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A New Analytic-Approximate Solution of Fokker-Planck Equation with Space-and Time-Fractional Derivatives

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Abstract. In this paper, the analytical solution of the space-and time-fractional Fokker-Planck equation was derived by means of the homotopy analysis method (HAM). The fractional derivatives are described in the Caputo sense. Some examples are given and comparisons are made, the comparisons show that the homotopy analysis method is very effective and convenient. An optimal value of the convergence control parameter is given through the square residual error. By minimizing the the square residual error, the optimal convergence-control parameters can be obtained. Several numerical examples are considered aiming to demonstrate the validity and applicability of the proposed techniques and to compare with the existing results.

AMS Subject Classification: 65Z05
Keywords and Phrases: Homotopy analysis method, caputo fractional derivative, fractional Fokker-Planck equation, optimal convergence-control parameter

1. Introduction

Many phenomena in engineering, physics, chemistry and other science can be described very successfully by models using the theory of derivatives and integrals of fractional order. Interest in the concept of differentiation and integration to noninteger order has existed since the development of the classical calculus [1, 2, 3]. By implication, mathematical modeling of many physical systems are governed by linear and

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nonlinear fractional differential equations in various applications in fluid mechanics, viscoelasticity, chemistry, physics, biology and engineering.
Since many fractional differential equations are nonlinear and do not have exact analytical solutions, various numerical and analytic methods have been used to solve these equations. The Adomain decomposition method (ADM) [4], the homotopy perturbation method (HPM)[5], the variational iteration method (VIM) [6] and other methods have been used to provide analytical approximation to linear and nonlinear problems. However, the convergence region of the corresponding results is rather small (see [22]).
In 1992, Liao employed the basic ideas of the homotopy in topology to propose a general analytic method for nonlinear problems, namely Homotopy Analysis Method (HAM), Liao [7, 11]. This method has been successfully applied to solve many types of nonlinear problems in science and engineering, such as the viscous flows of non-Newtonian fluids [12], the KdV-type equations [13], fractional foam drainage equation [19], fractional nonlinear coupled equations [20], and so on. The HAM contains a certain auxiliary parameter h which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.
The Fokker-Planck equation (FPE) arises in various fields in natural science, including solid-state physics, quantum optics, chemical physics, theoretical biology and circuit theory. The Fokker- Planck equation (FPE) was first used by Fokker and Plank [14] to describe the Brownian motion of particles. A FPE describes the change of probability of a random function in space and time; hence it is naturally used to describe solute transport. The general FPE for the motion of a concentration field $u(x,t)$ of one space variable x at time t has the form [14]

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x) + \frac{\partial^2}{\partial x^2} B(x) \right] u(x,t), \tag{1}$$

with the initial condition given by

$$u(x,0) = f(x), \qquad x \in \mathbb{R},$$

where $B(x) > 0$ is the diffusion coefficient and $A(x)$ is the drift coefficient. The drift and diffusion coefficients may also depend on time. Eq. (1)

is a linear second-order partial differential equation of parabolic type. There is a more general form of FPE which is called nonlinear Fokker-Planck equation. Nonlinear FPE has important applications in various areas such as plasma physics, surface physics, population dynamic, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology and marketing [15]. In one variable case, the nonlinear FPE is written in the following form

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial}{\partial x} A(x,t,u) + \frac{\partial^2}{\partial x^2} B(x,t,u) \right] u(x,t),$$

with the initial condition given by

$$u(x,0) = f(x), \qquad x \in \mathbb{R}.$$

In recent years there has been a great deal of interest in fractional diffusion equations. These equations arise in continuous time random walks, modelling of anomalous diffusive and subdiffusive systems, unification of diffusion and wave propagation phenomenon, and simplification of the results [16]. Our concern in this work is to consider the numerical solution of the nonlinear FPE with space-and time-fractional derivatives of the form:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta} A(x,t,u) + \frac{\partial^{2\beta}}{\partial x^{2\beta}} B(x,t,u) \right] u(x,t), \quad t > 0, \quad 0 < \alpha, \beta \leqslant 1,$$

where α and β are parameters describing the order of the fractional time-and space derivatives, respectively. The function $u(x,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t < 0$ and $x < 0$. The fractional derivatives are considered in the Caputo sense.

