-$Centralizers

We commence this section with the following result.

Theorem 3.1. *Let A be a normed algebra with a right identity u, and let $T \in C_{p,q}(A)$. Then the following assertions are equivalent.*
(a) *T maps A into uA.*
(b) *$T = \lambda_{a_0}$ for some $a_0 \in A$.*
(c) *$T = \lambda_{T(u)}$.*
(d) *$T(u) \in Z(A)$.*

Proof. Assume that $T \in C_{p,q}(A)$ maps A into uA. Then by Theorem 2.1, for every $a \in A$ we have

$$uT(a - ua) = T(u)(a - ua) = 0.$$

Thus

$$T(a - ua) \in \mathrm{ran}(A) \cap uA = \{0\}.$$

Hence $T(a) = T(ua) = T(u)a = \lambda_{T(u)}(a)$. Thus (a)⇒(b).

Let $T = \lambda_{a_0}$ for some $a_0 \in A$. Then for every $a \in A$, we have

$$\begin{aligned} a_0 &= T(u) = \rho_{T(u)}(u) \\ &= uT(u) = T(u)u \\ &= T(u). \end{aligned}$$

Hence (b)⇒(c). The implication (c)⇒(d) follows from Theorem 2.1. Finally, if $T(u) \in Z(A)$, then

$$T(a) = aT(u) = T(u)a = uT(a)$$

for all $a \in A$. Thus T maps A into uA. That is, (d)⇒(a). □

Proposition 3.2. *Let A be an algebra and $T \in C_{p,q}(A)$. Then $T^2 = 0$ if and only if the range of T is nilpotent. In this case, T maps A into the radical of A.*

Proof. For every $a \in A$, we have

$$\begin{aligned} 0 &= (p+q)T^2(a^2) \\ &= T(pT(a)a + qaT(a)) \\ &= p^2T^2(a)a + pqT(a)T(a) + pqT(a)T(a) + q^2aT^2(a) \\ &= 2pqT(a)^2. \end{aligned}$$

This proves the result. □

Let $\mathfrak{A}$ be a Banach algebra with a bounded approximate identity such that every proper closed ideal of $\mathfrak{A}$ is contained in a proper closed ideal with a bounded approximate identity. From Theorem 4.7 of [15] and Theorem 2.4 we have the following result.

Proposition 3.3. *Let $\mathfrak{A}$ be as above. Then $T \in C_{p,q}(\mathfrak{A})$ has a closed range if and only if there exist an idempotent $(p,q)-$centralizer T_1 of $\mathfrak{A}$ and an invertible $(p,q)-$centralizer T_2 of $\mathfrak{A}$ such that $T = T_1 \circ T_2 = T_2 \circ T_1$.*

4 $(p,q)-$centralizers on Group Algebras

Let G be a locally compact group with a left Haar measure. Let $L^1(G)$ be the group algebra of G. Then $L^1(G)$ with the convolution product "$*$" and the norm $\|.\|_1$ is a Banach algebra with a bounded approximate identity [3, 5]. Let $L^\infty(G)$ be the usual Lebesgue space as defined in [5] and $L_0^\infty(G)$ be the subspace of $L^\infty(G)$ consisting of all functions that vanish at infinity. Then $L^\infty(G)^*$ and $L_0^\infty(G)^*$ are Banach algebras with the first Arens product. One can prove that $L^\infty(G)^*$ and $L_0^\infty(G)^*$ have right identities [3, 7]; for more study see [9, 11, 12]. Let $M(G)$ be the measure algebra of G. Then $M(G)$ with the convolution product is a unital Banach algebra and

$$M(G) \cong C_0(G)^*,$$

where "$\cong$" denotes isometrically isomorphic as Banach algebras and $C_0(G)$ is the space of all complex-valued continuous functions on G that vanish at infinity [5]. Finally, let $C_b(G)$ be the space of all bounded continuous functions on G, and let $LUC(G)$ be the space of all $f \in C_b(G)$ such that the mapping

$$x \mapsto l_x f$$

from G into $C_b(G)$ is continuous, where $l_x f(y) = f(xy)$ for all $y \in G$. Let us remark that $LUC(G)^*$ with the product "$\cdot$" defined by

$$\langle m \cdot n, f\rangle = \langle m, nf\rangle \quad (m, n \in LUC(G)^*, f \in LUC(G)),$$

where

$$\langle nf, x\rangle = \langle n, l_x f\rangle \quad (x \in G)$$

is a unital Banach algebra [3]. These facts together with Corollary 2.3 show that $C_{p,q}(\mathfrak{A})$ is a Banach algebra and $C_{p,q}(\mathfrak{A}) \cong Z(\mathfrak{A})$, where $\mathfrak{A}$ is one of the Banach algebras $M(G)$ or $LUC(G)^*$.

Theorem 4.1. *Let G be a locally compact group. Then the following statements hold.*

(i) $C^b_{p,q}(L^1(G)) = \{\rho_\mu : \mu \in Z(M(G), L^1(G))\}$.

