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Original Research Paper

Some New Paranormed Sequence Spaces Derived by q-Second Difference Matrix

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Abstract. The goal of this research is to construct the extended versions of the original Maddox's paranormed sequence spaces, denoted by the notation $\ell(\nabla_q^2, p)$ and $\ell_\infty(\nabla_q^2, p)$. These spaces are linear isomorphic to the spaces $\ell(p)$ and $\ell_\infty(p)$, respectively. The next step is to build the Schauder basis for the $\ell(\nabla_q^2, p)$ space. After that, the topological features of the alpha, beta, and gamma duals of $\ell(\nabla_q^2, p)$ and $\ell_\infty(\nabla_q^2, p)$ are investigated. Finally, some matrix classes are characterized.

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1 Introduction

The expression $[\mathfrak{h}]_q$ defines a q-number [18], as given by the equation:

$$[\mathfrak{h}]_q = \begin{cases} \displaystyle\sum_{s=0}^{\mathfrak{h}-1} q^s, & \mathfrak{h} = 1, 2, 3, ..., \\ 0, & \mathfrak{h} = 0. \end{cases}$$

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It is possible to hypothesize that if $q \to 1^-$ then $[\mathfrak{h}]_q \to \mathfrak{h}$. The following equation gives the definition of the q-binomial coefficient:

$$\begin{bmatrix} \mathfrak{h} \\ \mathfrak{f} \end{bmatrix}_q = \begin{cases} \dfrac{[\mathfrak{h}]_q!}{[\mathfrak{f}]_q![\mathfrak{h}-\mathfrak{f}]_q!}, & 0 \le \mathfrak{f} \le \mathfrak{h}, \\ 0, & otherwise, \end{cases} \tag{1}$$

where q-factorial of $\mathfrak{h}$ is given by the product of $[\mathfrak{h}]_q$ for $\mathfrak{h} = 1, 2, 3, ...$, and is equal to 1 for $\mathfrak{h} = 0$. There is a wide range of work on studies of the q-analogue of sequence spaces; you can refer to the references [1, 3, 10, 22, 27, 28, 29, 30, 31, 32].

The traditional notation for the space of all real-valued sequences is ω. A sequence space is any vector subspace of ω. The most prevalent and often utilized spaces that are all null, convergent, and bounded sequences, respectively, are c_0, c, and ℓ_∞.

The domain $\mathcal{H}_A$ of the matrix A in the space $\mathcal{H}$ is a sequence space. It is defined by

$$\mathcal{H}_A = \{\mathfrak{h} \in \omega : A\mathfrak{h} \in \mathcal{H}\}$$

In the literature, the approach of constructing a new sequence spaces on the paranormed spaces by means of the matrix domain of a particular limitation method has recently been employed by several authors. For example, see [4, 5, 6, 8, 9, 14, 16, 24, 26, 33].

The sequences $c_0(p), c(p), \ell_\infty(p)$ and $\ell(p)$ were established by Maddox [20], Simons [25] and Nakano [21] in the following order:

$$\begin{aligned} c_0(p) &= \{\mathfrak{h} = (\mathfrak{h}_s) \in \omega : \lim_{s\to\infty} |\mathfrak{h}_s|^{p_s} = 0\}, \\ c(p) &= \{\mathfrak{h} = (\mathfrak{h}_s) \in \omega : \lim_{s\to\infty} |\mathfrak{h}_s - l|^{p_s} = 0 \text{ for some } l \in \mathbb{R}\}, \\ \ell_\infty(p) &= \{\mathfrak{h} = (\mathfrak{h}_s) \in \omega : \sup_{s\in\mathbb{N}} |\mathfrak{h}_s|^{p_s} < \infty\}, \\ \ell(p) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in \omega : \sum_s |\mathfrak{h}_s|^{p_s} < \infty\right\}, \end{aligned}$$

which are the complete spaces paranormed by $g_1(\mathfrak{h}) = \sup_{s\in\mathbb{N}} |\mathfrak{h}_s|^{p_s/L}$ and $g_2(\mathfrak{h}) = \left(\sum_s |\mathfrak{h}_s|^{p_s}\right)^{1/L}$, respectively, where $p = (p_s)$ is a bounded sequence of strictly positive real numbers with $S = \sup_s p_s$ and $L = \max\{1, S\}$.

Kızmaz [19] first proposed the difference sequence spaces

$$\mathfrak{D}(\Delta) = \{\mathfrak{h} = (\mathfrak{h}_s) : \Delta\mathfrak{h} \in \mathfrak{D}\}$$

for $\mathfrak{D} = \{\ell_\infty, c_0, c\}$. Then, several writers began to pay attention to the difference sequence spaces in various ways, including [2, 7, 11, 12, 13, 23].

