Journal of Mathematical Extension
Vol. 17, No. 11, (2023) (2)1-17
URL: https://doi.org/10.30495/JME.2023.2805
ISSN: 1735-8299
Original Research Paper

Upper Level Qualifications and Optimality in Multiobjective Convex Generalized Semi-Infinite Problems

G. Edalat
Payame Noor University

N. Kanzi*
Payame Noor University

M. Adib
Payame Noor University

Abstract. In this paper we consider the multiobjective generalized semi-infinite optimization problems with nondifferentiable convex data, whereas Soroush (Journal of Mathematical Extension 16(9): 1-14, 2022) investigated them in single-objective case. We introduce some upper-level qualification conditions for the problems, and based on these qualifications, we demonstrate some first-order necessary optimality conditions at weakly efficient and efficient solutions of the considered problem.

AMS Subject Classification: 90C34; 90C40; 49J52.
Keywords and Phrases: Multiobjective GSIP, Constraint qualification, Necessary condition, Convex Subdifferential.

1 Introduction

In the present paper we consider a multiobjective generalized semi-infinite programming problem (MGSIP in brief), defined as follows:

$$(P): \qquad \min_{x\in S} \big(f_1(x), \ldots, f_p(x)\big),$$

Received: July 2023; Accepted: November 2023
*Corresponding Author

with the feasible set S,

$$S := \{x \in \mathbb{R}^n \mid g(x,y) \geq 0,\ y \in Y(x)\},$$

and the index set,

$$Y(x) := \{y \in \mathbb{R}^m \mid h_t(x,y) \leq 0,\ t \in T\},$$

where the appearing functions $f_i : \mathbb{R}^n \to \mathbb{R}$ as $i \in I := \{1, \ldots, p\}$ and $g, h_t : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ as $t \in T$ are convex, the index set T is finite, and the set-valued mapping $x \mapsto Y(x)$ is uniformly bounded, i.e., for each $x_0 \in S$ there exists a neighborhood U of x_0 such that the set $\bigcup_{x \in U} Y(x)$ is bounded. The latter assumption implies that the mapping $x \mapsto Y(x)$ is compact valued and upper semi-continuous at each $x_0 \in S$ (cf. [15]).

If the index set $Y(x)$ is constant and independent of x, MGSIP coincides with "multiobjective semi-infinite programming problem", denoted by MSIP. Necessary conditions for optimality of linear, convex, and non-convex MSIPs have been studied in several articles; see, e.g., [1, 2, 8, 9, 10] and the references therein.

If $p = 1$, MGSIP coincides with the "generalized semi-infinite prorgramming problem (GSIP)" which is an active field in optimization theory. In almost all existing literature on GSIP theory, in order to establish optimality conditions for problem (P), several kinds of lower-level constraint qualifications (CQ, briefly) are introduced. Extensive references to these CQs and optimality conditions, as well as their applications and historical notes, in the case that all appearing functions are continuously differentiable (while not necessarily convex), can be found in the book by Stein [18]. These CQs and optimality conditions have been extended to the GSIPs with locally Lipschitz and DC (difference of convex functions) data by Kanzi and Nobakhtian [11] and by Kanzi [6, 12], respectively.

More recently, Soroush [19] considered GSIPs with nondifferentiable convex functions, and introduced a Mangasarian-Fromovitz type CQ and some optimality conditions for the considered problems. The first aim of this paper is to extension of [19] to multiobjective case. In the case when all appearing functions of MGSIP are continuously differentiable, some necessary first-order conditions have been given in [3], but according to our latest information for the nonsmooth case nothing has been done so far. The second aim of this paper is to fill this gap as the first task. It should be noted that in [3] only one constraint qualification of the Mangasarian-Fromovitz type is considered and under this some necessary conditions are present at weakly efficient solutions and properly efficient solutions of smooth MGSIPs. In this paper, we introduce several (constraint and data) qualification conditions of the Abadie and Guignard types (that are weaker than Mangasarian-Fromovitz type) and express some optimality conditions in weakly efficient solutions and efficient solutions of nonsmooth MGSIPs.

The structure of subsequent sections of this paper is as follows: In Sec. 2, we establish the definitions and preliminary results which are required thereafter. Section 3, which is devoted to the main results, introduces some qualification conditions, expressing the relationships between these qualification conditions, and setting several necessary optimality conditions for nondifferentiable convex MGSIPs.

2 Preliminaries

In this section, we briefly address some notations, basic definitions, and standard preliminaries which are used in the sequel, from [4, 16].

The standard inner product of $x, y \in \mathbb{R}^n$ and the zero vector of $\mathbb{R}^n$ are denoted by $\langle x, y\rangle$ and 0_n, respectively. We will use symbols $\mathbb{R}_+$ (respectively $\mathbb{R}_{++}$) to represent the set of non-negative (respectively positive real) real numbers.

