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Original Research Paper

A New Block by Block Scheme via Quadrature Rule of Lobatto-Gaussian for Nonlinear Volterra Integral Equations

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Abstract. In this paper, a multistage technique called Block by Block technique is proposed to solve nonlinear Volterra integral equations by combining quadrature rule of Lobatto-Gaussian. This procedure gets automatically calculates several values of unknown functions at once and it is the most appropriate method which has the ability to show high accuracy for entire points of intervals, especially at the end points of large intervals. Also, the convergence of the presented method via the Gronwall inequality is proven and it is shown that the rate of convergence is at least $O(h^8)$. Some numerical experiments report the ability and accuracy of the proposed method.

AMS Subject Classification: 65R20; 65D30; 45G10
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1 Introduction

The integral equations are one of the important subjects of applied mathematics which widely used for solving many problems in engineering, mathematical physics, economics, biology and etc. The process of solving linear and nonlinear integral equations analytically is very hard. However, some numerical techniques are used to solve these equations.
In recent decades, many attempts have been made to solve linear and nonlinear Volterra integral equations by many researchers using numerical methods. For example, Collocation method[7, 8, 19], Galerkin method [46, 33], Bernstein's approximation method [25], Chebyshev operational vector method [24], Wavelet method [20, 21, 44, 43], Reproducing kernel hilbert space method [40], Polynomial approximation method [2], Triangular functions method [18], Newton–Kantorovich method [23, 1], Iterative multistep kernel based method [6], Modified homotopy perturbation method [17], Differential transformation method [41], Homotopy perturbation method [11], Adomian decomposition method [3], Block-Pulse functions method [29], Radial basis functions method [26], Generalized quadrature method [45], Hilfer-type fractional operator [35], Quantum calculus method [42], Asymptotically almost automorphic mild solutions [31], Multi-Step method [28] and Homotopy analysis transform method [9].
The main objective of present research is to propose a multistage technique called Block by Block technique for solving nonlinear Volterra integral equations based on the extrapolation procedure of the Gauss–Lobatto quadrature rule. This technique calculates different values of the unknown function simultaneously without needing specific starting procedures and shows high accuracy for entire points of intervals, especially at the end points of large intervals. Moreover, the convergence of the presented method via the Gronwall inequality is proved and the rate of convergence was achieved.
On the other hand, application of the multistage technique has been used to solve various problems in recent years. The concept of this approach was suggested for the first time by Young in 1968. Linz [34] presented two-block method for nonlinear Volterra integral equations (VIEs) of the second kind. Then, AL-Asdi [4] studied two and three blocks for Hammersetien Volterra integral equations of the second kind.

Also, Saify [39] exerted Block by Block scheme to solve linear systems of second kind VIEs , constructed by several blocks. While, the another applications of the Block by Block method can be seen in the researches of Katani and Shahmorad [36, 37, 38] for different classes of integral equations. Beside, Afiatdoust et al [12, 13, 14, 15], utilized with this approach have done considerable studies.
To crystallize the presentation of the current work, the remainder of sections have been introduced as follows: Section 2, describes the numerical technique in general. The result of convergence procedure by using Gronwall inequality is shown in Section 3. Finally, Section 4, illustrates accuracy and convergence results by presenting some numerical examples.

2 Description of Numerical Technique

The general form of a nonlinear Volterra Integral Equations (VIEs) is as follows:

$$y(t) = x(t) + \int_0^t k(t, s, y(s))ds, \qquad 0 \le t \le T, \tag{1}$$

Where x is continuous on the interval $[0, +\infty)$ and k is continuous on:

$$D := \{(t, s, y)|\ 0 < s < t < +\infty,\ y \in R\}\,.$$

2.1 Block by Block method for four-dimensional blocks

Let the basic interval $[0, T]$ of integration divided into M parts with step size $h = \dfrac{T}{M}$ such that $t_i = ih,\ i = 0, 1, ..., M$ and the dimention of the blocks(M) must be multiple of 4. For $t = t_{4n+q}$, $(n = 0, 1, ..., M/4 - 1\ \mathit{and}\ q = 1, 2, 3, 4)$, we can write the equation (1) as

$$\begin{aligned} Y_{4n+q} \simeq y(t_{4n+q}) = x(t_{4n+q}) + \int_0^{t_{4n+q}} k(t_{4n+q}, s, y(s))ds = x(t_{4n+q}) \\ + \int_0^{t_{4n}} k(t_{4n+q}, s, y(s))ds + \int_{t_{4n}}^{t_{4n+q}} k(t_{4n+q}, s, y(s))ds, \end{aligned} \tag{2}$$