The definition of the difference operator ∇_q^2 [1], for a q-number is

$$(\nabla_q^2)_{rs} = \begin{cases} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q & , \quad 0 \le s \le r, \\ 0 & , \quad s > r, \end{cases}$$

equivalently, we may write

$$\nabla_q^2 = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -(1+q) & 1 & 0 & 0 & \cdots \\ q & -(1+q) & 1 & 0 & \cdots \\ 0 & q & -(1+q) & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

The inverse of the operator ∇_q^2 is derived as $\nabla_q^{-2} = ((\nabla_q^{-2})_{rs})$

$$(\nabla_q^{-2})_{rs} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q & , \quad 0 \le s \le r, \\ 0 & , \quad s > r. \end{cases}$$

It is well known that paranormed spaces have more general properties then normed spaces. In this article, we generalize the normed sequence spaces defined by Yaying et al. [1].

2 Main Results

The theory of q-calculus frequently uses the formula for ∇_q^2 to define new sequence spaces. Using the q-difference matrix ∇_q^2 of second order, Yaying et al. [1, 27] recently studied the sequence spaces as follows:

$$\begin{aligned} c_0(\nabla_q^2) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \nabla_q^2\mathfrak{h} \in c_0\right\}, \\ c(\nabla_q^2) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \nabla_q^2\mathfrak{h} \in c\right\}, \\ \ell_p(\nabla_q^2) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \nabla_q^2\mathfrak{h} \in \ell_p\right\}, \\ \ell_\infty(\nabla_q^2) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \nabla_q^2\mathfrak{h} \in \ell_\infty\right\}. \end{aligned}$$

In this study, we now present the q-paranorm difference sequence spaces of second order $\ell(\nabla_q^2, p)$ and $\ell_\infty(\nabla_q^2, p)$ by

$$\begin{aligned}\ell(\nabla_q^2, p) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \sum_{s=0}^{\infty} \left|(-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s\right|^{p_s} < \infty\right\}, \\ \ell_\infty(\nabla_q^2, p) &= \left\{\mathfrak{h} = (\mathfrak{h}_s) \in w : \sup_{s\in\mathbb{N}} \left|(-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s\right|^{p_s} < \infty\right\}.\end{aligned}$$

Following that, the aforementioned spaces may be stated in this manner:

$$\ell(\nabla_q^2, p) = [\ell(p)]_{\nabla_q^2} \text{ and } \ell_\infty(\nabla_q^2, p) = [\ell_\infty(p)]_{\nabla_q^2}.$$

Consequently, for $p = (p_s) = e$, the foregoing sequences simplify to $\ell_p(\nabla_q^2)$ and $\ell_\infty(\nabla_q^2)$, which were presented by Yaying et al. [1].

In order to define the sequence $\mathfrak{f} = (\mathfrak{f}_r)$, we must apply the ∇_q^2-transform of $\mathfrak{h} = (\mathfrak{h}_r)$.

$$\mathfrak{f}_r = \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s \quad \text{for all } r \in \mathbb{N}. \tag{2}$$

Then, it is possible to do a straightforward calculation (2) to establish that

$$\mathfrak{h}_r = \sum_{s=0}^{r} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q \mathfrak{f}_s \quad \text{for all } r \in \mathbb{N}. \tag{3}$$

The next theorem, which is important for the work, is where we may now start.

Theorem 2.1. *The sequence spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$ are each defined as a complete linear metric spaces in the following functions:*

$$g(\mathfrak{h}) = \sup_{r\in\mathbb{N}} \left|\sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s\right|^{p_r/L}, \tag{4}$$

$$\tilde{g}(\mathfrak{h}) = \left(\sum_{r=0}^{\infty} \left|\sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q x_s\right|^{p_r}\right)^{1/M}. \tag{5}$$

Proof. We simply present the proof for $\ell_\infty(\nabla_q^2, p)$ to avoid repeating the same statements. It is obvious that $g(\theta) = 0$ holds true and for all values of $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$, $g(-\mathfrak{h}) = g(\mathfrak{h})$. The next inequalities hold for $\mathfrak{h}, \mathfrak{t} \in \ell_\infty(\nabla_q^2, p)$ and $\alpha_1, \alpha_2 \in \mathbb{R}$

$$\begin{aligned} g(\alpha_1\mathfrak{h} + \alpha_2\mathfrak{t}) \qquad &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q (\alpha_1\mathfrak{h}_s + \alpha_2\mathfrak{t}_s) \right|^{p_r/L} \\ &\leq \max\{1, |\alpha_1|\} \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s \right|^{p_r/L} + \\ &+ \max\{1, |\alpha_2|\} \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{t}_s \right|^{p_r/L} \\ &= \max\{1, |\alpha_1|\} g(\mathfrak{h}) + \max\{1, |\alpha_2|\} g(\mathfrak{t}) \end{aligned}$$

can be used to demonstrate the linearity of g with regard to scalar multiplication and coordinatewise addition. This demonstrates that the space $\ell_\infty(\nabla_q^2, p)$ is linear.

