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Locally Closed Sets, Submaximal Spaces and Some Other Related Concepts

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Abstract. A subset A of a topological space X is called locally closed if $A = G \cap B$, where G is an open subset and B is a closed subset of X; X is called submaximal if every subset of X is locally closed. In this paper, we show that if βX, the Stone-Čech compactification of X, is a submaximal space, then X is a compact space and hence $\beta X = X$. We observe that every submaximal Hausdorff space is an *ncd*-space (a space in which does not have a nonempty compact and dense in itself subset). It turns out that every dense in itself Hausdorff space is pseudo-finite if and only if it is a (cei, f)-space (a space in which every compact subspace of X with empty interior is finite). A new characterization for submaximal spaces is given. Given a topological space $(X, \mathcal{T})$, the collection of all locally closed subsets of X forms a base for a topology on X which is denoted by $\mathcal{T}_l$. We study some relations between $(X, \mathcal{T})$ and $(X, \mathcal{T}_l)$. For example, we show that $(X, \mathcal{T})$ is a locally indiscrete space if and only if $\mathcal{T} = \mathcal{T}_l$.

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1 Introduction

Throughout this paper, we consider topological spaces on which no separation axioms are assumed unless explicitly stated. The topology of a space is denoted by $\mathcal{T}$ and $(X,\mathcal{T})$ will be replaced by X if there is no chance for confusion. For a subset A of $(X,\mathcal{T})$, the closure, the interior, the boundary and the set of accumulation points of A are denoted by $\mathrm{cl}_{\mathcal{T}}(A)$ or $\mathrm{cl}_X(A)$, $\mathrm{int}_{\mathcal{T}}(A)$ or $\mathrm{int}_X(A)$, $\mathrm{Fr}_{\mathcal{T}}(A)$ or $\mathrm{Fr}_X(A)$ and $\mathrm{l}_X(A)$, respectively. In places where there is no chance for confusion $\overline{A}$, A°, ∂A and A' stands for $\mathrm{cl}_{\mathcal{T}}(A)$, $\mathrm{int}_{\mathcal{T}}(A)$, $\mathrm{Fr}_X(A)$ and $\mathrm{l}_X(A)$, respectively. A subset A of a topological space X is called locally closed if $A = G\cap B$, where G is an open subset and B is a closed subset of X. This concept was introduced in Bourbaki [11] and then studied in [17]. Of course, according to the word local, the following definition seems more accurate. We say that A is locally closed at $x\in A$ if for every open subset U which containing x, there exists an open subset V, containing x, such that $V\cap A$ is closed in V. It is easily seen that the set A is locally closed at a point $x\in A$, if there exists an open subset U, containing x, such that $U\cap A$ is closed in U. We say A is locally closed if it is locally closed at every point $x\in A$. We will denote the collections of all locally closed sets of (X,τ) by $LC(X,\tau)$ or briefly $LC(X)$. If $A\subseteq X$ is open, closed or locally closed respect to the topology $\mathcal{T}$ on X, then we write sometimes to avoid confusion, $\mathcal{T}$-open, $\mathcal{T}$-closed or $\mathcal{T}$-locally closed, respectively. The space X is called a submaximal space if every subset of X is locally closed, see [4]. Since the intersection of two locally closed sets is locally closed, the family of $\mathcal{T}$-locally closed sets forms a base for a finer topology $\mathcal{T}_l$ on X. For a Tychonoff (completely regular Hausdorff) space X, βX is the Stone-Čech compactification and υX is the Hewitt realcompactification of X. It is well-known that X is compact, pseudocompact or realcompact if and only if $\beta X = X$, $\beta X = \upsilon X$ or $\upsilon X = X$, respectively. For a Tychonoff space the symbol $C(X)$ (resp. $C^*(X)$) denotes the ring of all continuous real valued (resp. all bounded continuous real valued) functions defined on X.

A space X is said to be

a) a door space if every set is either open or closed;

b) a $T_{\frac{1}{2}}$-space if every singleton is either open or closed;

c) a T_D-space if every singleton is locally closed, see [10] and [15];

d) an Alexandroff space, if every intersection of open sets is open. Equivalently, X is Alexandroff if and only if every $x \in X$ has a least open neighborhood;

e) a locally indiscrete if every open set is closed.

For details about Tychonoff P-spacs, see [Exercise 4J, 18]. A door space is submaximal and a submaximal space is $T_{\frac{1}{2}}$. Every T_1-space with finite number of nonisolated points is submaximal. A nonempty resolvable space never submaximal.

For more information about the locally closed sets, see [4,12, 17], about the submaximal spaces, see [20, 10, 1, 4, 12, 9, 3], about the door spaces, see [9, 14], about the $T_{\frac{1}{2}}$-spaces, see [5] and about Alexandroff space, see [1, 28]. For details about βX and υX, see [18, 29, 19, 24] and about other concepts of general topology, see [16, 31].

Section 2 of this paper is devoted to investigating some separation properties between $(X, \mathcal{T})$ and $(X, \mathcal{T}_l)$. For example, we show that $(X, \mathcal{T}_l)$ is discrete if and only if $(X, \mathcal{T})$ is a T_D-space. In Section 3, we show that if βX is submaximal, then X is compact and therefore, in this case, we conclude that $X = \beta X$. We observe that every submaximal Hausdorff space is an *ncd*-space (a space which does not have a nonempty compact and dense in itself subset). In this section, a new characterization for submaximal spaces is given. In Section 4, we study and investigate the behavior of locally indiscrete spaces. Furthermore, we introduce some lc-properties such as lc-regular and lc-completely regular and compare them with the concepts regular and completely regular. We prove that every clopen subset of an lc-compact space is lc-compact.