In this paper, we extend the application of HAM to obtain analytic solutions to the space-and time-fractional Fokker-Planck equations. The paper has been organized as follows. Notations and basic definitions are given in Section 2. In Section 3 the homotopy analysis method is described. In Section 4 we extend the method to solve the space-and time-fractional Fokker-Planck equations. Discussion and conclusions are presented in Section 5.

2. Description On the Fractional Calculus

Definition 2.1. *A real function $f(t), t > 0$ is said to be in the space $C_\mu, \mu \in \mathrm{R}$ if there exists a real number $p > \mu$, such that $f(t) = t^p f_1(t)$ where $f_1(t) \in C[0,\infty)$, and it is said to be in the space C_n^μ if and only if $f^{(n)}(t) \in C_\mu, n \in N$. Clearly $C_\mu \subset C_\nu$ if $\nu \leqslant \mu$.*

Definition 2.2. *The Riemann-Liouville fractional integral operator (J^α) of order $\alpha \geqslant 0$, of a function $f \in C_\mu, \mu \geqslant -1$, is defined as*

$$\begin{aligned} &J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \qquad \alpha > 0, x > 0. \\ &J^0 f(x) = f(x). \end{aligned}$$

$\Gamma(\alpha)$ is the well-known Gamma function. Some of the properties of the operator J^α, which we will need here, are as follows:
For $f \in C_\mu, \mu \geqslant -1, \alpha, \beta \geqslant 0$ and $\gamma \geqslant -1$

$$\begin{aligned} &J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x), \\ &J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x), \\ &J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned}$$

For the concept of fractional derivative, there exist many mathematical definitions [2, 17, 18, 21]. In this paper, the two most commonly used definitions: the Caputo derivative and its reverse operator Riemann-Liouville integral are adopted. That is because Caputo fractional derivative [2] allows the traditional assumption of initial and boundary conditions. The Caputo fractional derivative is defined as

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n u(x,t)}{\partial t^n} d\tau, & n-1 < \alpha < n, \\ \frac{\partial^n u(x,t)}{\partial t^n}, & \alpha = n \in N. \end{cases} \tag{2}$$

Here, we also need two basic properties about them:

$$\begin{aligned} &D^\alpha J^\alpha f(x) = f(x), \\ &J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{x^k}{k!}, \qquad x > 0. \end{aligned}$$

3. Basic Idea of HAM

To describe the basic ideas of the HAM, We consider the following differential equation:

$$D_t^{\alpha} u = 0, \qquad t > 0,$$

where the operator D_t^{α} stand for the fractional derivative and is defined as in Eq. (7), t denote an independent parameter and $u(t)$ is an unknown function.
By means of generalizing the traditional homotopy method, Liao [7] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(t;q) - u_0(t)] = q\,hH(t)D_t^{\alpha}[\phi(t;q)], \tag{3}$$

where $q \in [0,1]$ is the embedding parameter, $h \neq 0$ is a non-zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(t)$ is initial guesse of $u(t)$, $u(t;q)$ is unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(t;0) = u_0(t), \phi(t;1) = u(t),$$

respectively. Thus, as q increases from 0 to 1, the solution $u(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $u(t;q)$ in Taylor series with respect to q, we have

$$\phi(t;q) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t)q^m, \tag{4}$$

where

$$u_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}.$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (4) converges at $q = 1$, then we have

$$u(t) = u_0(t) + \sum_{m=1}^{+\infty} u_m(t),$$

which must be one of solutions of original nonlinear equation, as proved by Liao [9]. As $h = -1$ and $H(t) = 1$, Eq. (3) becomes

$$(1-q)L[\phi_1(t;q) - u_0(t)] + q\ N[\phi_1(t;q)] = 0,$$

which is used mostly in the homotopy perturbation method [23], where as the solution obtained directly, without using Taylor series. According to the definition (5), the governing equation can be deduced from the zero-order deformation equation (3). Define the vector