Assume that $\{\mathfrak{h}^r\}$ is any sequence of points $\mathfrak{h}^r \in \ell_\infty(\nabla_q^2, p)$ such that $g(\mathfrak{h}^r - \mathfrak{h}) \to 0$ and (α_r) is any sequence of scalars such that $\alpha_r \to \alpha$. Due to the subadditivity of g, the inequality

$$g(\mathfrak{h}^r) \leq g(\mathfrak{h}) + g(\mathfrak{h}^r - \mathfrak{h})$$

holds, and as a result $\{g(\mathfrak{h}^r)\}$ is bounded. Consequently, we have

$$\begin{aligned} g(\alpha_r\mathfrak{h}^r - \alpha\mathfrak{h}) \quad &= \quad \sup_{r\in\mathbb{N}} \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q (\alpha_r\mathfrak{h}_s^r - \alpha\mathfrak{h}_s) \right|^{p_r/L} \\ &\leq \quad |\alpha_r - \alpha|^{p_r/L} g(\mathfrak{h}^r) + |\alpha|^{p_r/L} g(\mathfrak{h}^r - \mathfrak{h}) \end{aligned} \tag{6}$$

which tends to zero as $i \to \infty$. Accordingly, the scalar multiplication is continuous. In light of this, g is a paranorm on the space $\ell_\infty(\nabla_q^2, p)$.

It is still necessary to demonstrate that the space $\ell_\infty(\nabla_q^2, p)$ is complete. In the space $\ell_\infty(\nabla_q^2, p)$, let $\{\mathfrak{h}^r\}$ be any Cauchy sequence , where $\mathfrak{h}^r = \{\mathfrak{h}_0^{(r)}, \mathfrak{h}_1^{(r)}, \mathfrak{h}_2^{(r)}, \ldots\}$. Then, for a given value of $\epsilon > 0$, $n_0(\epsilon)$ exists such that

$$g(\mathfrak{h}^r - \mathfrak{h}^s) < \frac{\epsilon}{2}$$

for all $s, r > n_0(\epsilon)$. For each constant $k \in \mathbb{N}$, we find using the definition of g that

$$|(\nabla_q^2 \mathfrak{h}^r)_k - (\nabla_q^2 \mathfrak{h}^s)_k|^{p_k/L} \leq \sup_{k \in \mathbb{N}} |(\nabla_q^2 \mathfrak{h}^r)_k - (\nabla_q^2 \mathfrak{h}^s)_k|^{p_k/L} < \frac{\epsilon}{2} \qquad (7)$$

for every $s, r > n_0(\epsilon)$. As a result, for every fixed $k \in \mathbb{N}$, $\{(\nabla_q^2 \mathfrak{h}^0)_k, (\nabla_q^2 \mathfrak{h}^1)_k, (\nabla_q^2 \mathfrak{h}^2)_k, \ldots\}$ is a Cauchy sequence of real numbers. It converges because $\mathbb{R}$ is complete, which means that $(\nabla_q^2 \mathfrak{h}^r)_k \to (\nabla_q^2 \mathfrak{h})_k$ as $k \to \infty$.

Now we define the sequence $\{(\nabla_q^2 \mathfrak{h})_0, (\nabla_q^2 \mathfrak{h})_1, \ldots\}$ using these infinitely many limits $(\nabla_q^2 \mathfrak{h})_0, (\nabla_q^2 \mathfrak{h})_1, \ldots$. For every fixed $k \in \mathbb{N}$, we have

$$|(\nabla_q^2 \mathfrak{h}^r)_k - (\nabla_q^2 \mathfrak{h})_k|^{p_k/L} \leq \frac{\epsilon}{2} \ (r > n_0(\epsilon)) \qquad (8)$$

from (7) with $s \to \infty$. Since $\mathfrak{h}^r = \{\mathfrak{h}_k^{(r)}\} \in \ell_\infty(\nabla_q^2, p)$ for each $k \in \mathbb{N}$,

$$|(\nabla_q^2 \mathfrak{h}^r)_k|^{p_k/L} < \frac{\epsilon}{2} \qquad (9)$$

for every $r \geq n_0(\epsilon)$ and for each fixed $k \in \mathbb{N}$. For this reason, assuming a constant $r > n_0(\epsilon)$ we derive by (8) and (9) that