Let $\vartheta : \mathbb{R}^n \to \mathbb{R}$ be a convex function, i.e.,

$$\vartheta\big(\lambda x + (1-\lambda)y\big) \leq \lambda\vartheta(x) + (1-\lambda)\vartheta(y), \qquad \forall x, y \in \mathbb{R}^n,\ \lambda \in [0, 1].$$

The subdifferential of ϑ at $x_0 \in \mathbb{R}^n$ is defined by

$$\partial\vartheta(x_0) := \{\xi \in \mathbb{R}^n \mid \vartheta(x) - \vartheta(x_0) \geq \langle \xi, x - x_0\rangle, \quad \forall x \in \mathbb{R}^n\}.$$

As we know from [4] that if $\vartheta : \mathbb{R}^n \to \mathbb{R}$ is convex, its classic directional differential $\vartheta'(x_0; d)$, defined by

$$\vartheta'(x_0; d) := \lim_{\varepsilon \to 0} \frac{\vartheta(x_0 + \varepsilon d) - \vartheta(x_0)}{\varepsilon},$$

exists, and we have

$$\partial\vartheta(x_0) = \{\xi \in \mathbb{R}^n \mid \vartheta'(x_0; d) \geq \langle \xi, d\rangle, \quad \forall d \in \mathbb{R}^n\}. \tag{1}$$

Also, we know that $\partial\vartheta(x_0)$ is always a non-empty compact convex set in $\mathbb{R}^n$, and if ϑ is differentiable at x_0, then $\partial\vartheta(x_0) = \{\nabla\vartheta(x_0)\}$, in which $\nabla\vartheta(x_0)$ denotes the gradient of ϑ at x_0. The following equality will be used in sequel:

$$\vartheta'(x_0; d) = \max\{\langle \xi, d\rangle \mid \xi \in \partial\vartheta(x_0)\}.$$

For a convex function $\psi : \mathbb{R}^{n+m} \to \mathbb{R}$ and a point $(x_0, y_0) \in \mathbb{R}^{n+m}$, let $\partial_x\psi(x_0, y_0) \subseteq \mathbb{R}^n$ and $\partial_y\psi(x_0, y_0) \subseteq \mathbb{R}^m$ denote the partial subdifferentials of $\psi(\cdot, \cdot)$ at (x_0, y_0), which are defined as $\partial\psi(\cdot, y_0)(x_0)$ and $\partial\psi(x_0, \cdot)(y_0)$, respectively.

Given a nonempty set $D \subseteq \mathbb{R}^n$, the notations $\overline{D}$, $int(D)$, $conv(D)$, $cone(D)$, $\overline{conv}(D)$, $\overline{cone}(D)$, and $span(D)$ denote the closure of D, the interior of D, the convex hull of D, the convex cone generated by D (containing the origin), the closed convex hull of D, the closed convex cone generated by D and the linear space spanned by D, respectively. Also, the positive polar cone, the strictly positive polar set, the negative polar cone, and the strictly negative polar set of D are respectively defined as

$$\begin{aligned} D^{\geq} &:= \big\{x \in \mathbb{R}^n \mid \langle x, d\rangle \geq 0, \quad \forall d \in D\big\}, \\ D^{>} &:= \big\{x \in \mathbb{R}^n \mid \langle x, d\rangle > 0, \quad \forall d \in D\big\}, \\ D^{\leq} &:= -D^{\geq}, \qquad \text{and} \qquad D^{<} := -D^{>}. \end{aligned}$$

Recall that if $D = \emptyset$, then each of the above four sets will be equal to $\mathbb{R}^n$ by definition.

It should be mentioned [16, Theorem 6.9] that if $\Pi := \{C_\omega \mid \omega \in \Omega\}$ is a collection of convex sets in $\mathbb{R}^n$, then:

$$cone\Big(\bigcup_{\omega\in\Omega} C_\omega\Big) = \bigcup_{\{C_{\omega_1},\ldots,C_{\omega_n}\}\subseteq\Pi} \ \bigcup_{(\lambda_1,\ldots,\lambda_n)\in\mathbb{R}^n_+} \sum_{\nu=1}^{n} \lambda_\nu C_{\omega_\nu}, \tag{2}$$

$$conv\Big(\bigcup_{\omega\in\Omega} C_\omega\Big) = \bigcup_{\{C_{\omega_1},\ldots,C_{\omega_{n+1}}\}\subseteq\Pi} \ \bigcup_{(\lambda_1,\ldots,\lambda_{n+1})\in\triangle^{n+1}_+} \sum_{\nu=1}^{n+1} \lambda_\nu C_{\omega_\nu}, \tag{3}$$

$$ri\Big(conv\Big(\bigcup_{\omega\in\Omega} C_\omega\Big)\Big) \subseteq \bigcup_{\{C_{\omega_1},\ldots,C_{\omega_{n+1}}\}\subseteq\Pi} \ \bigcup_{(\lambda_1,\ldots,\lambda_{n+1})\in\triangle^{n+1}_{++}} \sum_{\nu=1}^{n+1} \lambda_\nu C_{\omega_\nu}, \tag{4}$$

in which $ri(C)$ denotes the relative interior of convex set $C \subseteq \mathbb{R}^n$, and $\triangle_+^{n+1}$ and $\triangle_{++}^{n+1}$ are defined as

$$\triangle_+^{n+1} := \big\{(\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}_+^{n+1} \mid \sum_{\nu=1}^{n+1} \lambda_\nu = 1\big\},$$

$$\triangle_{++}^{n+1} := \big\{(\lambda_1, \dots, \lambda_{n+1}) \in \mathbb{R}_{++}^{n+1} \mid \sum_{\nu=1}^{n+1} \lambda_\nu = 1\big\}.$$

Theorem 2.1. *[4, Theorem 1.4.3] Let $D \subseteq \mathbb{R}^n$ be a compact set. Then, $conv(D)$ is compact.*

We recall that for $D \subseteq \mathbb{R}^n$ and $x_0 \in \overline{D}$, the contingent cone (or Bouligand tangent cone) of D at x_0, denoted by $\Gamma(D, x_0)$, is defined as the set of all vectors $u \in \mathbb{R}^n$ that can find two sequences $\{\varepsilon^\ell\} \downarrow 0$ and $\{u^\ell\} \to u$ in such a way $x_0 + \varepsilon^\ell u^\ell \in D$ for all $\ell \in \mathbb{N}$. Notice that $\Gamma(D, x_0)$ is always a closed cone (generally non-convex) in $\mathbb{R}^n$.