$$\vec{u}_n = \{u_0(t), u_1(t), \ldots, u_n(t)\},$$

Differentiating equation (3) m times with respect to the embedding parameter q and then setting $q = 0$ and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$L[u_m(t) - \chi_m u_{m-1}(t)] = h\,H(t) R_m(\vec{u}_{m-1}), \tag{5}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} D_t^{\alpha}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0},$$

and

$$\chi_m = \begin{cases} 0, & m \leqslant 1, \\ 1, & m > 1. \end{cases}$$

Applying the Riemann-Liouville integral operator J^{α} on both side of Eq. (5), we have

$$u_m(t) = \chi_m u_{m-1}(t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{i}(0^{+}) \frac{t^i}{i!} + hH(t) J^{\alpha} R_m(\vec{u}_{m-1}),$$

For the convergence of the above method we refer the reader to Liao's work.

Remark 3.1. *The parameters α and β can be arbitrarily chosen as, integer or fraction, bigger or smaller than 1. When the parameter is bigger than 1, we will need more initial and boundary conditions such as*

$u'_0(x,0), u''_0(x,0), \cdots$ *and the calculations will become more complicated correspondingly. In order to illustrate the problem and make it convenient for the readers, we only confine the parameter to* $[0,1]$ *to discuss.*

Remark 3.2. *In 2007, Yabushita et al. [25] applied the HAM to solve two coupled nonlinear ODEs, and suggested the so-called optimization method to find out optimal convergence-control parameter by means of the minimum of the square residual error integrated in the whole region having physical meanings. Their approach is based on the square residual error*

$$\Delta(h) = \int_\Omega \left(N\left[\sum_{k=0}^{M} u_k(r)\right]\right)^2 d\Omega, \tag{6}$$

of a nonlinear equation $N[u(r)] = 0$*, where* $\sum_{k=0}^{M} u_k(r)$ *gives the* M*th-order HAM approximation. Obviously,* $\Delta(h) \to 0$ *(as* $M \to +\infty$*) corresponds to a convergent series solution. For given order* M *of approximation, the optimal value of* h *is given by a nonlinear algebraic equation*

$$\frac{d\Delta(h)}{dh} = 0.$$

We use exact square residual error (6) integrated in the whole region of interest Ω, at the order of approximation M.

4. Application

In this section we apply this method for solving linear space fractional, nonlinear time-fractional and linear space-and time-fractional FPE.

Example 4.1. Consider the linear space fractional FPE

$$\frac{\partial u}{\partial t} = \left[-\frac{\partial^\beta}{\partial x^\beta}.x + \frac{\partial^{2\beta}}{\partial x^{2\beta}}.\frac{x^2}{2}\right] u(x,t), \quad t > 0, \quad x > 0, \tag{7}$$

where $0 < \beta \leqslant 1$, subject to the initial condition

$$u(x,0) = x, \tag{8}$$

The exact solutions of Eq. (7) for the special case: $\beta = 1$ is

$$u(x,t) = xe^t,$$

For application of homotopy analysis method, in view of Eq. (7) and the initial condition given in Eq. (8), it in convenient to choose

$$u_0(x,t) = x,$$

as the initial approximate. We choose the linear operators

$$L[\phi(t;q)] = \frac{\partial \phi(x,t;q)}{\partial t},$$

with the property $L(c) = 0$, where c is constant of integration. Furthermore, we define a nonlinear operator as

$$N[\phi(t;q)] = \frac{\partial \phi(x,t;q)}{\partial t} + \left[\frac{\partial^\beta}{\partial x^\beta}(x.\phi(x,t;q)) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{2}.\phi(x,t;q))\right].$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} + \left[\frac{\partial^\beta}{\partial x^\beta}(x.u_{m-1}) - \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{2}.u_{m-1})\right].$$