$$|(\nabla_q^2 \mathfrak{h})_k|^{p_k/L} \leq |(\nabla_q^2 \mathfrak{h})_k - (\nabla_q^2 \mathfrak{h}^r)_k|^{p_k/L} + |(\nabla_q^2 \mathfrak{h}^r)_k|^{p_k/L} < \epsilon$$

for every $s > s_0(\epsilon)$. This demonstrates that $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$. The space $\ell_\infty(\nabla_q^2, p)$ is complete, and this closes the proof because $\{\mathfrak{h}^r\}$ was an arbitrary Cauchy sequence. □

The sequence spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$ are in fact of the non-absolute type since there exists at least one sequence in them such that $g(\mathfrak{h}) \neq g(|\mathfrak{h}|)$, where $|\mathfrak{h}| = (|\mathfrak{h}_s|)$.

Theorem 2.2. *The two spaces, $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$, are linearly isomorphic to $\ell_\infty(p)$ and $\ell(p)$, respectively, where $0 < p_s \leq H < \infty$.*

Proof. In order to avoid reiterating identical claims, we only provide the evidence for $\ell_\infty(\nabla_q^2, p)$. The space $\ell_\infty(\nabla_q^2, p)$ should be shown as linearly bijective. Defining the transformation $\mathfrak{T}$ of $\ell_\infty(\nabla_q^2, p)$ to $\ell_\infty(p)$ by $\mathfrak{h} \mapsto \mathfrak{f} = \mathfrak{T}\mathfrak{h}$ use the notation (2). The linearity of $\mathfrak{T}$ is trivial.

Additionally, it is clear that $\mathfrak{T}$ is injective because $\mathfrak{h} = \theta$ whenever $\mathfrak{T}\mathfrak{h} = \theta$.

Let $\mathfrak{f} \in \ell_\infty(p)$ with (3), we have

$$\begin{aligned} g(\mathfrak{h}) &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \mathfrak{h}_s \right|^{p_r/L} \\ &= \sup_r \left| \sum_{s=0}^{r} (-1)^{r-s} q^{\binom{r-s}{2}} \begin{bmatrix} 2 \\ r-s \end{bmatrix}_q \sum_{s=0}^{j} \begin{bmatrix} j-s+1 \\ j-s \end{bmatrix} \mathfrak{f}_s \right|^{p_r/L} \\ &= \sup_r \left| \sum_{s=0}^{r} \delta_{rs} y_s \right|^{p_r/L} \\ &= \sup_r |y_r|^{p_r/L} < \infty, \end{aligned}$$

where

$$\delta_{rs} = \begin{cases} 1 & , \quad s = r, \\ 0 & , \quad s \neq r. \end{cases}$$

Therefore, we get $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$ and $\mathfrak{T}$ is a surjective. As a result, $\mathfrak{T}$ is a linear bijection, indicating that there is the necessary linear isomorphism between the spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell_\infty(p)$. □

We will now finish by giving the Schauder basis of space $\ell(\nabla_q^2, p)$. First, let us recall the definition of the Schauder basis. The description of a paranormed space is $(\mathcal{H}, g)$, where $\mathcal{H}$ is a set and g is a paranorm. There exist is a unique sequence of scalars (α_s) such that

$$g\left(\mathfrak{h} - \sum_{s=0}^{r} \alpha_s \beta_s\right) \to 0 \text{ as } r \to \infty$$

if and only if the sequence (β_s) of the elements of $\mathcal{H}$ is called a basis for $\mathcal{H}$.

Let $\mathcal{H}$ is a sequence space and $\mathcal{A}$ is a triangle. Due to the fact (cf. [17, Remark 2.4]) that $\mathcal{H}_\mathcal{A}$ has a basis when $\mathcal{H}$ has a basis, we get the following theorem.

Theorem 2.3. *For all $s \in \mathbb{N}$ and $0 < p_s \leq H < \infty$, let $\mathcal{H}_s = (\nabla_q^2 \mathfrak{h})_s$. The sequence $b^{(s)} = \{b^{(s)}\}_{s\in\mathbb{N}}$ of the elements of the space $\ell(\nabla_q^2, p)$ is defined by*

$$b_r^{(s)} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q & , \quad 0 \leq s < r \\ 0 & , \quad s \geq r \end{cases}$$

for every fixed $s \in \mathbb{N}$. Then, the sequence $\{b^{(s)}\}_{s\in\mathbb{N}}$ is a basis for the space $\ell(\nabla_q^2, p)$, and any $\mathfrak{h} \in \ell(\nabla_q^2, p)$ has a unique representation of the form

$$\mathfrak{h} = \sum_s \mathcal{H}_s b^{(s)}.$$

3 The $\alpha-$, β- and γ-Duals of the Spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$

Our most important theorems, which establish the α-, β- and γ-duals of our new sequence spaces, are presented and demonstrated in this section. The following will assume that $p^\star$ is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p^\star} = 1$, and designate the collection of all finite subsets of $\mathbb{N}$ by $\mathcal{N}$.