(ii) *If $T \in C_{p,q}(L^\infty_0(G)^*)$, then $T = \rho_\mu$ for some $\mu \in Z(M(G), L^1(G))$.*

Proof. Let $\mathfrak{B}$ be one of the Banach algebras $L^1(G)$ or $L^\infty_0(G)^*$. Assume that $T \in C^b_{p,q}(\mathfrak{B})$. Then by Theorems 2.1 and 2.4, $T \in C_{ts}(\mathfrak{B})$. So T is a right multiplier on $\mathfrak{B}$. Note that if $\mathfrak{B} = L^\infty_0(G)^*$, then $T = \rho_n$, where

$$n = T(u)$$

and u is a right identity for $L^\infty_0(G)^*$ with $\|u\| = 1$. By Lemma 2.2 of [7], $T = \rho_\mu$ for some $\mu \in M(G)$. If $\mathfrak{B} = L^1(G)$, then by [17], $T = \rho_\mu$ for some $\mu \in M(G)$. Thus $T = \rho_\mu$ on $\mathfrak{B}$. Now, let $(e_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity of $L^1(G)$. Then for every $\phi \in L^1(G)$ and $\alpha \in \Lambda$, we have

$$\begin{aligned} e_\alpha * \phi * \mu &= T(e_\alpha * \phi) \\ &= T(e_\alpha) * \phi = e_\alpha * \mu * \phi. \end{aligned}$$

Since $L^1(G)$ is an ideal of $M(G)$, it follows that $\phi * \mu = \mu * \phi$. Thus $\mu \in Z(M(G), L^1(G))$. □

Let $C^w_{p,q}(A)$ be the space of all weakly compact $(p, q)-$centralizers of a Banach algebra A.

Corollary 4.2. *Let G be a compact group. Then the following statements hold.*

(i) $C^w_{p,q}(L^1(G)) = \{\rho_\phi : \phi \in Z(L^1(G))\}$.

(ii) $C^w_{p,q}(M(G)) = \{\rho_\phi : \phi \in Z(L^1(G), M(G))\}$.

(iii) $C^w_{p,q}(L^\infty(G)^*) = \{\rho_\phi : \phi \in Z(L^1(G))\}$.

(iv) $C^w_{p,q}(LUC(G)^*) = \{\rho_\phi : \phi \in Z(L^1(G))\}$.

Proof. (i) Let $T \in C^w_{p,q}(L^1(G))$. Then $T = \rho_\mu$ for some $\mu \in Z(M(G), L^1(G))$. Since G is compact, every right centralizer of $L^1(G)$ is weakly compact and it is of the form ρ_ϕ for some $\phi \in L^1(G)$; see [1]. Thus $\mu \in L^1(G)$ and so $\mu \in Z(L^1(G))$.

(ii) Let $T \in C^w_{p,q}(M(G))$. Then $T = \rho_\mu$ for some $\mu \in Z(M(G))$. Thus ρ_μ is a weakly compact right multiplier on $L^1(G)$ which implies that $\mu \in L^1(G)$. So,

$$\mu \in Z(L^1(G), M(G)).$$

(iii) Let $T \in C^w_{p,q}(L^\infty(G)^*)$. Then T is a weakly compact right multiplier on $L^\infty(G)^*$. So $T = \rho_\phi$ for some $\phi \in L^1(G)$; see [13]. A similar argument to the proof of Theorem 4.1 shows that $\phi \in Z(L^1(G))$.

(iv) This follows from Theorem 2.1 and Corollary 2.3 of [10]. □

Now, we prove the main result of this section.

Theorem 4.3. *Let G be a locally compact group. Then the following assertions are equivalent.*

(a) $C^w_{p,q}(L^\infty_0(G)^*) \neq \{0\}$.
(b) $C^w_{p,q}(L^1(G)) \neq \{0\}$.
(c) $C^w_{p,q}(L^\infty(G)^*) \neq \{0\}$.
(d) $C^w_{p,q}(LUC(G)^*) \neq \{0\}$.
(e) *G is compact and* $Z(L^1(G)) \neq \{0\}$.

Proof. Let $T \in C^w_{p,q}(L^\infty_0(G)^*)$ be a non-zero. By Theorem 2.1, $T = \rho_n$ for some $n \in L^\infty_0(G)^*$. Since $L^1(G)$ is an ideal of $L^\infty_0(G)^*$, it follows that

$$T|_{L^1(G)} \in C^w_{p,q}(L^1(G)).$$

From weak* density of $L^1(G)$ into $L^\infty_0(G)^*$ we infer that $T|_{L^1(G)}$ is non-zero. So (a)⇒(b).

Let $T \in C^w_{p,q}(L^1(G))$ be non-zero. According to Theorem 2.4, T is a non-zero weakly compact right centralizer of $L^1(G)$. Hence G is compact; see [14]. By Corollary 4.2,

$$Z(L^1(G)) \neq \{0\}.$$

Thus (b)⇒(e).