We now successively obtain

$$\begin{aligned}
u_1(t) &= 2\frac{hx^{(-\beta+2)}t}{\Gamma(-\beta+3)} - 3\frac{hx^{(-2\beta+3)}t}{\Gamma(-2\beta+4)},\\
u_2(t) &= \frac{3}{2}\frac{h^2x^{(-\beta+2)}t}{\Gamma(-\beta+3)} - 3\frac{hx^{(-2\beta+3)}t}{\Gamma(-2\beta+4)}2\frac{hx^{(-\beta+2)}t}{\Gamma(-\beta+3)} - 3\frac{hx^{(-2\beta+3)}t}{\Gamma(-2\beta+4)} + \cdots\\
&\vdots
\end{aligned}$$

The optimal value of h is determined by the minimum of Δ_3, corresponding to the nonlinear algebraic equation $\frac{d\Delta_3}{dh} = 0$. The results are shown in Table 1.

Table 1: Optimal value of h at different order of β.

β	Optimal value of h	Minimum value of Δ_m
1	-1	2.2031e-023
$\frac{1}{2}$	-0.9782	3.8914e-020
$\frac{3}{4}$	-1.0867	1.1865e-019

By taking $\beta = 1,\ h = -1$, we reproduce the solution of problem as follows

$$u(x,t) = x(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots),$$

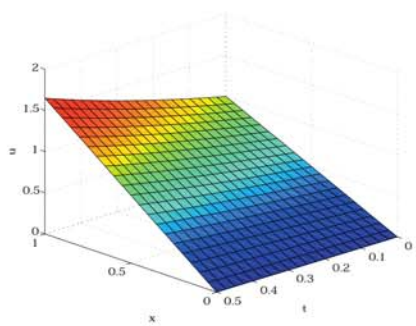

Fig 1: HAM solution with $h = -1, \beta = 1$

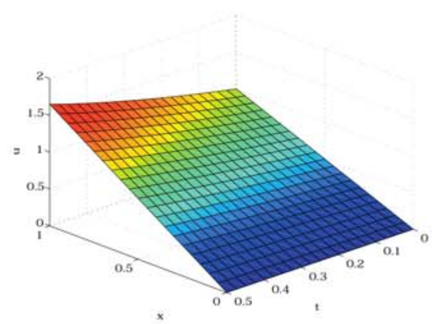

Fig 2: Exact solution.

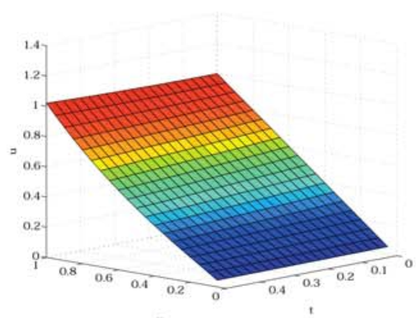

Fig 3: HAM solution with $h = -0.9782,\ \beta = 0.5$

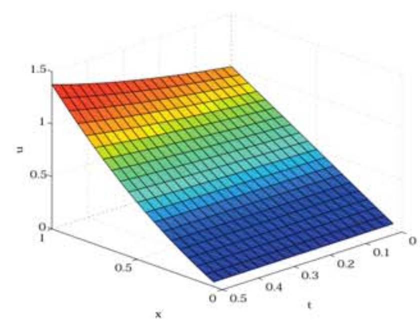

Fig 4: HAM solution with $h = -1.0867, \beta = 0.75$.

This solution is equivalent to the exact solution in a closed form $u(x,t) = xe^t$. Fig. 1 and Fig. 2, shown the exact and HAM solutions linear space fractional FPE with $h = -1, n = 3, \beta = 1$. It is obvious that, when $\beta = 1$, the solution is nearly identical with the exact solution. Fig. 3 and Fig. 4, shown the approximate solutions linear space fractional FPE with $h = -0.9782, n = 3, \beta = 0.5$ and $h = -1.0867, \beta = 0.75$, respectively. By HAM, it is easy to discover the valid region of h, which corresponds to the line segments nearly parallel to the horizontal axis. To find a proper value of h the h-curve of $u(0.1, 0.1)$ given by the 4th-order HAM approximation is drawn in Fig. 5.