Let Y_i be the approximate value of $y(t)$ at $t = t_i$, $(i = 0, \ldots M)$ and let $Y_0 = x(t_0)$. If $Y_0, Y_1, \ldots, Y_{4n}$ are known, we can approximate the integral over $[0, 4n]$ in (2) utilizing standard five point integration rules. Moreover, for calculating the integral over $[4n, 4n+q]$, we use the Gauss-Lobatto quadrature rule at the points $t_{4n},\ t_{4n+1},\ t_{4n+2},\ t_{4n+3},\ t_{4n+4}$. Therefore, we have a system with four simultaneous equations by solving the system in each block, four values of Y are obtained. To simplify notation, we set $k_i = k(t_{4n+q}, t_i, Y_i)$ and by exerting five points quadrature rule of Lobatto-Gaussian for integral over $[4n, 4n+q]$, we define

$$\begin{aligned}
&\int_{t_{4n}}^{t_{4n+q}} k\left(t_{4n+q}, s, y(s)\right) ds \simeq \frac{t_{4n+q} - t_{4n}}{2} \sum_{\tau=0}^{4} \gamma_\tau \\
&k\left(t_{4n+q}, \frac{t_{4n+q} - t_{4n}}{2} \bar{y}_\tau + \frac{t_{4n+q} + t_{4n}}{2}, y\left(\frac{t_{4n+q} - t_{4n}}{2} \bar{y}_\tau + \frac{t_{4n+q} + t_{4n}}{2}\right)\right) \\
&= \frac{t_q}{2}\left[\frac{1}{10}\left(k_{4n+r_0} + k_{4n+r_4}\right) + \frac{32}{45} k_{4n+r_2} + \frac{49}{90}\left(K_{4n+r_1} + k_{4n+r_3}\right)\right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q = 1, 2, 3, 4. \quad (3)
\end{aligned}$$

where, five points $\bar{y}_i$ and five weights γ_i in five points quadrature rule of Lobatto-Gaussian for $i = 0, 1, \ldots, 4$ are as follows:

$$\bar{y}_0 = -1, \quad \bar{y}_1 = -\sqrt{\frac{3}{7}}, \quad \bar{y}_2 = 0, \quad \bar{y}_3 = \sqrt{\frac{3}{7}}, \quad \bar{y}_4 = 1.$$

$$\gamma_0 = \frac{1}{10}, \quad \gamma_1 = \frac{49}{90}, \quad \gamma_2 = \frac{32}{45}, \quad \gamma_3 = \frac{49}{90}, \quad \gamma_4 = \frac{1}{10},$$

and

$$\begin{aligned}
r_0 &= \frac{(\overline{y_0} + 1)q}{2}, \quad r_1 = \frac{(\overline{y_1} + 1)q}{2}, \quad r_2 = \frac{(\overline{y_2} + 1)q}{2}, \\
r_3 &= \frac{(\overline{y_3} + 1)q}{2}, \quad r_4 = \frac{(\overline{y_4} + 1)q}{2}.
\end{aligned}$$

If $r_i, i = 1, 2, 3$ are not integers, then Y_{4n+r_i} in (3) will be unknown. In this case, they are obtained via interpolation by Lagrange formula utilizing any five points $t_{4n+r_i},\ i = 0, 1, 2, 3, 4$. Thus, we have

$$Y_{4n+r_i} \approx \mathcal{P}(t_{4n} + r_i h) = \sum_{j'=0}^{4} L_{j'}(r_i) Y_{4n+j'}, \quad i = 1, 2, 3,$$