3 Necessary Conditions

As the beginning of this section, we recall the following definition.

Definition 3.1. Let $\hat{x} \in S$ be a feasible point for (P).

(i) $\hat{x}$ is said to be a weakly efficient solution for (P) whenever there is no $x \in S$ satisfying $f_i(x) < f_i(\hat{x})$ for all $i \in I$.

(ii) $\hat{x}$ is said to be an efficient solution for (P) whenever there is no $x \in S$ satisfying $f_i(x) \leq f_i(\hat{x})$ for all $i \in I$, and $f_k(x) < f_k(\hat{x})$ for some $k \in I$.

For each $x_0 \in S$, we define the index set of active constraints and the lower level problem at x_0, respectively as

$$Y_0(x_0) := \{y \in Y(x_0) \mid g(x_0, y) = 0\},$$

$$LL(x_0) : \qquad \min\ g(x_0, y), \quad \text{s.t.} \quad y \in Y(x_0).$$

Also, the set (probably empty) of active inequalities of $LL(x_0)$ at each $y_0 \in Y(x_0)$ is denoted by $T_0(x_0, y_0)$,

$$T_0(x_0, y_0) := \{t \in T \mid h_t(x_0, y_0) = 0\}.$$

If $y_0 \in Y_0(x_0)$, then the Fritz-John (FJ) multipliers set of $LL(x_0)$ at $y_0 \in Y_0(x_0)$ is denoted by

$$F(x_0, y_0) := \left\{ (\alpha, \beta) \in \mathbb{R}_+ \times \mathbb{R}_+^{|T_0(x_0,y_0)|} \;\middle|\; \begin{array}{l} \alpha + \sum_{t \in T_0(x_0,y_0)} \beta_t = 1 \\ \\ 0_m \in \partial_y \mathcal{L}_{y_0}^{x_0}(x_0, y_0, \alpha, \beta) \end{array} \right\},$$

where $\mathcal{L}_{y_0}^{x_0}$ refers to Lagrangian function, defined as

$$\mathcal{L}_{y_0}^{x_0}(x, y, \alpha, \beta) := \alpha g(x, y) + \sum_{t \in T_0(x_0,y_0)} \beta_t h_t(x, y).$$

For each $k \in I$ and $x_0 \in S$, put

$$\mathfrak{F}(x_0) := \bigcup_{i=1}^{p} \partial f_i(x_0), \qquad \mathfrak{F}_k(x_0) := \bigcup_{i \in I \setminus \{k\}} \partial f_i(x_0),$$

$$\mathfrak{D}(x_0) := \bigcup_{y \in Y_0(x_0)} \Bigg(\bigcup_{(\alpha,\beta) \in F(x_0,y)} \partial_x \mathcal{L}_y^{x_0}(x_0, y, \alpha, \beta) \Bigg).$$

We should mention that $\mathfrak{D}(x_0)$ is nonsmooth counterpart of $V(x_0)$ that is defined in [5, 3].

Let $x_0 \in S$ be a feasible point for (P). For each $k \in I$ set,

$$Q_k(x_0) := S \cap \Big\{ x \in \mathbb{R}^n \mid f_i(x) \leq f_i(x_0), \quad \forall i \in I \setminus \{k\} \Big\},$$

with the convention that $Q_1(x_0) = S$ when $p = 1$.

The following lemma from [13] will be used below.

Lemma 3.2. *The following assertions are always true*

(i): *If $\hat{x}$ is a weakly efficient solution for (P), then*

$$\mathfrak{F}^{<}(\hat{x}) \cap \Gamma(S, \hat{x}) = \emptyset.$$

(ii): *If $\hat{x}$ is an efficient solution for (P), then*

$$\big(\partial f_i(\hat{x})\big)^{<} \cap \Gamma\big(Q_i(\hat{x}), \hat{x}\big) = \emptyset, \quad \forall i \in I.$$

Now, we define four data qualifications in the Abadie type for (P).

Definition 3.3. We say that (P) satisfies

- the `first Abadie data qualification`, denoted by FADQ, at $x_0 \in S$ if
$$\mathfrak{F}^{<}(x_0) \cap \mathfrak{D}^{\geq}(x_0) \subseteq \Gamma(S, x_0).$$
- the `second Abadie data qualification`, denoted by SADQ, at $x_0 \in S$ if
$$\mathfrak{F}^{\leq}(x_0) \cap \mathfrak{D}^{\geq}(x_0) \subseteq \bigcap_{i=1}^{p} \Gamma\big(Q_i(x_0), x_0\big).$$
- the `first Guignard data qualification`, denoted by FGDQ, at $x_0 \in S$ if
$$\mathfrak{F}^{<}(x_0) \cap \mathfrak{D}^{\geq}(x_0) \subseteq \overline{conv}\big(\Gamma(S, x_0)\big).$$
- the `second Guignard data qualification`, denoted by SGDQ, at $x_0 \in S$ if
$$\mathfrak{F}^{\leq}(x_0) \cap \mathfrak{D}^{\geq}(x_0) \subseteq \bigcap_{i=1}^{p} \overline{conv}\Big(\Gamma\big(Q_i(x_0), x_0\big)\Big).$$