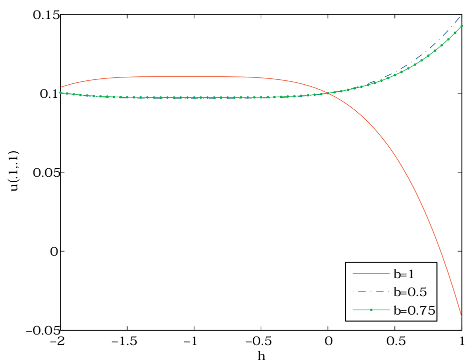

Fig 5: The h-curve of $u(0.1, 0.1)$ based on the 4th-order HAM.

Example 4.2. Consider the nonlinear time-fractional FPE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}(\frac{4u}{x} - \frac{x}{3}) + \frac{\partial^2}{\partial x^2}(u)\right] u(x,t), \quad t > 0, \quad x > 0, \tag{9}$$

where $0 < \alpha \leqslant 1$, subject to the initial condition

$$u(x, 0) = x^2. \tag{10}$$

The exact solutions of Eq. (9) for the special case: $\beta = 1$ is

$$u(x,t) = x^2 e^t,$$

For application of homotopy analysis method, in view of Eq. (9) and the initial condition given in Eq. (10), it in convenient to choose

$$u_0(x,t) = x^2,$$

as the initial approximate. We choose the linear operators

$$L[\phi(x;q)] = D_t^\alpha[\phi(x;q)].$$

Furthermore, we define a nonlinear operator as

$$N[\phi(t;q)] = \frac{\partial^\alpha \phi(x,t;q)}{\partial t^\alpha} = \left[-\frac{\partial}{\partial x}(\frac{4\phi(x,t;q)}{x} - \frac{x}{3}) + \frac{\partial^2}{\partial x^2}.(\phi(x,t;q))\right]\phi(x,t;q).$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{u}_{m-1}) = \frac{\partial^\alpha u_{m-1}}{\partial t^\alpha} + 4\frac{\partial}{\partial x}\frac{1}{x}(\sum_{k=0}^{m-1} u_k u_{m-1-k}) + \frac{x u_{m-1}}{x} - \frac{\partial^2}{\partial x^2}(\sum_{k=0}^{m-1} u_k u_{m-1-k}).$$

We now successively obtain

$$\begin{aligned}
u_1(t) &= -h\frac{x^2 t^\alpha}{\Gamma(\alpha+1)},\\
u_2(t) &= \frac{x^2}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}(-ht^\alpha\Gamma(2\alpha+1) - h^2t^\alpha\Gamma(2\alpha+1)\\
&\quad -\Gamma(\alpha+1)\Gamma(2\alpha+1) - h\Gamma(\alpha+1)\Gamma(2\alpha+1) + h^2t^{2\alpha}\Gamma(\alpha+1))\\
&\vdots
\end{aligned}$$

The optimal value of h is determined by the minimum of Δ_3, corresponding to the nonlinear algebraic equation $\frac{d\Delta_3}{dh} = 0$. The results are shown in Table 2.

Table 2: Optimal value of h at different order of β.

β	Optimal value of h	Minimum value of Δ_m
1	-1	1.0352e-021
$\frac{1}{2}$	-1.0921	1.0985e-020
$\frac{3}{4}$	-1.0702	2.1205e-020

By taking $\alpha = 1, h = -1$, we reproduce the solution of problem as follows

$$u(x,t) = x^2(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots).$$

This solution is equivalent to the exact solution in a closed form $u(x,t) = x^2e^t$. Fig. 6 and Fig. 7, shown the exact and HAM solutions linear space fractional FPE with $h = -1, n = 3, \alpha = 1$. It is obvious that, when $\alpha = 1$, the solution is nearly identical with the exact solution. Fig. 8 and Fig. 9, shown the approximate solutions linear space fractional FPE with $h = -1.0921, n = 3, \alpha = 0.5$ and $h = -1.0702, \alpha = 0.75$, respectively. To find a proper value of h the h-curve of $u(0.1, 0.1)$ given by the 4th-order HAM approximation is drawn in Fig. 10.