The α-, β- and γ-duals of a sequence space $\mathcal{H}$ are denoted by $\mathcal{H}^\alpha$, $\mathcal{H}^\beta$ and $\mathcal{H}^\gamma$, respectively, and are defined by

$$\mathcal{H}^\alpha = \{\mathfrak{t} = (\mathfrak{t}_s) \in w : \mathfrak{h}\mathfrak{t} = (\mathfrak{h}_s\mathfrak{t}_s) \in \ell_1 \text{ for all } \mathfrak{h} = (\mathfrak{h}_s) \in \mathcal{H}\}$$
$$\mathcal{H}^\beta = \{\mathfrak{t} = (\mathfrak{t}_s) \in w : \mathfrak{h}\mathfrak{t} = (\mathfrak{h}_s\mathfrak{t}_s) \in cs \text{ for all } \mathfrak{h} = (\mathfrak{h}_s) \in \mathcal{H}\}$$
$$\mathcal{H}^\gamma = \{\mathfrak{t} = (\mathfrak{t}_s) \in w : \mathfrak{h}\mathfrak{t} = (\mathfrak{h}_s\mathfrak{t}_s) \in bs \text{ for all } \mathfrak{h} = (\mathfrak{h}_s) \in \mathcal{H}\}$$

Lemma 3.1. *[15, $t_r = 1$] The following statements hold for an infinite matrix $\mathcal{A} = (a_{rs})$:*

(i) *$\mathcal{A} \in (\ell_\infty(p) : \ell(t))$ iff*

$$\forall M, \sup_K \sum_r \left| \sum_{s\in K} a_{rs} M^{1/p_s} \right|^{t_r} < \infty. \tag{10}$$

(ii) $\mathcal{A} \in (\ell_\infty(p) : c(t))$ *iff*

$$\forall M, \sup_r \sum_s |a_{rs}| M^{1/p_s} < \infty, \tag{11}$$

$$\exists (\alpha_s) \subset \mathbb{R} \ni \forall M, \lim_r \left(\sum_s |a_{rs} - \alpha_s| \, M^{1/p_s} \right)^{t_r} = 0. \tag{12}$$

(iii) $\mathcal{A} \in (\ell_\infty(p) : \ell_\infty(t))$ *iff*

$$\forall M, \sup_r \left(\sum_s |a_{rs}| M^{1/p_s} \right)^{t_r} < \infty. \tag{13}$$

(iv) $\mathcal{A} \in (\ell(p) : \ell_1)$ *iff*

(a) *For each* $s \in \mathbb{N}$*, set* $0 < p_s \leq 1$*. Next*

$$\sup_{N \in \mathcal{F}} \sup_{s \in \mathbb{N}} \left| \sum_{r \in N} a_{rs} \right|^{p_s} < \infty. \tag{14}$$

(b) *For each* $s \in \mathbb{N}$*, set* $1 < p_s \leq S < \infty$*. Next, if* M *is an integer greater than 1, then*

$$\sup_{N \in \mathcal{F}} \sum_s \left| \sum_{r \in N} a_{rs} M^{-1} \right|^{p_s^\star} < \infty. \tag{15}$$

Lemma 3.2. *[15] The following statements hold for an infinite matrix* $\mathcal{A} = (a_{rs})$*:*

(i) $\mathcal{A} \in (\ell(p) : \ell_\infty)$ *iff*

(a) *For each* $s \in \mathbb{N}$*, set* $0 < p_s \leq 1$*. Next*

$$\sup_{r,s \in \mathbb{N}} |a_{rs}|^{p_s} < \infty. \tag{16}$$

(b) *For each* $s \in \mathbb{N}$*, set* $1 < p_s \leq S < \infty$*. Next, if* M *is an integer greater than 1, then*

$$\sup_{r \in \mathbb{N}} \sum_s \left| a_{rs} M^{-1} \right|^{p_s^\star} < \infty. \tag{17}$$