2 Locally Closed Sets and $\mathcal{T}_l$-Topology

Every submaximal space is a T_D-space, see [Corollary 3.5, 10]. Recall that the space X is submaximal if and only if every subset of X is locally closed, see [Theorem 4.2, 12] and also every subspace of a submaximal space is submaximal, see [Theorem 1.1, 12].

The proof of the following proposition is straightforward.

Proposition 2.1. *The following statements are equivalent, for a subset A of the space X.*
a) A is a locally closed subset in X.
b) For every $x \in A$ there is an open set $U \subseteq X$ such that $x \in U$ and $A \cap U$ is closed in U.
c) $A = H \cap \overline{A}$, where H is an open subset of X.
d) $A = E - F$, where E and F are closed subsets of X.
e) $\overline{A} - A$ is a closed subset in X.
f) $A \subseteq (A \cup (X - \overline{A}))^\circ$.
g) $A \cup (X - \overline{A})$ is an open subset of X.
h) A is an open subset in $\overline{A}$.

Remark 2.2. a) The complement of a locally closed (called co-locally closed) set is not necessarily locally closed. For example, $A = \{\frac{1}{n} : n = 1, \cdots\}$ is a locally closed set in $\mathbb{R}$ while by Proposition 2.1, $\mathbb{R} - A$ is not locally closed. For another example we consider the topology $\mathcal{T} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on $\mathbb{R}$. Now $A = (1, 2]$ is a locally closed set in $\mathbb{R}$ while $\mathbb{R} - A$ is not locally closed.
b) Let A and B be subsets of a space X. We say that A and B are separated if $A \cap \overline{B} = B \cap \overline{A} = \emptyset$. The union of two separated locally closed sets is locally closed, see [Proposition 4, 17].
c) If A is preopen, (that is, $A \subseteq \mathrm{int}(\mathrm{cl}(A))$) then it is open if and only if it is locally closed.
d) Let X be a topological space and Y a subspace of X. A set $A \subseteq Y$ is locally closed in Y if and only if $A = Y \cap B$, where B is locally closed in X.
e) In a locally compact Hausdorff space, a subset is locally closed if and only if it is a locally compact subset, see [Theorem 18.4, 31] or [Corollary 3.3.10, 16].

Given a topological space $(X, \mathcal{T})$, the collection of all locally closed subsets of X forms a base for a topology on X which is denotes by $\mathcal{T}_l$. It is clear that $\mathcal{T} \subseteq \mathcal{T}_l$ and in locally indiscrete spaces we have $\mathcal{T} = \mathcal{T}_l$, see Proposition 4.3. Also, $(X, \mathcal{T})$ is a indiscrete space if and only if $(X, \mathcal{T}_l)$ is a indiscrete space if and only if $(X, \mathcal{T}_l)$ is a connected space. In the sequel, we study some relations between $(X, \mathcal{T})$ and $(X, \mathcal{T}_l)$.

Proposition 2.3. *Let X be a topological space. The following statements hold.*

a) $(X,\mathcal{T})$ is a T_0-space if and only if $(X,\mathcal{T}_l)$ is a T_0-space.

b) $(X,\mathcal{T})$ is a T_D-space if and only if $(X,\mathcal{T}_l)$ is discrete.

Proof. $(a \Rightarrow)$ It is trivial.
$(a \Leftarrow)$ Let $x, y \in X$ and $x \neq y$. Hence there is $G \in \mathcal{T}_l$, say $x \in G$ and $y \notin G$. Suppose that $G = U \cap C$, where $U, X - C \in \mathcal{T}$. Now, if $y \notin U$, then we are done; otherwise, $y \in X - C$ and $x \notin X - C$.

$(b \Rightarrow)$ Suppose that $x \in X$. By hypothesis, $\{x\}$ is locally closed and so $\{x\} \in \mathcal{T}_l$. Therefore, $(X,\mathcal{T}_l)$ is discrete.

$(b \Leftarrow)$ Assume that $x \in X$. By hypothesis, $\{x\} \in (X,\mathcal{T}_l)$ and so $\{x\}$ is a union of locally closed sets in X. It follows that $\{x\}$ is locally closed. □

Clearly, if $(X,\mathcal{T})$ is a T_1-space then $(X,\mathcal{T}_l)$ is discrete. Hence, in Proposition 2.3, we cannot put T_1 instead of T_0.

Remark 2.4. Every T_D-space is T_0. The converse is false. For example, we consider the topology $\mathcal{T} = \{(a,\infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$ on $\mathbb{R}$.

Recall that if $\mathcal{T}$ be the Alexandroff topology on X, then every element $x \in X$ has a minimal open neighborhood $M_x = \bigcap\{G \in \mathcal{T} : x \in G\}$.

Lemma 2.5. *Every Alexandroff T_0-space is T_D.*

Proof. It follows from the fact that $\{x\} = M_x \cap \overline{\{x\}}$, for every $x \in X$. □

Remark 2.6. a) Let $(X,\mathcal{T})$ be an Alexandroff space. It is clear that every intersection of locally closed sets is locally closed.
b) If $(X,\mathcal{T})$ is an Alexandroff space, then $(X,\mathcal{T}_l)$ is Alexandroff. The converse is false. For example, $(\mathbb{R},\mathcal{T}_l)$ is Alexandroff but $(\mathbb{R},\mathcal{T}_u)$, where $\mathcal{T}_u$ denotes usual topology on $\mathbb{R}$, is not Alexandroff.