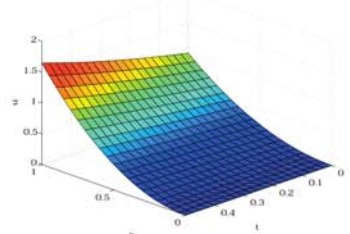

Fig 6: HAM solution with $h = -1, \alpha = 1$,

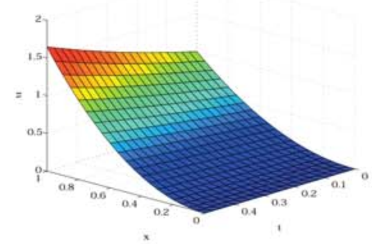

Fig 7: Exact solution.

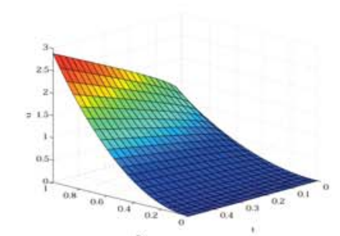

Fig 8: HAM solution with $h = -1.0921, \alpha = 0.5$,

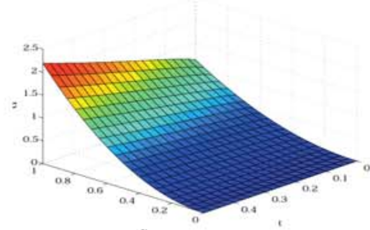

Fig 9: HAM solution with $h = -1.0702$, $\alpha = 0.75$.

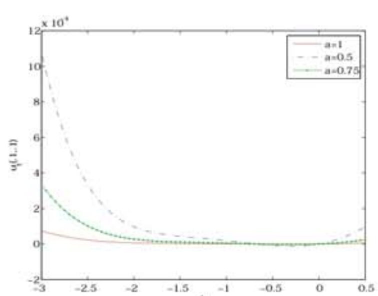

Fig 10: The h-curve of $u_t(0.1, 0.1)$ based on the 4th-order HAM.

Example 4.3. Consider the linear space- and time-fractional FPE

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \left[-\frac{\partial^\beta}{\partial x^\beta}(\frac{x}{6}) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{12})\right] u(x,t), \quad t > 0, \quad x > 0, \tag{11}$$

where $0 < \alpha, \beta \leqslant 1$, subject to the initial condition

$$u(x,0) = x^2. \tag{12}$$

The exact solutions of Eq. (11) for the special case: $\beta = 1$ is

$$u(x,t) = xe^{\frac{t}{2}}.$$

For application of homotopy analysis method, in view of Eq. (11) and the initial condition given in Eq. (12), it in convenient to choose

$$u_0(x,t) = x^2,$$

as the initial approximate. We choose the linear operators

$$L[\phi(x;q)] = D_t^\alpha[\phi(x;q)].$$

Furthermore, we define a nonlinear operator as

$$N[\phi(t;q)] = \frac{\partial\phi(x,t;q)}{\partial t} - \left[-\frac{\partial^\beta}{\partial x^\beta}(\frac{x}{6}) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{12})\right]\phi(x,t;q).$$

We construct the zeroth-order and the mth-order deformation equations where

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}}{\partial t} - \left[-\frac{\partial^\beta}{\partial x^\beta}(\frac{x}{6}) + \frac{\partial^{2\beta}}{\partial x^{2\beta}}(\frac{x^2}{12})\right] u_{m-1}.$$

We now successively obtain

$$\begin{aligned}
u_1(t) &= \frac{-2hx^{(4-2\beta)}t^\alpha}{\Gamma(5-2\beta)\Gamma(\alpha+1)} + \frac{hx^{(3-\beta)}t^\alpha}{\Gamma(4-\beta)\Gamma(\alpha+1)},\\
u_2(t) &= \frac{-hx^{(4-2\beta)}t^{2\alpha}\Gamma(5-\beta)}{6\Gamma(5-\beta)\Gamma(4-\beta)\Gamma(2\alpha+1)} + \cdots,\\
&\vdots
\end{aligned}$$

The optimal value of h is determined by the minimum of Δ_3, corresponding to the nonlinear algebraic equation $\frac{d\Delta_3}{dh} = 0$. The results are shown in Table 3.