where

$$L_{j'}(r_i) := \prod_{\substack{p=0\\p\neq j'}}^{4} \frac{r_i - p}{j' - p}.$$

Given require the use of a multiple 4, so we write

$$\begin{aligned}\Phi &:= \int_0^{t_{4n}} k(t_{4n+q}, s, y(s))ds = \sum_{\tau=1}^{n}\Big(\int_{t_{4(\tau-1)}}^{t_{4\tau}} k(t_{4n+q}, s, y(s))ds\Big)\\
&= \sum_{\tau=1}^{n}\Big(\frac{t_{4\tau} - t_{4(\tau-1)}}{2}\Big[\frac{1}{10}(k_{4(\tau-1)} + k_{4\tau})\\
&+ \frac{49}{90}(k_{4(\tau-1)+r_1} + k_{4(\tau-1)+r_3}) + \frac{32}{45}k_{4(\tau-1)+2}\Big]\Big).\end{aligned} \tag{4}$$

substituting (4) and (3) in (2), we have

$$\begin{aligned}&Y_{4n+q} = x(t_{4n+q}) + \Phi + \frac{t_q}{2}\Big[\frac{1}{10}(k_{4n+r_0} + k_{4n+r_4})\\
&+ \frac{49}{90}\Big(k(t_{4n+q}, t_{4n+r_1}, \mathcal{P}(t_{4n} + r_1 h)) + k(t_{4n+q}, t_{4n+r_3}, \mathcal{P}(t_{4n} + r_3 h))\Big)\\
&+ \frac{32}{45}\Big(k(t_{4n+q}, t_{4n+r_2}, \mathcal{P}(t_{4n} + r_2 h))\Big)\Big].\end{aligned} \tag{5}$$

Consequently, at each step from relation (5), we can solve a system of equations for the unknowns Y_{4n+1}, Y_{4n+2}, Y_{4n+3} and Y_{4n+4}, which, this system of equations can be linear or nonlinear and due to the linear or nonlinear type of equations, can be used a direct method or Newton's iterative method to solve, respectively.

2.2 Block by Block method for six-dimensional blocks

Suppose $0 = t_0 < t_1 < \ldots < t_M = T$ be a partition of $[0, T]$ with step size $h = \dfrac{T}{M}$ such that $t_i = t_0 + ih, i = 1, 2, ..., M$ where M is a multiple of 6. For $t = t_{6n+q}$, the equation (1) can be written as

$$Y_{6n+q} \simeq y(t_{6n+q}) = x(t_{6n+q}) + \int_0^{t_{6n+q}} k(t_{6n+q}, s, y(s))ds = x(t_{6n+q}) + \int_0^{t_{6n}} k(t_{6n+q}, s, y(s))ds + \int_{t_{6n}}^{t_{6n+q}} k(t_{6n+q}, s, y(s))ds, \tag{6}$$

For $n = 0, 1, ..., M/6 - 1$ and $q = 1, 2, ..., 6$. Let Y_i be the approximate value of $y(t)$ at $t = t_i$ for $i = 0, \dots, M$ and let $Y_0 = x(t_0)$.
If $Y_0, Y_1, \dots, Y_{6n}$ are known, then for calculating the integral over $[0, 6n]$ in (6) using a standard seven points integration rule. Moreover, for computing the integral over $[6n, 6n + q]$, we can use the Gauss-Lobatto quadrature rule at the points t_{6n}, t_{6n+1},...,t_{6n+6}. Therefore, we have a system with six simultaneous equations to be solved. To simplify our notation, we set $k_i = k(t_{6n+q}, t_i, Y_i)$ and by exerting seven points quadrature rule of Lobatto-Gaussian for integral over $[6n, 6n + q]$, we define

$$\begin{aligned}
&\int_{t_{6n}}^{t_{6n+q}} k(t_{6n+q}, s, y(s))ds \simeq \frac{t_{6n+q} - t_{6n}}{2} \sum_{\tau=0}^{6} \gamma_\tau \\
&k\left(t_{6n+q}, \frac{t_{6n+q} - t_{6n}}{2}\bar{y}_\tau + \frac{t_{6n+q} + t_{6n}}{2}, y\left(\frac{t_{6n+q} - t_{6n}}{2}\bar{y}_\tau + \frac{t_{6n+q} + t_{6n}}{2}\right)\right) \\
&= \frac{t_q}{2}\left[\frac{1}{21}(k_{6n+r_0} + k_{6n+r_6}) + \frac{124 - 7\sqrt{15}}{350}(k_{6n+r_1} + k_{6n+r5}) + \frac{256}{525}k_{6n+r_3}\right. \\
&\left. + \frac{124 + 7\sqrt{15}}{350}(k_{6n+r_2} + k_{6n+r4})\right], \qquad q = 1, 2, 3, 4, 5, 6,
\end{aligned} \tag{7}$$