Example 2.7. A continuous image of a T_D-space need not be T_D-space. Let X be a space with cardinality ≥ 2 equipped with the indiscrete topology and $f : \mathbb{R} \to X$ be a surjective map.

If $\prod_{\alpha\in I} X_\alpha$ is a T_D-space, then X_α is a T_D-space for each $\alpha \in I$. The converse is true if I is finite. In the next example we see that an arbitrary product of T_D-spaces need not be T_D.

Example 2.8. Let $X_n = \{a, b\}$ and $\mathcal{T}_n = \{\emptyset, \{a\}, X\}$, for any $n \in \mathbb{N}$ and let $X = \prod_{n\in\mathbb{N}} X_n$. Suppose that $x = (x_n) \in X$, where $x_n = a$ for every $n \in \mathbb{N}$. It is clear that x belongs to every nonempty open set in X. We claim that $\{x\}$ is not a locally closed set. Otherwise, there is an open set G and a closed set F such that $\{x\} = G \cap F$. If $F \neq X$, then $x \in F \cap (X - F)$ which is not true. If $F = X$, then $\{x\} = G$. But $G = \prod_{n\in\mathbb{N}} G_n$, where $G_n = X_n$, for every $n \notin I$, which $I \subseteq \mathbb{N}$ is finite and this is impossible. It consequence that $\{x\}$ is not locally closed.

Every $T_{\frac{1}{2}}$-space is T_D. The converse is not true. See the next example.

Example 2.9. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{a, b\}, X\}$. Clearly, X is a T_D-space while the set $\{b\}$ neither open nor closed. For another example, let $X = \mathbb{N}$ and $\mathcal{T} = \{E_n : n = 1, \cdots\} \cup \{\emptyset\}$, where $E_n = \{n, n + 1, \cdots\}$, for any $n \in \mathbb{N}$. Since $\{n\} = E_n \cap (\mathbb{N} - E_{n+1})$, we infer that $\mathbb{N}$ is a T_D-space. It is clear that the set $\{2\}$ neither open nor closed, so $\mathbb{N}$ is not a $T_{\frac{1}{2}}$-space.

In the next example, we see that even if we add one cluster point to a locally closed set, the resulting set may not be locally closed. In other words, if A is locally closed and $x \in \overline{A} - A$, then $A \cup \{x\}$ is not necessarily locally closed.

Example 2.10. Let $0 \notin A \subseteq \mathbb{R}$ be such that $A' = \{0\}$ and $B = (0, +\infty) - A$. Clearly, B is an open subset and so is locally closed. But $0 \in B'$ and $B \cup \{0\}$ is not locally closed.

3 Submaximal and Pseudo-finite Spaces

A space X is called a pseudo-finite (resp. pseudo-discrete) space if every compact subspace of X is finite (resp. has finite interior). The term cf space for this topological space were introduced in [25] and the term pseudo-finite has also been used to describe such spaces, see [30]. In this work, we prefer to use the terms "(c,f)-space" and "(c,fi)-space" instead of the terms "pseudo-finite" and "pseudo-discrete", respectively. Every (c,f)-space is (c,fi)-sapce, but not conversely. For example, we consider the space $\mathbb{Q}$ of rational numbers. To see a nontrivial example see the following example. Recall that $C(X)$ endowed with the m-topology is denoted by $C_m(X)$.

Example 3.1. Let X be an infinite space. Then [Corollary 4.2, 7] conclude that $C_m(X)$ is a (c,fi)-space. On the other hand the subset $A=\{0,1,\frac{1}{2},\cdots,\}$ as a set of constant functions is an infinite compact subset of $C_m(X)$. This shows that $C_m(X)$ is not a (c,f)-space. Also, $C(X)$ under the uniform topology, is a (c,fi)-space which is not a (c,f)-space.

Recall that a space X is said to be dense in itself or crowded if it has no isolated points, see [16] and [13]. Also, a topological space is said to be an MI-space if it crowded and submaximal space, see [20, 23]. In the main theorem of [23], it is shown that each MI-space is a (c,f)-space. We also recall that a space X is said to be scattered if every subset of X has an isolated point.

Definition 3.2. *A space X is called*
a) a (cd,f)-space, if every compact dense in itself subspace of X is finite.
b) a (cei,f)-space, if every compact subspace of X with empty interior is finite.
c) an ncd-space, if it does not have a nonempty compact and dense in itself subset.
d) a (cl,f)-space, if every compact subset A of X with condition $A\subseteq X'$ is finite.
e) a (c,ei)-space, if every compact subset of X has an empty interior.
f) an (s,nd)-space, if every singleton subset of X is nowhere dense.