Table 3: Optimal value of h at different order of β.

β	Optimal value of h	Minimum value of Δ_m
1	-1	1.0029e-021
$\frac{1}{2}$	-0.9431	2.0326e-019
$\frac{3}{4}$	-1.0301	1.7062e-019

By taking $\alpha, \beta = 1,\ h = -1$, we reproduce the solution of problem as follows

$$u(x,t) = x^2(1 + \frac{t}{2} + \frac{(\frac{t}{2})^2}{2!} + \frac{(\frac{t}{2})^3}{3!} + \cdots).$$

This solution is equivalent to the exact solution in a closed form $u(x,t) = x^2e^t$. Fig. 11 and Fig. 12, shown the exact and HAM solutions linear space fractional FPE with $h = -1, n = 3, \alpha, \beta = 1$. It is obvious that, when $\alpha, \beta = 1$, the solution is nearly identical with the exact solution. Fig. 13 and Fig. 14, shown the approximate solutions linear space fractional FPE with $h = -0.9431, n = 3, \alpha, \beta = 0.5$ and $h = -1.0301, \alpha, \beta = 0.75$, respectively. To find a proper value of h the h-curve of $u(0.1, 0.1)$ given by the 4th-order HAM approximation is drawn in Fig. 15.

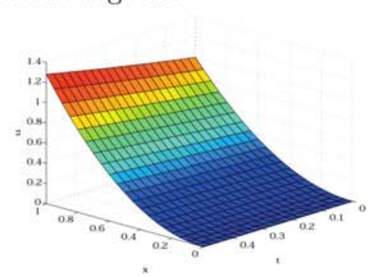

Fig 11: HAM solution with $h = -1, \alpha, \beta = 1$,

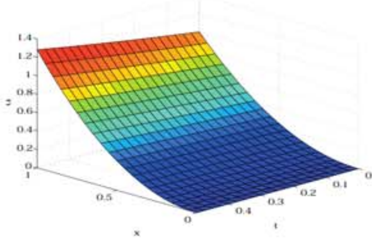

Fig 12: Exact solution .

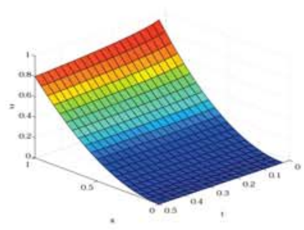

Fig 13: HAM solution with $h = -0.9431, \alpha, \beta = 0.5$,

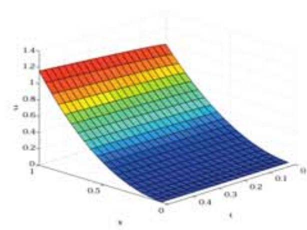

Fig 14: HAM solution with $h = -1.0301$,

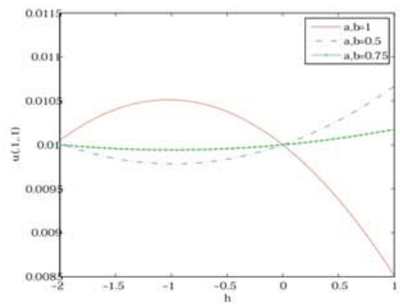

Fig. 15. The h-curve of $u(0.1, 0.1)$ based on the 4th-order HAM.

Remark 4.4. *This examples has been solved using homotopy perturbation method [24]. The graphs drawn by* $h = -1$ *are in excellenst agreement with that graphs drawn with HPM. Therefore, the HAM logically contains the HPM.*

5. Conclusion

In this paper, based on the symbolic computation Matlab, we have successfully developed HAM for solving space- and time-fractional Fokker-Planck equation. HAM provides us with a convenient way to control the convergence of approximation series by adapting h, which is a fundamental qualitative difference in analysis between HAM and other methods. The obtained results demonstrate the reliability of the HAM and its wider applicability to fractional differential equation. It, therefore, provides more realistic series solutions that generally converge very rapidly in real physical problems.

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