(ii) *For each $s \in \mathbb{N}$, set $0 < p_s \leq H < \infty$. Next, $\mathcal{A} = (a_{rs}) \in (\ell(p) : c)$ iff (16) and (17) hold, and*

$$\lim_{r\to\infty} a_{rs} = \beta_s, \ \forall s \in \mathbb{N}. \tag{18}$$

Theorem 3.3. *Let $J \in \mathcal{F}$ and $J^* = \{s \in \mathbb{N} : r \geq s\} \cap J$ for $J \in \mathcal{F}$. Define the sets c_1, c_2, c_3, c_4, c_5, c_6, c_7 and c_8 as follows:*

$$c_1 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \sup_J \sum_r \left| \sum_{s\in J} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r M^{1/p_s} \right| < \infty \right\},$$

$$c_2 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \sup_r \sum_s \left| \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r \right| M^{1/p_s} < \infty \right\},$$

$$c_3 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \exists (\alpha_s) \subset \mathbb{R} \ni \lim_r \sum_s \left| \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r - \alpha_s \right| M^{1/p_s} = 0 \right\},$$

$$c_4 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \sup_{N\in\mathcal{F}} \sum_s \left| \sum_{r\in N} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r M^{-1} \right|^{p_s^\star} < \infty, \right\},$$

$$c_5 = \left\{ a = (a_s) \in w : \sup_{N\in\mathcal{F}} \sup_{s\in\mathbb{N}} \left| \sum_{r\in N} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r \right|^{p_s} < \infty \right\},$$

$$c_6 = \bigcup_{M>1} \left\{ a = (a_s) \in w : \sup_{r\in\mathbb{N}} \sum_s \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z M^{-1} \right|^{p_s^\star} < \infty \right\},$$

$$c_7 = \left\{ a = (a_s) \in w : \lim_{r\to\infty} \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \text{ exists for each } s \in \mathbb{N} \right\},$$

$$c_8 = \left\{ a = (a_s) \in w : \sup_{r,s\in\mathbb{N}} \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right|^{p_s} < \infty \right\}.$$

Then, $[\ell_\infty(\nabla_q^2, p)]^\alpha = [\ell_\infty(\nabla_q^2, p)]^\gamma = c_1$, $[\ell_\infty(\nabla_q^2, p)]^\beta = c_2 \cap c_3$,

$$[\ell(\nabla_q^2, p)]^\alpha = \begin{cases} c_4 & , \quad 1 < p_s \leq H < \infty, \forall s \in \mathbb{N}, \\ c_5 & , \quad 0 < p_s \leq 1, \forall s \in \mathbb{N}. \end{cases}$$

$$[\ell(\nabla_q^2, p)]^\beta = \begin{cases} c_6 \cap c_7 & , \quad 1 < p_s \leq H < \infty, \forall s \in \mathbb{N}, \\ c_7 \cap c_8 & , \quad 0 < p_s \leq 1, \forall s \in \mathbb{N}. \end{cases}$$

$$[\ell(\nabla_q^2, p)]^\gamma = \begin{cases} c_6 & , \quad 1 < p_s \leq H < \infty, \forall s \in \mathbb{N}, \\ c_8 & , \quad 0 < p_s \leq 1, \forall s \in \mathbb{N}. \end{cases}$$

Proof. To prove α- dual, take the sequence $a = (a_s) \in w$ and $\mathfrak{h} = (\mathfrak{h}_s)$ as in defined in (3), then

$$a_r \mathfrak{h}_r = \sum_{s=0}^{r} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r \mathfrak{f}_s = (C\mathfrak{f})_r, \tag{19}$$

for all $r \in \mathbb{N}$, where the matrix $C = (c_{rs})$ defined by

$$c_{rs} = \begin{cases} \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_r & , \quad 0 \leq s \leq r, \\ 0 & , \quad s > r. \end{cases}$$

It follows from (19) that $a\mathfrak{h} = (a_r \mathfrak{h}_r) \in \ell_1$ whenever $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$ iff $C\mathfrak{f} \in \ell_1$ whenever $\mathfrak{f} \in \ell_\infty(p)$. This means that $a = (a_r) \in [\ell_\infty(\nabla_q^2, p)]^\alpha$ iff $C \in (\ell_\infty(p) : \ell_1)$. Then we derive by (10) with $t_r = 1$ for all $r \in \mathbb{N}$ that $[\ell_\infty(\nabla_q^2, p)]^\alpha = c_1$.

Also, using the (14),(15) and (19), we can prove

$$[\ell(\nabla_q^2, p)]^\alpha = \begin{cases} c_4 & , \quad 1 < p_s \leq H < \infty, \forall s \in \mathbb{N}, \\ c_5 & , \quad 0 < p_s \leq 1, \forall s \in \mathbb{N}, \end{cases}$$

can also be obtained in a similar way.