Assume that $Z(L^1(G)) \neq \{0\}$ and G is compact. It follows from Corollary 4.2 (iii) that $C^w_{p,q}(L^\infty(G)^*) \neq \{0\}$. Hence (e)⇒(a).

By [8] and Corollary 4.2, the implication (c)⇒(e) holds. The converse follows from the fact that

$$L^\infty(G)^* = L^\infty_0(G)^*$$

when G is compact.

Finally, Theorem 2.1 of [10] together with Corollary 4.2 (iv) proves the implication (d)⇒(e); also the implication (e)⇒(d) holds by Corollary 2.3 of [10] and Corollary 4.2 (iv). □

Let us remark that if G is a locally compact abelian group, then $Z(L^1(G)) = L^1(G)$.

Corollary 4.4. *Let G be a locally compact abelian group. Then the following assertions are equivalent.*

(a) $C^w_{p,q}(L^\infty_0(G)^*) \neq \{0\}$.
(b) $C^w_{p,q}(L^1(G)) \neq \{0\}$.
(c) $C^w_{p,q}(L^\infty(G)^*) \neq \{0\}$.
(d) $C^w_{p,q}(LUC(G)^*) \neq \{0\}$.
(e) *G is compact.*

5 $(p, q)-$Jordan Centralizers

We commence this section with the following result.

Theorem 5.1. *Let A be a commutative algebra. Then $C^J_{p,q}(A) = C_{1,1}(A) = C_{ts}(A)$.*

Proof. First, assume that m, n are distinct positive integers and $T \in C_{m,n}(A)$. Then for every $a, b \in A$, we have

$$\begin{aligned} mT(a)b + naT(b) &= (m+n)T(ab) \\ &= (m+n)T(ba) \\ &= mT(b)a + nbT(a) \end{aligned}$$

So $T(a)b = aT(b)$. Hence for every $a, b \in A$,

$$T(ab) = T(a)b = aT(b).$$

Thus $C_{m,n}(A) = C_{ts}(A)$ for all distinct positive integers m, n. This shows that

$$C_{1,1}(A) = C_{ts}(A).$$

Now, let $T \in C^J_{p,q}(A)$. Then for every $a, b \in A$

$$(p+q)T(ab+ba) = pT(a)b + pT(b)a + qaT(b) + qbT(a).$$

Since A is commutative, it follows that

$$2T(ab) = T(a)b + aT(b).$$

Consequently, $T \in C_{1,1}(A)$ and therefore, $C^J_{p,q}(A) = C_{1,1}(A)$. □

We now give a characterization of $(p,q)-$Jordan centralizers of an algebra with a right identity.

Theorem 5.2. *Let A be an algebra with a right identity u. If $T \in C^J_{p,q}(A)$, then*

$$T(a) = (a - ua)T(u) + uT(a)$$

for all $a \in A$.

Proof. Let $T \in C^J_{p,q}(A)$. Then for every $a, b \in A$

$$(p+q)T(ab+ba) = pT(a)b + pT(b)a + qaT(b) + qbT(a). \tag{2}$$

Put $a = b = u$ in (2). Then $T(u) = uT(u)$. Taking $b = u$ in (2), we get

$$qT(a) + (p+q)T(ua) = pT(u)a + qaT(u) + quT(a). \tag{3}$$

If we set $a = ua$ in (3), then

$$(p+2q)T(ua) = pT(u)a + quaT(u) + quT(ua). \tag{4}$$

From (3) we infer that

$$quT(a) + (p+q)uT(ua) = pT(u)a + quaT(u) + quT(a).$$

Thus

$$(p+q)uT(ua) = pT(u)a + quaT(u)$$

for all $a \in A$. This together with (4) shows that

$$\begin{aligned}(p+2q)T(ua) &= pT(u)a + quaT(u) + \frac{q}{p+q}(quaT(u) + pT(u)a) \\ &= \frac{p(p+2q)}{p+q}T(u)a + \frac{q(p+2q)}{p+q}uaT(u).\end{aligned}$$

Hence

$$(p+q)T(ua) = pT(u)a + quaT(u).$$

From this and (3) we see that

$$qT(a) + quaT(u) + pT(u)a = qaT(u) + pT(u)a + quT(a).$$

This implies that

$$T(a) = (a - ua)T(u) + uT(a)$$

for all $a \in A$. □

Theorem 5.3. *Let A be an algebra with identity 1_A and $T \in C^J_{p,q}(A)$. If $T(1_A) \in Z(A)$, then $T \in C_{ts}(A)$.*

Proof. Let $T \in C^J_{p,q}(A)$. Then for every $a, b \in A$

$$(p+q)T(ab+ba) = pT(a)b + pT(b)a + qaT(b) + qbT(a). \tag{5}$$

Put $b = 1_A$ in (5). Then

$$(p+q)T(a) = pT(1_A)a + qaT(1_A).$$

If $T(1_A) \in Z(A)$, then $T(a) = aT(1_A) = T(1_A)a$. Thus

$$T(ab) = abT(1_A) = aT(b) = aT(1_A)b = T(a)b$$

That is, $T \in C_{ts}(A)$.

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