Remark 3.3. a) Every (c,f)-space is (cd,f)-space, (cei,f)-space and (cl,f)-space.
b) Every (cl,f)-space is a (cd,f)-space.
c) $\mathbb{Q}$ is an ncd-space. To see this let A be a dense in itself compact subset of $\mathbb{Q}$. Then A is a dense in itself subset of $\mathbb{R}$ and hence it is an uncountable set which is not true.

d) $\mathbb{Q}$ is neither a (cl, f)-space nor a (cei, f)-space.
e) Every T_0-space is an ncd-space if and only if it is a (cd, f)-space. For the reverse, we mention that, every finite T_0-space has an isolated point.
f) The space $\mathbb{Q}$ is a (c, ei)-space.
g) Every (c, ei)-space is a (c, fi)-space. The converse is true if the space is a dense in itself T_1-space.
h) Any space that is both (c, ei)-space and (cei, f)-space is (c, f)-space.
i) Every dense in itself T_1-space is an (s, nd)-space.
j) Every (s, nd)-space is dense in itself.
k) The space $\mathbb{R}$ equipped with the topology introduce in Remark 2.4 is an (s, nd)-space.

Proposition 3.4. *A compact space is scattered if and only if it is an ncd-space.*

Proof. It is clear that every scattered space is a ncd-space. For the converse suppose that X is an ncd-space and $A \subseteq A'$ where A is a subset of X. Then A' is closed and hence it is a compact set. Also A' is a dense in itself and so we conclude that $A' = \emptyset$. This implies that $A = \emptyset$, that is X is scattered. □

The least cardinal number of a nonempty open subset of a topological space X is said to be the dispersion character of X. For details see [20].

Proposition 3.5. *[Page 321 of 20] Every T_0-space is dense in itself if and only if its dispersion character is infinite.*

Proof. Let X be a T_0-space and Y be a nonempty open set of X with least cardinal number. Clearly, Y is a T_0-space and $Y \subseteq Y'$. On the contrary assume that $Y = \{x_1, \cdots, x_n\}$. Suppose that U_{x_i} be the smallest open set of Y which containing x_i. Since $Y \subseteq Y'$, there is $x_i \neq x_j \in U_{x_i}$. Since Y is a T_0-space, we infer that $U_{x_j} \subsetneq U_{x_i}$. By continuing this process, we come to the conclusion that there is a $x_k \in Y$ such that $(U_{x_k} - \{x_k\}) \cap Y = \emptyset$ which is contradicts to $x_k \in Y'$. The converse is trivial. □

Question: Is every (cei, f)-space an ncd-space?

We give an affirmative answer to this question, in case the space is Hausdorff.

Proposition 3.6. *Every Hausdorff (cei, f)-space is an ncd-space.*

Proof. Let X be a Hausdorff (cei, f)-space and let K be a dense in itself compact subset of X. On the contrary assume that $K \neq \emptyset$, then by Proposition 3.5 K is infinite. Hence, K contains a copy of $\mathbb{N}$, namely N. Therefore, $\overline{N}$ is an infinite compact subset of K. We claim that $\overline{N}^{\circ} = \emptyset$, to reach a contradiction. Suppose that $x_0 \in \overline{N}^{\circ}$. Hence there is an open set $G \subseteq X$ such that $x_0 \in G \subseteq \overline{N}$. Now there exists an open set H in X such that $H \subseteq G$ and $H \cap N = \{t_0\}$, where $t_0 \in X$. We claim that $H \cap K = \{t_0\}$. Otherwise, assume that $t_0 \neq t_1 \in H \cap K$. Hence, there are two open subsets U and V in X such that $t_0 \in U$, $t_1 \in V$ and $U \cap V = \emptyset$. Put $W = H \cap V$. Then $t_1 \in W \subseteq \overline{N}$ and so $W \cap N \neq \emptyset$. On the other hand we have $W \cap N \subseteq H \cap N = \{t_0\}$. This implies that $W \cap N = \{t_0\}$. Hence, $t_0 \in W \subseteq V$ which is not true. Therefore $H \cap K = \{t_0\}$, that is, t_0 is an isolated point of K which is a contradiction. □

Every (c, f)-space is both (cei, f)-space and *ncd*-space. The converse is not true, in general. For example we consider the space $X = \{0, 1, \frac{1}{2}, \frac{1}{3}, \cdots\}$. In the following proposition we show that the converse is valid if the space is a dense in itself T_1-space.

Proposition 3.7. *Let X be a dense in itself T_1-space, (cei, f)-space and ncd-space. Then X is a (c, f)-space.*

Proof. Let K be a compact subset of X. Then $K - K^{\circ}$ is compact with empty interior. Hence by hypothesis is finite. Let I be the set of isolated points of K. Since X is dense in itself, then $I \subseteq K - K^{\circ}$ and hence I is finite. Also $K - I$ is compact. We claim that $K - I$ is dense in itself. On the contrary, suppose that there exists $x_0 \in K - I$ such that $x_0 \notin (K - I)'$. Therefore, there is an open set $H \subseteq X$ such that $x_0 \in H$ and $H \cap (K - I) = \{x_0\}$ and thus $(H \cap K) \cup I = I \cup \{x_0\}$. This implies that $H \cap K$ is finite and since X is a T_1-space, we infer that $H \cap K \subseteq I$. This conclude that $x_0 \in I$ which is a contradiction. Now by hypothesis $K - I = \emptyset$ and so K is finite. This complete the proof. □

By Propositions 3.6 and 3.7, the following is immediate.