Now, consider the equation

$$\begin{aligned} \sum_{s=0}^{r} a_s \mathfrak{h}_s &= \sum_{s=0}^{r} \left[\sum_{z=0}^{s} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q \mathfrak{f}_z \right] a_s \\ &= \sum_{s=0}^{r} \left[\sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z \right] \mathfrak{f}_s = (D\mathfrak{f})_r, \end{aligned} \tag{20}$$

where $D = (d_{rs})$ is a matrix defined by

$$d_{rs} = \begin{cases} \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_z & , \quad (0 \leq s \leq r) \\ 0 & , \quad (s > r) \end{cases}$$

Thus, we conclude from (20) that $a\mathfrak{h} = (a_s\mathfrak{h}_s) \in cs$ whenever $\mathfrak{h} = (\mathfrak{h}_s) \in \ell_\infty(\nabla_q^2, p)$ iff $D\mathfrak{f} \in \ell_\infty$ whenever $\mathfrak{f} = (\mathfrak{f}_s) \in \ell_\infty(p)$. That is to say that $a = (a_s) \in [\ell_\infty(\nabla_q^2, p)]^\beta$ iff $D \in (\ell_\infty(p) : c)$. Therefore, we derive from (11), (12) and (20) with $t_r = 1$ for all $r \in \mathbb{N}$ that $[\ell_\infty(\nabla_q^2, p)]^\beta = c_2 \cap c_3$.

Also, using the (16),(17), (18) and (20), the proofs of the

$$[\ell(\nabla_q^2, p)]^\beta = \begin{cases} c_6 \cap c_7 & , \quad 1 < p_s \leq H < \infty, \forall s \in \mathbb{N}, \\ c_7 \cap c_8 & , \quad 0 < p_s \leq 1, \forall s \in \mathbb{N} \end{cases}$$

can also be obtained in a similar way. □

4 Certain Matrix Mappings on the Sequence Spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$

This section describes the matrix transformations from any given sequence space $\mathcal{H}$ into the spaces $\ell_\infty(\nabla_q^2, p)$ and $\ell(\nabla_q^2, p)$ as well as from a given sequence space $\mathcal{H}$ into the two previously mentioned spaces: The equivalence "$\mathfrak{h} \in \mathcal{H}_\mathcal{A}$ holds for every triangle $\mathcal{A}$ and any sequence space $\mathcal{H}$ iff $\mathfrak{f} = \mathcal{A}\mathfrak{h} \in \mathcal{H}$". The reason for this is that $\mathcal{H}_\mathcal{A} \cong \mathcal{H}$.

Theorem 4.1. *Set $\mathcal{H} = \{\ell_\infty, c, \ell_1\}$. Then*

(i) $\mathcal{A} = (a_{rs}) \in (\ell_\infty(\nabla_q^2, p) : \mathcal{H})$ *iff*

$$\mathcal{A}_r \in \left\{\ell_\infty(\nabla_q^2, p)\right\}^\beta \text{ for all } r \in \mathbb{N} \tag{21}$$

$$B \in (\ell_\infty : \mathcal{H}), \tag{22}$$

(ii) $\mathcal{A} = (a_{rs}) \in (\ell(\nabla_q^2, p) : \mathcal{H})$ *iff*

$$\mathcal{A}_r \in \left\{\ell(\nabla_q^2, p)\right\}^\beta \text{ for all } r \in \mathbb{N} \tag{23}$$

$$B \in (\ell_p : \mathcal{H}), \tag{24}$$

where $B = (b_{rs})$ with $b_{rs} = \sum_{z=s}^\infty \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_{rz}$ for all $r, s \in \mathbb{N}$ and $\mathcal{A}_r$ denotes the r^{th} row of the infinite matrix $\mathcal{A}$.

Proof. We prove only part of (i). Let $\mathcal{A} = (a_{rs}) \in (\ell_\infty(\nabla_q^2, p) : \mathcal{H})$ and $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$ be any given sequence space. Then $\mathcal{A}_r \in \left\{\ell_\infty(\nabla_q^2, p)\right\}^\beta$

for all $r \in \mathbb{N}$. Taking $\mathfrak{f} = \nabla_q^2 \mathfrak{h}$, we conclude from $\mathcal{A}_r \in \left\{\ell_\infty(\nabla_q^2, p)\right\}^\beta$ that

$$(\mathcal{A}\mathfrak{h})_r = (B\mathfrak{f})_r \quad \text{for all } r \in \mathbb{N}. \tag{25}$$

Hence $B\mathfrak{f} \in \mathcal{H}$ for all $\mathfrak{f} \in \ell_\infty$. Thus $B \in (\ell_\infty : \mathcal{H})$.