Obviously, FADQ (resp. SADQ) is stronger than FGDQ (resp. SGDQ) at each feasible point. Also, the inclusions $\mathfrak{F}^{<}(x_0) \subseteq \mathfrak{F}^{\leq}(x_0)$ and $\bigcap_{i=1}^{p} \Gamma\big(Q_i(x_0), x_0\big) \subseteq \Gamma(S, x_0)$ and $\bigcap_{i=1}^{p} \overline{conv}\Big(\Gamma\big(Q_i(x_0), x_0\big)\Big) \subseteq \overline{conv}\big(\Gamma(S, x_0)\big)$ imply that FADQ (resp. FGDQ) is weaker than SADQ (resp. SGDQ)at x_0, i.e.,

$$\begin{array}{ccc} \text{SADQ} & \longrightarrow & \text{FADQ} \\ \downarrow & & \downarrow \\ \text{SGDQ} & \longrightarrow & \text{FGDQ} \end{array}.$$

Theorem 3.4. (KKT Necessary Condition Under FADQ): *Suppose that $\hat{x}$ is a weakly efficient solution for (P), and that FADQ holds at $\hat{x}$.*

(i): *Then, there exist some non-negative scalars $\lambda_i \in \mathbb{R}_+$ as $i \in I$, satisfying $\sum_{i=1}^{p} \lambda_i = 1$ and*

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \overline{cone}\big(\mathfrak{D}(\hat{x})\big).$$

(i): *If in addition, $cone\big(\mathfrak{D}(\hat{x})\big)$ is closed, there exist some $y^\nu \in Y_0(\hat{x})$, $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$, and $\mu_\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, as well as some non-negative numbers $\lambda_i \in \mathbb{R}_+$ as $i \in I$, satisfying $\sum_{i=1}^{p} \lambda_i = 1$ and*

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \mu_\nu \partial_x \mathcal{L}_{y^\nu}^{\hat{x}}(\hat{x}, y^\nu, \alpha^\nu, \beta^\nu).$$

Proof.
(i): If $conv\big(\mathfrak{F}(\hat{x})\big) \cap \overline{cone}\big(\mathfrak{D}(\hat{x})\big) = \emptyset$, then the compactness of $conv\big(\mathfrak{F}(\hat{x})\big)$ (by Theorem 2.1) and the closedness of $\overline{cone}\big(\mathfrak{D}(\hat{x})\big)$, allows us to use the strong separation theorem [16, Corollary 11.4.1]. Thus,

$$\Big(conv\big(\mathfrak{F}(\hat{x})\big)\Big)^{<} \cap \Big(\overline{cone}\big(\mathfrak{D}(\hat{x})\big)\Big)^{\geq} \neq \emptyset.$$

Since $\Big(conv\big(\mathfrak{F}(\hat{x})\big)\Big)^{<} = \mathfrak{F}^{<}(\hat{x})$ and $\Big(\overline{cone}\big(\mathfrak{D}(\hat{x})\big)\Big)^{\geq} = \mathfrak{D}^{\geq}(\hat{x})$, the above inclusion and the FADQ assumption imply that

$$\emptyset \neq \mathfrak{F}^{<}(\hat{x}) \cap \mathfrak{D}^{\geq}(\hat{x}) \subseteq \mathfrak{F}^{<}(\hat{x}) \cap \Gamma(S, \hat{x}), \tag{5}$$

which contradicts Lemma 3.2(i). Thus,

$$conv\big(\mathfrak{F}(\hat{x})\big) \cap \overline{cone}\big(\mathfrak{D}(\hat{x})\big) \neq \emptyset,$$

and so

$$0_n \in conv\big(\mathfrak{F}(\hat{x})\big) - \overline{cone}\big(\mathfrak{D}(\hat{x})\big).$$

This inclusion and (3) prove the result.

(ii): The result is a direct consequence of part (i) and the structure of convex cones (2). ☐

The following two theorems are required for presenting the KKT necessary condition under FGDQ.

Theorem 3.5. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a convex function and $x_0 \in \mathbb{R}^n$. Then, $\varphi'(x_0;\cdot)$ is a linear function (respect to direction) if and only if $\varphi(\cdot)$ is differentiable at x_0.*

Proof. If $\varphi(\cdot)$ is differentiable at x_0, we know $\varphi'(x_0;d) = \langle \nabla\varphi(x_0), d\rangle$ for all $d \in \mathbb{R}^n$, and so $\varphi'(x_0;\cdot)$ is linear. Conversely, If $\varphi'(x_0;\cdot)$ is linear function, then it is convex and differentiable at 0_n, and so $\partial\varphi'(x_0;\cdot)(0_n) = \{u\}$ for some $u \in \mathbb{R}^n$. Hence, by the definition of convex subdifferential we have

$$\begin{aligned}\{u\} &= \Big\{\xi \in \mathbb{R}^n \mid \varphi'(x_0;\cdot)(z) - \overbrace{\varphi'(x_0;\cdot)(0_n)}^{=0} \geq \langle \xi, z - 0_n\rangle, \quad \forall z \in \mathbb{R}^n\Big\} \\ &= \Big\{\xi \in \mathbb{R}^n \mid \varphi'(x_0;z) \geq \langle \xi, z\rangle, \quad \forall z \in \mathbb{R}^n\Big\} = \partial\varphi(x_0),\end{aligned}$$

where the last equality holds by (1). Consequently, $\partial\varphi(x_0)$ is singleton, and so, $\varphi(\cdot)$ is differentiable at x_0. ☐