Corollary 3.8. *Let X be a dense in itself Hausdorff space. Then X is a (c, f)-space if and only if it is a (cei, f)-space.*

Proposition 3.9. *Each MI-space which is also a T_0-space contains a copy of $\mathbb{N}$ which is closed.*

Proof. Let X be an MI-space which is also a T_0-space. By Proposition 3.5 we conclude that X is infinite. Then X contains an infinite subset A such that $A^\circ = \emptyset$, see [Lemma 1, 23]. Suppose that $E \subseteq A$ is countably infinite. Therefore $E^\circ = \emptyset$. Now Lemma 2 in [23] implies that E is a closed and discrete. $\square$

The following lemma is the same as [Lemma 4, 23] with slight correction.

Lemma 3.10. *Each nonempty MI-space is noncompact.*

Proposition 3.11. *Each MI-space is an ncd-space.*

Proof. It is trivial by Lemma 3.10. $\square$

Proposition 3.12. *Each MI-space is an (s, nd)-space.*

Proof. Suppose that X is an MI-space and $x \in X$. Then $Fr(\{x\}) = \overline{\{x\}}$ is discrete. Therefore, there exists an open subset H in X such that $x \in H$ and $\{x\} = H \cap \overline{\{x\}}$. On the contrary that $t \in \overline{\{x\}}^\circ$ exists, then there exists an open subset G in X such that $t \in G$ and $G \subseteq \overline{\{x\}}$. Clearly, $x \in G$ and $G \cap H = \{x\}$ which is a contradiction. $\square$

Corollary 3.13. *For a submaximal topological space X, the following statements are equivalent:*
a) X is an MI-space.
b) X is an (s, nd)-space.

In the next result we show that every submaximal Hausdorff space is a *ncd*-space.

Theorem 3.14. *Every submaximal Hausdorff space is an ncd-space.*

Proof. Let X be a submaximal Hausdorff space and let K be a dense in itself compact subspace of X. By [Proposition 3.2 of 1], K' is finite. Now if $K \neq \emptyset$, then by Proposition 3.5 K is infinite and so K' is infinite which is contradiction. $\square$

The converse of the above theorem is not true, in general. For example, $\mathbb{Q}$ is an *ncd*-space but it is not a submaximal space. Note that if $\mathbb{Q}$ is an MI-space, then must be (c, f)-space which is a contradiction.

An arbitrary product of Tychonoff, compact and submaximal spaces need not be submaximal. For example, assume that $X_n^* = X_n \cup \{\sigma_n\}$ be the one-point compactification of a discrete space X_n, for every $n \in \mathbb{N}$. Suppose that $X = \prod_{n\in\mathbb{N}} X_n^*$ and $A = \prod_{n\in\mathbb{N}} X_n$, then A is dense in X which is not open.

In the next proposition we obtain a new characterization for submaximal spaces.

Proposition 3.15. *A topological space X is submaximal if and only if for any $A, B \subseteq X$ with $A \cap B = \emptyset$ it follows that $\overline{A} \cap \overline{B}$ is discrete.*

Proof. $\Rightarrow$) Suppose that $x \in \overline{A} \cap \overline{B}$. Hence, $A \cup \{x\}$ is locally closed in X. Therefore, there exists an open set U in X such that $A \cup \{x\} = U \cap \overline{A}$. Similarly, there is an open set V in X such that $B \cup \{x\} = V \cap \overline{B}$. Now $\{x\} = (A \cup \{x\}) \cap (B \cup \{x\}) = (U \cap \overline{A}) \cap (V \cap \overline{B}) = (U \cap V) \cap (\overline{A} \cap \overline{B})$. This consequence that x is an isolated point of $\overline{A} \cap \overline{B}$.

$\Leftarrow$) By [Theorem 3.3, 12], it is clear. □

If X is a submaximal space and $A \subseteq X$, then $\mathrm{Fr}(A)$ is a discrete subset of X. The converse of this fact is also true, see [Theorem 3.3, 12]. Also if X is a submaximal space and A is a discrete subset of X, then $\overline{A'}$ is discrete and if in addition X is a T_1-space, then A' is discrete. In [Theorem 3.2, 1], proved that a compact Hausdorff space X is a submaximal space if and only if X has a finite number of accumulation points. In case of βX we have the following result.

Lemma 3.16. *If βX is the one-point compactification of a discrete space, then X is compact, that is $\beta X = X$.*

Proof. Suppose that $\beta X = D^*$ where D^* is the one-point compactification of discrete space D, i.e., $D^* = D \cup \{\sigma\}$. Since $\overline{X} = \beta X$, we infer that $D \subseteq X$. If $\sigma \notin X$, then $X = D$ and hence $\beta X = \beta D$. But $|\beta X| = |D|$ and $|\beta D| = 2^{2^{|D|}}$ (see [Theorem 9.2, 18]) which is a contradiction. This implies that $\sigma \in X$ and we are done. □

Theorem 3.17. *If βX is a submaximal space, then X is compact. In this case $\beta X = X$. Furthermore X is not a dense in itself space.*

Proof. Since βX is countably compact, by [Theorem 4.20, 3], βX is a finite disjoint union of one-point compactification of some discrete spaces. Suppose that $\beta X = \bigcup_{i=1}^{n} X_i^*$, where X_i^* is the one-point compactification of discrete space X_i, for every $i = 1, \cdots, n$. Assume that $X_i^* = X_i \cup \{\sigma_i\}$ and let $X \cap X_i^* = Y_i$, for every $i = 1, \cdots, n$. Then $X = \bigcup_{i=1}^{n} Y_i$ and hence $\beta X = \bigcup_{i=1}^{n} \beta Y_i$. Therefore, $\beta Y_i = X_i^*$, for any $i = 1, \cdots, n$. Now by the above lemma we conclude that $Y_i = X_i^*$, for any $i = 1, \cdots, n$ and so $X = \beta X$. Finally, since X is a submaximal compact space, then it is not dense in itself. □

We say that a subspace S of X is C-embedded (resp. C^*-embedded) in X if every function in $C(S)$ (resp. $C^*(S)$) can be extended to a function in $C(X)$ (resp.$C^*(X)$). Every C-embedded is C^*-embedded and every compact set in a Tychonoff space is C-embedded, see [3.11(c), 18]. It is well-known that S is C^*-embedded in X if and only if $\mathrm{cl}_{\beta X} S = \beta S$, see [6.9(a), 18].