where, seven points $\bar{y}_i$ and seven weights γ_i in seven points quadrature rule of Lobatto-Gaussian for $i = 0, 1, \dots, 6$ are as follows:

$$\overline{y_0} = -1, \quad \overline{y_1} = -\sqrt{\frac{5}{11} + \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \overline{y_2} = -\sqrt{\frac{5}{11} - \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \overline{y_3} = 0,$$
$$\overline{y_4} = \sqrt{\frac{5}{11} - \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \overline{y_5} = \sqrt{\frac{5}{11} + \frac{2}{11}\sqrt{\frac{5}{3}}}, \quad \overline{y_6} = 1,$$

$$\gamma_0 = \gamma_6 = \frac{1}{21}, \quad \gamma_1 = \gamma_5 = \frac{124 - 7\sqrt{15}}{350}, \quad \gamma_2 = \gamma_4 = \frac{124 + 7\sqrt{15}}{350}, \ \gamma_3 = \frac{256}{525},$$

and

$$r_0 = \frac{(\overline{y_0}+1)q}{2}, \quad r_1 = \frac{(\overline{y_1}+1)q}{2}, \quad r_2 = \frac{(\overline{y_2}+1)q}{2}, \quad r_3 = \frac{(\overline{y_3}+1)q}{2},$$
$$r_4 = \frac{(\overline{y_4}+1)q}{2}, \quad r_5 = \frac{(\overline{y_5}+1)q}{2}, \quad r_6 = \frac{(\overline{y_6}+1)q}{2}.$$

If $r_i, i = 1, 2, 3, 4, 5$ are not integers, then Y_{6m+r_i} in (7) will be unknown. In this case, we use the interpolation by Lagrange formula utilizing any seven points $t_{6n+r_i},\ i = 0, 1, 2, \ldots, 6$. Thus, we have

$$Y_{6n+r_i} \approx \mathcal{P}(t_{6n} + r_i h) = \sum_{j'=0}^{6} L_{j'}(r_i) Y_{6n+j'}, \qquad i = 1, 2, 3, 4, 5,$$

where

$$L_{j'}(r_i) := \prod_{\substack{p=0 \\ p\neq j'}}^{6} \frac{r_i - p}{j' - p}.$$

On the other hand, for calculate the integral over $[0, 6n]$, because $6n$ is a multiple of 6, so we define

$$\begin{aligned}
\Phi := \int_0^{t_{6n}} k(t_{6n+q}, s, y(s))ds &= \sum_{\tau=1}^{n}\left(\int_{t_{6(\tau-1)}}^{t_{6\tau}} k(t_{6n+q}, s, y(s))ds\right) \\
&= \sum_{\tau=1}^{n}\Bigg(\frac{t_{6\tau} - t_{6(\tau-1)}}{2}\Bigg[\frac{1}{21}(k_{6(\tau-1)} + k_{6\tau}) \\
&+ \frac{124 - 7\sqrt{15}}{350}(k_{6(\tau-1)+r_1} + k_{6(\tau-1)+r_5}) \\
&+ \frac{256}{525}k_{6(\tau-1)+3} \\
&+ \frac{124 + 7\sqrt{15}}{350}(k_{6(\tau-1)+r_2} + k_{6(\tau-1)+r_4})\Bigg]\Bigg).
\end{aligned} \tag{8}$$

Consequently, substituting (8) and (7) in (6), we have

$$\begin{aligned} Y_{6n+q} = x(t_{6n+q}) + \Phi + \frac{t_q}{2}\Bigg[\frac{1}{21}(k_{6n+r_0} + k_{6n+r_6}) + \frac{124-7\sqrt{15}}{350}\Bigg(\\ k(t_{6n+q}, t_{6n+r_1}, \mathcal{P}(t_{6n} + r_1 h)) + k(t_{6n+q}, t_{6n+r_5}, \mathcal{P}(t_{6n} + r_5 h))\Bigg) \\ + \frac{256}{525}\Bigg(k(t_{6n+q}, t_{6n+r_3}, \mathcal{P}(t_{6n} + r_3 h))\Bigg) + \frac{124+7\sqrt{15}}{350}\Bigg(\\ k(t_{6n+q}, t_{6n+r_2}, \mathcal{P}(t_{6n} + r_2 h)) + k(t_{6n+q}, t_{6n+r_4}, \mathcal{P}(t_{6n} + r_4 h))\Bigg)\Bigg]. \end{aligned} \tag{9}$$