Theorem 3.6. *Suppose that the f_i functions as $i \in I$ are differentiable. Then, the following assertions are always true:*

(i): *If $\hat{x}$ is a weakly efficient solution for (P), then*

$$\Big\{\nabla f_i(\hat{x}) \mid i \in I\Big\}^{<} \cap \overline{conv}\big(\Gamma(S,\hat{x})\big) = \emptyset.$$

(ii): *If $\hat{x}$ is an efficient solution for (P), then*

$$\Big\{\nabla f_i(\hat{x})\Big\}^{<} \cap \overline{conv}\Big(\Gamma\big(Q_i(\hat{x}),\hat{x}\big)\Big) = \emptyset, \quad \forall i \in I.$$

Proof.

(i): Suppose, on the contrary, that there exists a vector $d \in \mathbb{R}^n$ such that

$$d \in \Big\{\nabla f_i(\hat{x}) \mid i \in I\Big\}^{<} \cap \overline{conv}\big(\Gamma(S,\hat{x})\big). \tag{6}$$

Then, there exists a sequence $\{d_\ell\}_{\ell=1}^{\infty}$ in $conv\big(\Gamma(S,\hat{x})\big)$ converging to d. Also, for each $\ell \in \mathbb{N}$ we can find some vectors $d_\ell^\nu \in \Gamma(S,\hat{x})$ as

$\nu = 1, \dots, n_\ell$ and some non-negative numbers α_ℓ^ν as $\nu = 1, \dots, n_\ell$ such that

$$d_\ell = \sum_{\nu=1}^{n_\ell} \alpha_\ell^\nu d_\ell^\nu, \qquad \sum_{\nu=1}^{n_\ell} \alpha_\ell^\nu = 1.$$

Owing to Lemma 3.2(i), $d_\ell^\nu \in \Gamma(S, \hat{x})$ as $\nu = 1, \dots, n_\ell$, and the fact that $\mathfrak{F}(\hat{x}) = \{\nabla f_i(\hat{x}) \mid i \in I\}$, we conclude that $\langle \nabla f_i(\hat{x}), d_\ell^\nu \rangle \geq 0$ for all $\nu = 1, \dots, n_\ell$, for all $\ell \in \mathbb{N}$, and for all $i \in I$. Thus, for all $\ell \in \mathbb{N}$ and for all $i \in I$, we have

$$\langle \nabla f_i(\hat{x}), d_\ell \rangle = \langle \nabla f_i(\hat{x}), \sum_{\nu=1}^{n_\ell} \alpha_\ell^\nu d_\ell^\nu \rangle = \sum_{\nu=1}^{n_\ell} \alpha_\ell^\nu \langle \nabla f_i(\hat{x}), d_\ell^\nu \rangle \geq 0.$$

Consequently, for all $i \in I$, we obtain that

$$\langle \nabla f_i(\hat{x}), d \rangle = \langle \nabla f_i(\hat{x}), \lim_{\ell \to \infty} d_\ell \rangle = \lim_{\ell \to \infty} \langle \nabla f_i(\hat{x}), d_\ell \rangle \geq 0,$$

which contradicts (6). The proof is complete.

(ii): Based on Lemma 3.2(ii), the proof is same as above. □

Now, we can state the KKT necessary condition under FGDQ.

Theorem 3.7. (KKT Necessary Condition Under FGDQ): *Suppose that $\hat{x}$ is a weakly efficient solution for (P), that FADQ holds at $\hat{x}$, and that the f_i functions are differentiable as $i \in I$.*

(i) *Then, there exist some non-negative scalars $\lambda_i \in \mathbb{R}_+$ as $i \in I$, satisfying $\sum_{i=1}^p \lambda_i = 1$ and*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{x}) \in \overline{cone}\big(\mathfrak{D}(\hat{x})\big).$$

(ii) *If in addition, $cone\big(\mathfrak{D}(\hat{x})\big)$ is closed, there exist some $y^\nu \in Y_0(\hat{x})$, $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$, and $\mu_\nu \in \mathbb{R}_+$ as $\nu = 1, \dots, q$, as well as some non-negative numbers $\lambda_i \in \mathbb{R}_+$ as $i \in I$, satisfying $\sum_{i=1}^p \lambda_i = 1$ and*

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{x}) \in \sum_{\nu=1}^{q} \mu_\nu \partial_x \mathcal{L}_{y^\nu}^{\hat{x}}(\hat{x}, y^\nu, \alpha^\nu, \beta^\nu).$$

Proof.
(i): If $\{\nabla f_i(\hat{x}) \mid i \in I\} \cap \overline{cone}(\mathfrak{D}(\hat{x})) = \emptyset$, by repeating the proof of relation (5) in proof of Theorems 3.4 and 3.6(i), we get

$$\emptyset \neq \{\nabla f_i(\hat{x}) \mid i \in I\}^{<}(\hat{x}) \cap \mathfrak{D}^{\geq}(\hat{x}) \subseteq \mathfrak{F}^{<}(\hat{x}) \cap \overline{conv}\big(\Gamma(S, \hat{x})\big),$$

which contradicts Theorem 3.6(i). Thus, same as the proof of Theorem 3.4(i), we obtain that

$$0_n \in conv\Big(\{\nabla f_i(\hat{x}) \mid i \in I\}\Big) - \overline{cone}\big(\mathfrak{D}(\hat{x})\big).$$

The result is obtained from (3) and the above inclusion.