Corollary 3.18. *Let X be a compact and Hausdorff space. If X is submaximal, then every C^*-embedded subset of X is compact.*

Proof. Suppose that S is C^*-embedded in X. Then $\mathrm{cl}_{\beta X} S = \beta S \subseteq \beta X = X$. Hence, βS is submaximal and so by Theorem 3.17 we conclude that $\beta S = S$. This means that S is compact. □

Lemma 3.19. *Let T be a Hausdorff submaximal space. If X is a countably compact subspace and a dense subset of T, then $X = T$.*

Proof. Since X is a submaximal space, then by [Theorem 4.21, 3], X is compact and since T is Hausdorff, we infer that X is closed in T. Therefore, by density of X we have $X = T$. □

Remark 3.20. We denote the set of all isolated points of space X with $I(X)$. If X is open in T, then $I(X) \subseteq I(T)$ and if X is dense in T, then $I(T) \subseteq I(X)$. Hence, if X is an open dense set in T, then $I(X) = I(T)$.

Proposition 3.21. *Suppose that T is a first countable, Hausdorff and submaximal space and X is a dense set in T with $I(X) = \emptyset$. Then $X = T$.*

Proof. Let $p \in T - X$. Hence, there is an infinite sequence (x_n) contained in X which converges to p. Put $A = \{x_n : n = 1, \cdots\}$. Indeed it is clear that $A^\circ = \emptyset$ and thus $T - A$ is dense and also it is clear that $\overline{A} = A \cup \{p\}$ and thus $T - A$ is not open which is a contradiction. □

Corollary 3.22. *Let X be a compact submaximal space. Then $|I(X)|$ is finite if and only if X is finite.*

A (c, f)-space may not be submaximal even it is a Tychonoff and countably compact or dense in itself and T_4-space. See the next three examples.

Example 3.23. a) Consider the space X in [Example 2, 27]. This example shows that the Tychonoff space X is an infinite countably compact subset of $\beta\mathbb{N}$ which is also a (c, f)-space. We claim that X is not submaximal. Otherwise, by [Theorem 4.20, 3], X must be a compact space. In this case, since X is a (c, f)-space, it must be finite, which is not true.
b) For any $n \in \mathbb{N}$, suppose that $S_n = \{2n - 1, 2n\}$. The collection $\mathcal{S} = \{S_n : n \in \mathbb{N}\}$ is a subbase for a topology on $\mathbb{N}$ which is called odd-even topology. One can easily check that $\mathbb{N}$, with this topology, is a (c, f)-space dense in itself space, but it is not a submaximal. Also, $\mathbb{N}$ is second countable, separable and Lindelöf.
c) This example is based on the structure introduced in [2]. Let $T_0 = \Sigma$, where Σ be the space introduced in [4M.2, 18]. At each $x \in I(T_0)$, we attach a copy of Σ, namely Σ_x, to T_0 in x such that x is the only nonisolated point of Σ_x and $\Sigma_x \cap T_0 = \{x\}$. We do this in such a way that $\Sigma_x \cap \Sigma_y = \emptyset$, for any two distinct points x, y in $I(T_0)$. The resulting quotient space is denoted by T_1. We repeat this process for T_1 and resulting quotient space is denoted by T_2. In this way, by induction method, we find the space T_n, for any $n = 0, 1, \cdots$, such that $T_n \subseteq T_{n+1}$. We put $T = \bigcup_{n=0}^{\infty} T_n$ and $\tau = \{\bigcup_{n=0}^{\infty} G_n : G_n \in \tau_n\}$, where τ_n is the topology on T_n. In [2], it is shown that τ is a topology on T. Clearly, $T_0 = \Sigma$ is a (c, f)-space. By induction, we can show that the space

T_n is a (c,f)-space, for any $n \in \mathbb{N}$. We claim that T is a (c,f)-space. To see this let K be an infinite compact subset of T. It is enough to show that there is an $n_0 \in \mathbb{N}$ such that $K \subseteq T_{n_0}$. Otherwise, there exists an $a_n \in K \cap (T_n - T_m)$, for any $n, m \in \mathbb{N}$ and $m \geq n$. Now put $A = \{a_n : n \in \mathbb{N}\}$. One can easily see that $A' = \emptyset$ and this is a contradicts to compactness of K. Finally, it is not hard to check that T is a dense in itself and T_4-space.

The space $\mathbb{R}$ is a realcompact space which is not submaximal. See the following examples for two examples of a spaces that are submaximal but not realcompact.