Thus, at each block from relation (9), we can solve a system of equations for the unknowns Y_{6n+1}, Y_{6n+2}, Y_{6n+3}, Y_{6n+4}, Y_{6n+5} and Y_{6n+6}. This system of equations will be linear or non linear depends upon on the linear or nonlinear type of the integral equation. Then, we may use a direct method or Newton's iterative method to solve the corresponding system for the case of linear or nonlinear.

3 Convergence Analysis

In this section we will investigate the convergent of the our method.

Theorem 3.1. *Suppose that k and y are functions defined in (1) each having differentiable at least eight times. Therefore, the order of convergence of the method so presented in (5) is of precision at least 8.*

Proof. It is seen from (4) that

$$\Phi := \int_0^{t_{4n}} k(t_{4n+q}, s, y(s))ds \approx h\sum_{i=0}^{4n} \gamma_i k(t_{4n+q}, t_i, Y_i).$$

Let $q = 1$ (the process is similar for the other values of q). Then it follows from (2) and (5) that

$$\begin{aligned}
|e_{4n+q}| &= |y(t_{4n+q}) - Y_{4n+q}| \\
&= \Bigg| \int_0^{t_{4n+q}} k(t_{4n+q}, s, y(s))ds - h\sum_{i=0}^{4n} \gamma_i k(t_{4n+q}, t_i, Y_i) \\
&- \frac{t_q}{2}\Bigg[\frac{1}{10}k(t_{4n+q}, t_{4n+r_0}, Y_{4n}) + \frac{49}{90}k\left(t_{4n+q}, t_{4n+r_1}, \mathcal{P}\left(t_{4n} + r_1 h\right)\right) \\
&+ \frac{32}{45}k\left(t_{4n+q}, t_{4n+r_2}, \mathcal{P}\left(t_{4n} + r_2 h\right)\right) \\
&+ \frac{49}{90}k\left(t_{4n+q}, t_{4n+r_3}, \mathcal{P}\left(t_{4n} + r_3 h\right)\right) + \frac{1}{10}k(t_{4n+q}, t_{4n+r_4}, Y_{4n+q})\Bigg]\Bigg|.
\end{aligned}$$

In addition by enhancing and diminishing the terms we will have:

$$\begin{aligned}
& h\sum_{i=0}^{4n}\gamma_i k(t_{4n+q}, t_i, y(t_i)),\ \ \frac{49t_q}{180}k(t_{4n+q}, t_{4n+r_1}, \sum_{j'=0}^{4} L_{j'}(r_1)y(t_{4n+j'})), \\
& \frac{t_q}{20}k(t_{4n+q}, t_{4n+r_0}, y(t_{4n})),\ \ \frac{32t_q}{90}k(t_{4n+q}, t_{4n+r_2}, \sum_{j'=0}^{4} L_{j'}(r_2)y(t_{4n+j'})), \\
& \frac{49t_q}{180}k(t_{4n+q}, t_{4n+r_3}, \sum_{j'=0}^{4} L_{j'}(r_3)y(t_{4n+j'})),\ \ \frac{t_q}{20}k(t_{4n+q}, t_{4n+r_4}, y(t_{4n+q})).
\end{aligned}$$