(ii): Based on Theorem 3.6(ii), the proof is stated similar to previous part. □

Now, we define the optimal value function of $LL(x)$ as follows:

$$\Psi(x) := \begin{cases} \inf\{g(x, y) \mid y \in Y(x)\} & \text{if} \quad Y(x) \neq \emptyset, \\ +\infty & \text{if} \quad Y(x) = \emptyset. \end{cases}$$

It should be noted that due to the importance of function $\Psi(\cdot)$, which is called "marginal function", many researches have worked on its properties and the upper estimate of its nonconvex subdifferentials see, e.g., [14, 17] and the references therein. Soroush [19] proved $\Psi(\cdot)$ is a convex function, and so, $\partial\Psi(x_0)$ is well-defined.

In many situations, we obtain positive KKT multiplier associated with vector-valued objective function $\big(f_1(x), \ldots, f_p(x)\big)$, some of the multipliers may be equal to zero. We say that strong KKT condition holds for (P), when the KKT multipliers are positive for all components of the objective function. The aim of next theorems is to derive the strong KKT types necessary optimality conditions for the (P).

As we noted in Section 1, if the index set $Y(x)$ is independent of x, the problem (P) increases to an MSIP. It is noteworthy that the following definition is a direct generalization of the PLV property that is provided for MSIPs (see [2, 7]).

Definition 3.8. We say that the `generalized PLV` (GPLV) property holds at $x_0 \in S$ when $\partial\Psi(x_0) \subseteq conv\big(\mathfrak{D}(x_0)\big)$.

Now, the following theorem states a strong KKT necessary condition at an efficient (not at a weakly efficient) solution of (P) under SADQ.

Theorem 3.9. (Strong KKT Necessary Condition Under SADQ): *Suppose that $\hat{x}$ is an efficient solution for (P), that the GPLV property and SADQ hold at $\hat{x}$. If the condition*

$$\mathfrak{F}^{\leq}(\hat{x}) \subseteq \{0_n\} \cup \bigcup_{i=1}^{p} \big(\partial^c f_i(\hat{x})\big)^{<}, \tag{7}$$

is met, then there exist some $y^\nu \in Y_0(\hat{x})$, $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$, and $\mu_\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, as well as some positive numbers $\lambda_i \in \mathbb{R}_{++}$ as $i \in I$, satisfying $\sum_{i=1}^{p} \lambda_i = 1$ and

$$0_n \in \sum_{i=1}^{p} \lambda_i \partial f_i(\hat{x}) - \sum_{\nu=1}^{q} \mu_\nu \partial_x \mathcal{L}^{\hat{x}}_{y^\nu}(\hat{x}, y^\nu, \alpha^\nu, \beta^\nu).$$

Proof. We claim that

$$ri\Big(conv\big(\mathfrak{F}(\hat{x})\big)\Big) \cap cone\big(\mathfrak{D}(\hat{x})\big) \neq \emptyset. \tag{8}$$

By contradiction, we suppose that (8) does not hold. Thus, by [16, Theorem 11.7] and the proper separation theorem [16, Theorem 11.3] and noting that $cone\big(\mathfrak{D}(\hat{x})\big)$ is a convex cone, it follows that there is a hyperplane $H_u := \{x \in \mathbb{R}^n \mid \langle x, u \rangle = 0\}$ for some $u \in \mathbb{R}^n \setminus \{0_n\}$ separating $conv\big(\mathfrak{F}(\hat{x})\big)$ and $cone\big(\mathfrak{D}(\hat{x})\big)$ properly. In other words, there exists a vector $u \in \mathbb{R}^n$ satisfying

$$0_n \neq u \in \Big(conv\big(\mathfrak{F}(\hat{x})\big)\Big)^{\geq} \cap \Big(cone\big(\mathfrak{D}(\hat{x})\big)\Big)^{\geq} = \mathfrak{F}^{\geq}(\hat{x}) \cap \mathfrak{D}^{\geq}(\hat{x}).$$

Thus, owning to SADQ and condition (7), we conclude that

$$\Big(\bigcup_{i=1}^{p} \big(\partial f_i(\hat{x})\big)^{<}\Big) \cap \Big(\bigcap_{i=1}^{p} \Gamma\big(Q_i(\hat{x}), \hat{x}\big)\Big) \neq \emptyset.$$

This relation together with

$$\Big(\bigcup_{i=1}^{p} \big(\partial f_i(\hat{x})\big)^{<}\Big) \cap \Big(\bigcap_{i=1}^{p} \Gamma\big(Q_i(\hat{x}), \hat{x}\big)\Big) = \bigcup_{i=1}^{p} \Big[\big(\partial f_i(\hat{x})\big)^{<} \cap \Big(\bigcap_{j=1}^{p} \Gamma\big(Q_j(\hat{x}), \hat{x}\big)\Big)\Big],$$

obtains a $k \in I$ such that

$$\big(\partial f_k(\hat{x})\big)^{<} \cap \Big(\bigcap_{j=1}^{p} \Gamma\big(Q_j(\hat{x}), \hat{x}\big)\Big) \neq \emptyset.$$