We have $E\mathfrak{f} = \mathcal{A}\mathfrak{h}$ which leads us to the consequence $E \in (c_0(p) : \mu)$. To obtain a contradiction, suppose that (21) and (22) hold. Then (25) again holds. It follows that $\mathcal{A}\mathfrak{h} \in \mathcal{H}$ for all $\mathfrak{h} \in \ell_\infty(\nabla_q^2, p)$. □

Corollary 4.2. *The following statements hold for an infinite matrix $\mathcal{A} = (a_{rs})$:*

(i) *$\mathcal{A} \in (\ell_\infty(\nabla_q^2, p) : \ell_1(q))$ iff*

$$\sup_r \sum_s \left| \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_{rs} \right| M^{1/p_s} < \infty, \tag{26}$$

$$\exists (\alpha_s) \subset \mathbb{R} \ni \lim_r \sum_s \left| \begin{bmatrix} r-s+1 \\ r-s \end{bmatrix}_q a_{rs} - \alpha_s \right| M^{1/p_s} = 0 \tag{27}$$

$$\sup_J \sum_r \left| \sum_{s \in J} b_{rs} M^{1/p_s} \right|^{p_r^\star} < \infty,$$

(ii) *$\mathcal{A} \in (\ell_\infty(\nabla_q^2, p) : \ell_\infty(q))$ iff (26) and (27) hold, and*

$$\sup_r \left(\sum_s |b_{rs}| M^{1/p_s} \right)^{p_r^\star} < \infty,$$

(iii) *$\mathcal{A} \in (\ell_\infty(\nabla_q^2, p) : c(q))$ iff (26) and (27) hold, and*

$$\sup_r \sum_s |b_{rs}| M^{1/p_s} < \infty,$$

$$\lim_r \sum_s |b_{rs} - \alpha_s| \, M^{1/p_s} = 0$$

where $B = (b_{rs})$ is defined as in Theorem 4.1.

Corollary 4.3. *The following statements hold for an infinite matrix* $\mathcal{A} = (a_{rs})$*:*

(i) $\mathcal{A} \in (\ell(\nabla_q^2, p) : \ell_1)$ *iff*

(a) *For* $1 < p_s \leq S < \infty$,

$$\lim_{r\to\infty} \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_{rz} \text{ for all } r \in \mathbb{N}, \tag{28}$$

$$\sup_{r\in\mathbb{N}} \sum_s \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_{rz} M^{-1} \right|^{p_s^\star} < \infty \tag{29}$$

$$\sup_{N\in\mathcal{F}} \sum_s \left| \sum_{r\in N} b_{rs} M^{-1} \right|^{p_s^\star} < \infty. \tag{30}$$

(b) *For* $0 < p_s < 1$, *(28) holds and*

$$\sup_{t,s\in\mathbb{N}} \left| \sum_{z=s}^{r} \begin{bmatrix} s-z+1 \\ s-z \end{bmatrix}_q a_{rz} \right|^{p_s} < \infty \tag{31}$$

$$\sup_{N\in\mathcal{F}} \sup_{s\in\mathbb{N}} \left| \sum_{r\in N} b_{rs} \right|^{p_s} < \infty \tag{32}$$

(ii) $\mathcal{A} \in (\ell(\nabla_q^2, p) : \ell_\infty)$ *iff*

(a) *For* $1 < p_s \leq S < \infty$, *(28) and (29) hold, and*

$$\sup_{r\in\mathbb{N}} \sum_s \left| b_{rs} M^{-1} \right|^{p_s^\star} < \infty. \tag{33}$$

(b) *For* $0 < p_s < 1$, *(28) and (31) hold, and*

$$\sup_{r,s\in\mathbb{N}} |b_{rs}|^{p_s} < \infty. \tag{34}$$

(iii) $\mathcal{A} \in (\ell(\nabla_q^2, p) : c)$ *iff*

(a) *For* $1 < p_s \leq S < \infty$, *(28), (29), (33) and (34) hold, and*

$$\lim_{r\to\infty} b_{rs} = \beta_s, \ \forall s \in \mathbb{N}. \tag{35}$$

(b) *For* $0 < p_s < 1$, *(28), (31), (33), (34) and (35) hold.*

5 Conclusions

A q-analogue is a mathematical notion that adds a parameter q to classical mathematical structures or functions to make them more broad. Existing mathematical and physical theories are expanded upon or altered using q-analogues to make them relevant to a larger variety of circumstances.

In this study, we use the q-analogue version of the second difference matrix, thus providing new results. For example, we defined new paranormed sequence spaces and examined some of the algebraical and topological properties of these sequence spaces.

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