Example 3.24. a) We consider the space $\mathbf{\Psi}$ in [5I, 18]. By 5I. 5, the space $\mathbf{\Psi}$ is pseudocompact which it is not a realcompact space. It is not hard to see that $\mathbf{\Psi}$ is submaximal. Furthermore, since $\mathbf{\Psi}$ is not compact, by Theorem 3.17, $\beta\mathbf{\Psi}$ is not submaximal. Now $\upsilon\mathbf{\Psi} = \beta\mathbf{\Psi}$ shows that $\upsilon\mathbf{\Psi}$ is not a submaximal space.

b) We consider the space $\mathbf{W} = W(\omega_1) = \{\sigma : \sigma < \omega_1\}$ of all countable ordinals and $\mathbf{W}^* = W(\omega_1+1) = \{\sigma : \sigma \leq \omega_1\}$, where ω_1 denotes the first uncountable ordinal. It is well known that $\mathbf{W}$ is a pseudocompact locally compact space which neither compact nor realcompact. Clearly, $\upsilon\mathbf{W} = \beta\mathbf{W} = \mathbf{W}^*$. Since $\mathbf{W}$ is not compact, then by Theorem 3.17, $\upsilon\mathbf{W}$ is not submaximal.

4 Locally Indiscrete Spaces and lc-Properties

Recall that a space X is called locally indiscrete if every open set is closed or equivalently if every closed set is open. Every discrete space and every partition topology (each partition of any set X into disjoint subsets, together with $\emptyset$, is a basis for a topology on X, known as partition topology) is locally indiscrete. For another nontrivial example, let R be a principal ideal ring. Then the space $\mathrm{Min}(R)$ with Zariski topology is a locally indiscrete space. Also, every locally indiscrete space is an Alexandroff space. The converse is not true in general. For example, let $X = \mathbb{N}$ and $\mathcal{T} = \{E_n : n = 1, \cdots\} \cup \{\emptyset\}$, where $E_n = \{n, n+1, \cdots\}$, for any $n \in \mathbb{N}$. It is easy to check that every Alexandroff T_1-space is indiscrete.

Proposition 4.1. *For a topological space X the following conditions are equivalent.*
a) X is locally indiscrete.
b) Every subset of X is preopen.
c) Every singleton in X is preopen.
d) Every closed subset of X is preopen.
e) Every locally closed subset of X is open.
f) Every locally closed subset of X is closed.
g) The closure of every locally closed set is open.
h) X is the only dense subset of itself.

Proof. All implications are straightforward. We only show $(h \Rightarrow a)$. Suppose that A is an open set in X and consider $B = A \cup (X - A)^\circ$. Then $\overline{B} = \overline{A} \cup \overline{(X - A)^\circ} = \overline{A} \cup \overline{(X - \overline{A})} = \overline{A} \cup (X - \overline{A}^\circ) = X$. Hence, by hypothesis $B = X$ and so $\overline{A} \cap (X - A) = \emptyset$. This consequence that A is closed and we are done. □

Proposition 4.2. *For a topological T_1-space X the following conditions are equivalent.*
a) X is discrete.
b) X is locally indiscrete.
c) X is Alexandroff.
d) Every open set is regular open.

Proof. It is straightforward. □

If X is a $T_{\frac{1}{2}}$-space, then it is locally indiscrete if and only if it is discrete. A locally indiscrete space need not be submaximal. For instance, let $X = \{a, b, c\}$ and $\mathcal{T} = \{\emptyset, \{a\}, \{b, c\}, X\}$. A submaximal space need not be a locally indiscrete space. For example, we consider the space Σ of 4M in [18].

Proposition 4.3. *The space $(X, \mathcal{T})$ is a locally indiscrete space if and only if $\mathcal{T} = \mathcal{T}_l$.*

Proof. $(\Rightarrow)$ Let $G \in \mathcal{T}_l$. Then $G = \bigcup_{\alpha \in \Lambda} A_\alpha$, where A_α is $\mathcal{T}$-locally closed, for any $\alpha \in \Lambda$. By hypothesis, A_α is $\mathcal{T}$-open, hence $G \in \mathcal{T}$ and

we are done.
($\Leftarrow$) Let A is a $\mathcal{T}$-locally closed subset of X. Hence $A \in \mathcal{T}_l$ and therefore $A \in \mathcal{T}$. This shows that X is a locally indiscrete space. □

Proposition 4.4. *Let X be a locally indiscrete space. The following conditions are equivalent.*
a) X is a T_1-space.
b) X is a $T_{\frac{1}{2}}$-space.
c) X is a T_D-space.
d) X is a T_0-space.
e) X is a submaximal space.
f) X is a discrete space.

Proof. All implications are obvious. We only show ($d \Rightarrow a$). Let $x \in X$ and on the contrary suppose that $y \in \overline{\{x\}}$ and $y \neq x$. Since X is T_0, it follows that there is an open set H such that $x \in H$ and $y \notin H$. Thus, $y \in X - H$ and since $X - H$ is open we infer that $(X - H) \cap \{x\} \neq \emptyset$ and this is a contradiction. □

Here, by using the locally closed sets, we introduce some separation axioms. We begin with the following definition.

Definition 4.5. *A space X is called*
a) lc-regular if for each locally closed set A and for each point $x \notin A$, there are disjoint open sets U and V with $x \in U$ and $A \subseteq V$.
b) lc-completely regular if for each locally closed set A and for each point $x \notin A$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = \{1\}$.
c) lc-normal if for every two disjoint locally closed sets A and B, there are disjoint open sets U and V with $A \subseteq U$ and $B \subseteq V$.