Thus, for some $k \in I$ we have

$$\big(\partial f_k(\hat{x})\big)^{<} \cap \Gamma\big(Q_k(\hat{x}), \hat{x}\big) \neq \emptyset, \tag{9}$$

which contracts Lemma (3.2)(ii). This contradiction proves the claimed (8), and hence

$$0_n \in ri\Big(conv\big(\mathfrak{F}(\hat{x})\big)\Big) - cone\big(\mathfrak{D}(\hat{x})\big).$$

The above inclusion together with (2) and (4) justifies the result. □

Theorem 3.10. (Strong KKT Necessary Condition Under SGDQ): *Suppose that $\hat{x}$ is an efficient solution for (P), that the GPLV property and SADQ hold at $\hat{x}$, and the f_i functions are differentiable at $\hat{x}$ as $i \in I$. If the condition*

$$span\big(\{\nabla f_i(\hat{x}) \mid i \in I\}\big) = \mathbb{R}^n, \tag{10}$$

is met, then there exist some $y^\nu \in Y_0(\hat{x})$, $(\alpha^\nu, \beta^\nu) \in F(\hat{x}, y^\nu)$, and $\mu_\nu \in \mathbb{R}_+$ as $\nu = 1, \ldots, q$, as well as some positive numbers $\lambda_i \in \mathbb{R}_{++}$ as $i \in I$, satisfying $\sum_{i=1}^{p} \lambda_i = 1$ and

$$\sum_{i=1}^{p} \lambda_i \nabla f_i(\hat{x}) \in \sum_{\nu=1}^{q} \mu_\nu \partial_x \mathcal{L}_{y^\nu}^{\hat{x}}(\hat{x}, y^\nu, \alpha^\nu, \beta^\nu).$$

Proof. At the first step of proof, we claim that the condition (10) implies the condition (7). It is easy to see that

$$\begin{aligned}
& span\big(\{\nabla f_i(\hat{x}) \mid i \in I\}\big) = \mathbb{R}^n \quad \Longleftrightarrow \\
& cone\big(\{\nabla f_i(\hat{x}), -\nabla f_i(\hat{x}) \mid i \in I\}\big) = \mathbb{R}^n \quad \Longrightarrow \\
& \Big(cone\big(\{\nabla f_i(\hat{x}), -\nabla f_i(\hat{x}) \mid i \in I\}\big)\Big)^{\leq} = \{0_n\}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\{0_n\} &= \big\{\nabla f_i(\hat{x}), -\nabla f_i(\hat{x}) \mid i \in I\big\}^{\leq} \\
&= \big\{\nabla f_i(\hat{x}) \mid i \in I\big\}^{\leq} \cap \big\{\nabla f_i(\hat{x}) \mid i \in I\big\}^{\geq}.
\end{aligned}$$

Consequently,

$$\mathfrak{F}^{\leq}(\hat{x}) \subseteq \mathbb{R}^n \setminus \big(\mathfrak{F}^{\geq}(\hat{x}) \setminus \{0_n\}\big) = \{0_n\} \cup \big(\mathbb{R}^n \setminus \mathfrak{F}^{\geq}(\hat{x})\big) =$$

$$\{0_n\} \cup \{\nabla f_1(\hat{x})\}^{<} \cup \ldots \cup \{\nabla f_p(\hat{x})\}^{<} = \{0_n\} \cup \bigcup_{i=1}^{p} \big(\partial f_i(\hat{x})\big)^{<},$$

and hence, (7) holds. Now, we claim that (8) holds. Otherwise, by same as proofs of relation (9) and Theorem 3.6 we conclude that

$$\big\{\nabla f_i(\hat{x})\big\}^{<} \cap \overline{conv}\Big(\Gamma\big(Q_i(\hat{x}), \hat{x}\big)\Big) = \emptyset, \quad \text{for all } i \in I,$$

in which contradicts Theorem 3.6(ii). Thus, (8) holds. The rest of the proof is same as the proof of Theorem 3.9. □

The following example shows both the truth of Theorem 3.4 and the impossibility of removing condition (7) from Theorem 3.9.

Example 3.11. Put in problem (P), $h_1(x, y) = |x_1| + |x_2| + |y| - 1$, $h_2(x, y) = -y$, $g(x, y) = x_1 + y$, $f_1(x) = -x_1$, and

$$f_2(x) = \sup\big\{x_1u_1 + x_2u_2 \mid u_1^2 + u_2^2 + 2u_2 \leq 0\big\}.$$

In fact, $f_2(\cdot)$ is the support function of convex set

$$\mathcal{U} = \big\{(u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 + (u_2 + 1)^2 \leq 1\big\}.$$