Every lc-regular (resp. lc-completely regular, lc-normal) space is a regular (resp. completely regular, normal) space. The converse is hold in locally indiscrete spaces, but is not true, in general. See the following example.

Example 4.6. The space $\mathbb{R}$ with usual topology is not a lc-regular space. To see this let $A = [0, 1)$. Then A is locally closed and $1 \notin A$. But A cannot be separated from 1 by disjoint open sets.

Remark 4.7. a) Every lc-completely regular T_D-space is lc-normal.
b) Every lc-regular T_D-space is Hausdorff.

Definition 4.8. *A space X is called lc-compact if each locally closed cover of X has a finite subcover.*

Every lc-compact space is compact. An infinite compact T_1-space is never lc-compact. In locally indiscrete spaces, the two concepts of lc-compactness and compactness coincide. One can easily see that $(X, \mathcal{T})$ is lc-compact if and only if $(X, \mathcal{T}_l)$ is compact. See the next example to see an lc-compact.

Example 4.9. Let τ_X be the co-finite topology on X and $\sigma \notin X$. Suppose that $Y = X \cup \{\sigma\}$. Then $\tau_Y = \{A \cup \{\sigma\} : A \in \tau_X\} \cup \{\emptyset\}$ is a topology on Y. It is clear that $LC(Y) = \{U \in \tau_Y\} \cup \{F \subseteq X : F\ is finite\}$. We claim that Y is lc-compact. To see this let $Y = \bigcup_{\alpha \in I}(G_\alpha \cap F_\alpha)$, where $G_\alpha, X - F_\alpha \in \tau_Y$, for every α. Therefore, there exists an $\alpha_0 \in I$ such that $\sigma \in G_{\alpha_0} \cap F_{\alpha_0}$. Hence, $Y - G_{\alpha_0} = \{x_1, x_2, \cdots, x_n\}$ and since the only closed subset Y which contain σ is itself Y we infer that $F_{\alpha_0} = Y$. On the other hand there is an $\alpha_i \in I$ such that $x_i \in G_{\alpha_i} \cap F_{\alpha_i}$, for $i = 1, 2, \cdots, n$. This mean that $Y = \bigcup_{i=0}^{n}(G_{\alpha_i} \cap F_{\alpha_i})$, that is Y is lc-compact.

Proposition 4.10. *Every clopen subset of an lc-compact space is lc-compact.*

Proof. Suppose that $Y \subseteq \bigcup_{\alpha \in \Lambda} A_\alpha \in \Lambda$, where A_α is a locally closed set in Y, for each $\alpha \in \Lambda$. Hence, $A_\alpha = B_\alpha \cap Y$, which B_α is a locally closed set in X, for each $\alpha \in \Lambda$. Therefore there is an open set G_α and a closed set F_α in X which $B_\alpha = G_\alpha \cap F_\alpha$. Now it is clear that $X = \bigcup_{\alpha \in \Lambda}((G_\alpha \cup (X - Y)) \cap (F_\alpha \cup (X - Y)))$. Since X is an lc-compact space we infer that $X = \bigcup_{k=1}^{n}((G_{\alpha_k} \cup (X - Y)) \cap (F_{\alpha_k} \cup (X - Y)))$, for a natural number n and $\alpha_i \in \Lambda$ for $i = 1, \cdots, n$. It implies that $Y \subseteq \bigcup_{k=1}^{n} A_{\alpha_k}$, that is, Y is an lc-compact space. □

In the above proposition, the condition clopen cannot be replaced by locally closed subset. For example, in any T_D-space, lc-compact subsets are necessarily finite, while locally closed subsets are not necessarily finite.

Definition 4.11. *A function $f : X \to Y$ is called lc-continuous if the converse image of any open set in Y is locally closed in X.*

Every continuous function is lc-continuous. The converse is not true. For example, let $X = \{a, b\}$ and $\mathcal{T}_1 = \{\emptyset, \{a\}, X\}$ and $\mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, X\}$ are two topology on X. We define $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ by $f(x) = x$. Then f is lc-continuous but it is not a continuous function. For more details about lc-continuous functions, see [17].

Remark 4.12. Let $f : X \to Y$ be an onto and X be an lc-compact space. Then
a) if f is continuous then Y is lc-compact;
b) if f is lc-continuous then Y is compact.

Remark 4.13. Suppose that $f : \mathbb{R} \to Y = \{0, 1\}$ with $f(\mathbb{Q}) = \{0\}$ and $f(\mathbb{R} - \mathbb{Q}) = \{1\}$, where Y is equipped with discrete topology. If we consider $\mathbb{R}$ equipped with usual topology $\mathcal{T}$, then f is not lc-continuous. But if we consider $\mathbb{R}$ equipped with $\mathcal{T}_l$, then f is continuous.

Remark 4.14. Let $f : X \to Y$ be continuous. The following statement are hold.
a) If $B \subseteq Y$ is locally closed then $f^{-1}(B) \subseteq X$ is locally closed.
b) If $A \subseteq X$ is locally closed then $f(A)$ is not necessarily locally closed. For example, let $X = \{a, b\}$, $\mathcal{T}_1$ be discrete topology and $\mathcal{T}_2$ be indiscrete topology on X. We define $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ by $f(x) = x$. Then $A = \{a\}$ is a locally closed set in $(X, \mathcal{T}_1)$ but $f(A) = \{a\}$ is not a locally closed set in $(X, \mathcal{T}_2)$.

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