It is easy to see that $\hat{x} = 0_2$ is an efficient point of the problem. Also, a short calculation shows that $Y_0(0_2) = \{0\}$, $T_0(0_2, 0) = \{2\}$, $F(0_2, 0) = \{(\frac{1}{2}, \frac{1}{2})\}$, $\mathfrak{D}^{\geq}(0_2) = \{(\frac{1}{2}, 0)\}^{\geq} = \mathbb{R}_+ \times \mathbb{R}$, $\mathfrak{F}^{\leq}(0_2) = \{0\} \times \mathbb{R}_+$, $Q_1(0_2) = \{0\} \times \mathbb{R}$, and $Q_2(0_2) = \mathbb{R}_+ \times \mathbb{R}$. It should be observed that the condition (7) fails whereas SADQ (and hence, FADQ) holds at $\hat{x}$. The GPLV property is satisfied at $\hat{x}$, clearly. It is not hard to see that there are not $\lambda_1, \lambda_2 \in \mathbb{R}_{++}$ and $\mu_1, \alpha, \beta_2 \in \mathbb{R}_+$ satisfying

$$0_2 \in \lambda_1\{(-1, 0)\} + \lambda_2\mathcal{U} - \mu_1\big(\alpha\{(1, 0)\} + \beta_2\{0_2\}\big).$$

Also, we can see the above inclusion holds with $\lambda_1 = 0$, $\lambda_2 = 1$, $\mu_1 = 0$, and $\alpha = \beta_2 = 1$.

Acknowledgements

The authors express their gratitude to the referees of this paper, which their valuable suggestions led to the clarification and improvement of several points in the paper.

References

[1] G. Caristi and N. Kanzi, Karush-Kuhn-Tuker Type Conditions for Optimality of Non-Smooth Multiobjective Semi-Infinite Programming. *International Journal of Mathematical Analysis.* **9** (2015), 1929-1938.

[2] M.A. Goberna and N. Kanzi, Optimality conditions in convex multiobjective SIP. *Mathematical Programming*, **164** (2017), 167-191.

[3] F. Guerra-Vázquez, L.A. Hernández-Rebollar, and J.J. Rückmann, On vector generalized semi-infinite programming, Investigación Operacional, **39**(3), 341-352.

[4] J.B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms*, Springer, Berlin, Heidelberg (1991).

[5] H.T. Jongen, J.J. Rückmann, and O. Stein, Generalized semi-infinite optimization: a first order optimality condition and examples. *Mathematical Programming*, **83** (1-3), (1998) 145-158.

[6] N. Kanzi, Fritz-John type necessary conditions for optimality of convex generalized semi-Infinite optimization problems. *Journal of Mathematical Extension*, **70** (2013), 51-60.

[7] N. Kanzi, Constraint qualifications in semi-infinite systems and their applications in nonsmooth semi-infinite problems with mixed constraints. *SIAM Journal on Optimization*, **24** (2014), 559-572.

[8] N. Kanzi, On strong KKT optimality conditions for multiobjective semi-infinite programming problems with Lipschitzian data. *Optimization Letters*, **9** (2015), 1121-1129.

[9] N. Kanzi, J. Shaker Ardekani and G. Caristi, Optimality, scalarization and duality in linear vector semi-infinite programming. *Optimization*, **67** (2018), 523-536.

[10] N. Kanzi, Necessary and Sufficient Conditions for (Weakly) Efficient of Non-Differentiable Multi-Objective Semi-Infinite Programming Problems. *Iranian Journal of Science and Technology, Transaction A, Science*, **42** (2018), 1537-1544.

[11] N. Kanzi and S. Nobakhtian, Necessary optimality conditions for nonsmooth generalized semi-infinite programming problems. *European Journal of Operational Research*, **205** (2010), 253-261.

[12] N. Kanzi, Lagrange multiplier rules for non-differentiable DC generalized semi-infinite programming problems. *Journal of Global Optimization*, **56** (2013), 417-430.

[13] X.F. Li, Constraint qualifcations in nonsmooth multiobjective optimization. *Journal of Optimization Theory and Applications*, **106** (2000), 373-398.

[14] B.S . Mordukhovich, N.M. Nam and N.D. Yen, Subgradients of marginal functions in parametric mathematical programming. *Mathematical Programming*, **116** (2009), 369-396.

[15] S.B. Regina and A. Iusem, *Set-Valued Mapping and Enlargements of Monotone Operatores*, Spring, New York (2008).

[16] R.T. Rockafellar, *Convex Analysis.* Princeton University Press, Princeton, NJ (1970).

[17] R.T. Rockafellar, Differentiability properties of the minimum value in an optimization frobiem depending on parameters. *Proceedings of International Congoress of Mathematics*, **19** (1983), 16-24.

[18] O. Stein, *Bi-level strategies in semi-infinite programming* (Vol. 71). Springer Science & Business Media, (2003).

[19] H. Soroush, Necessary Conditions in Generalized Semi-Infinite Optimization with Nondifferentiable Convex Data. *Journal of Mathematical Extension*, **16** (2022), 1-14.

Golnaz Edalat
Department of Mathematics
PHD student of Mathematics
Payame Noor University, P.O. Box, 19395-3697
Tehran, Iran
E-mail: golnaz.edalat@student.pnu.ac.ir

Nader Kanzi
Department of Mathematics
Associate Professor of Mathematics
Payame Noor University, P.O. Box, 19395-3697
Tehran, Iran
E-mail: n.kanzi@pnu.ac.ir nad.kanzi@gmail.com

Marjan Adib
Department of Mathematics
Assistant Professor of Mathematics
Payame Noor University, P.O. Box, 19395-3697
Tehran, Iran
E-mail: m.adib@pnu.ac.ir