By using the Lipschitz condition, we have:

$$\begin{aligned}
|e_{4n+q}| \le h&\sum_{i=0}^{4n}\gamma_i \varrho(t_{4n+q}, t_i)|e_i| \\
&+ \frac{t_q}{20}\varrho(t_{4n+q}, t_{4n})|e_{4n}| + \frac{t_q}{20}\varrho(t_{4n+q}, t_{4n+q})|e_{4n+p}| \\
&+ \frac{49t_q}{180}\varrho(t_{4n+q}, t_{4n+r_1})|\mathcal{P}(t_{4n} + r_1 h) - \sum_{j'}^{4} L_{j'}(r_1)y(t_{4n+j'})| \\
&+ \frac{32t_q}{90}\varrho(t_{4n+q}, t_{4n+r_2})|\mathcal{P}(t_{4n} + r_2 h) - \sum_{j'}^{4} L_{j'}(r_2)y(t_{4n+j'})| \\
&+ \frac{49t_q}{180}\varrho(t_{4n+q}, t_{4n+r_3})|\mathcal{P}(t_{4n} + r_3 h) - \sum_{j'}^{4} L_{j'}(r_3)y(t_{4n+j'})| + R_1 + R_2,
\end{aligned}$$

where $R = R_1 + R_2$ is the error of numerical integration. Therefore, assuming that ϱ is continuous and bounded on $[0, T]$. We have:

$$\begin{aligned}
|e_{4n+q}| \leq & h \sum_{i=0}^{4n} \gamma_i l_i |e_i| + \frac{q}{20} h l_{4n} |e_{4n}| + \frac{q}{20} h l_{4n+q} |e_{4n+q}| \\
& + \frac{49q}{180} h l_{4n+r_1} \max_{j'} \{L_{j'}(r_1)\} \sum_{j'=0}^{4} |e_{4n+j'}| \\
& + \frac{32q}{90} h l_{4n+r_2} \max_{j'} \{L_{j'}(r_2)\} \sum_{j'=0}^{4} |e_{4n+j'}| \\
& + \frac{49q}{180} h l_{4n+r_3} \max_{j'} \{L_{j'}(r_3)\} \sum_{\bar{j}=0}^{4} |e_{4n+j'}| + R \\
& \leq hc_1 \sum_{i=0}^{4n} |e_i| + hc_2 |e_{4n+1}| + hc_3 |e_{4n+2}| + hc_4 |e_{4n+3}| \\
& \quad + hc_5 |e_{4n+4}| + R.
\end{aligned}$$

Let $\|e_j\|_\infty = \max_{j=4n+1,4n+2,\ldots,4n+4} |e_j| = |e_{4n+q}|$. Hence, we end up with the following inequality,

$$|e_{4n+q}| \leq hc \sum_{i=0}^{4n+q-1} |e_i| + hc' |e_{4n+q}| + R,$$

where c, c' are constants. Thus:

$$\|e_j\|_\infty \leq \frac{hc}{1 - hc'} \sum_{i=0}^{4n+q-1} |e_i| + \frac{R}{1 - hc'}.$$

Therefor, from the Gronwall inequality, we have:

$$\|e_j\|_\infty \leq \frac{R}{1 - hc'} e^{\frac{c}{1 - hc'} Mh}.$$

hence $\|e_j\|_\infty \to 0$ as $h \to 0$ since R tends to zero as $h \to 0$. Thus, the R order is of precision at least $O(h^8)$ and, hence,

$$\|e_j\|_\infty = O(h^8).$$

□

Corollary 3.2. *According to theorem (3.1) the approximation method presented in (9) for the six- dimensional blocks is also convergent and the convergence order of the method is of precision at least* 12.

4 Numerical Results

This section deals with some examples to verify the convergence and error bounds of the our method. All computations of this paper are performed by MAPLE 2016 software on the computer with an Intel(R) Core(TM) i7-6700K CPU@ 4.00 GHz 4.00 GHz and 16.00 GB RAM.

Example 4.1. ([1, 5, 23]) Consider the following problem:

$$y(t) = \sin(t) - \frac{t}{2} + \frac{1}{4}\sin(2t) + \int_0^t y^2(s)ds \qquad , 0 \leq t \leq 1,$$

with analytic solution $y(t) = \sin(t)$. The comparison among the absolute errors of our method with four- dimensional blocks for $n = 9$ with Newton–Kantorovich quadrature method [23], Newton–Kantorovich and Haar Wavelets Methods[1] and Block-pulse method [5] are reported in Table 1, which confirms the proposed method is more accurate than the methods given in [1, 5, 23]. Moreover, the computational time in the last row of Table 1 shows that the presented method has fewer calculations . Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 9$ is depicted in Fig 1.

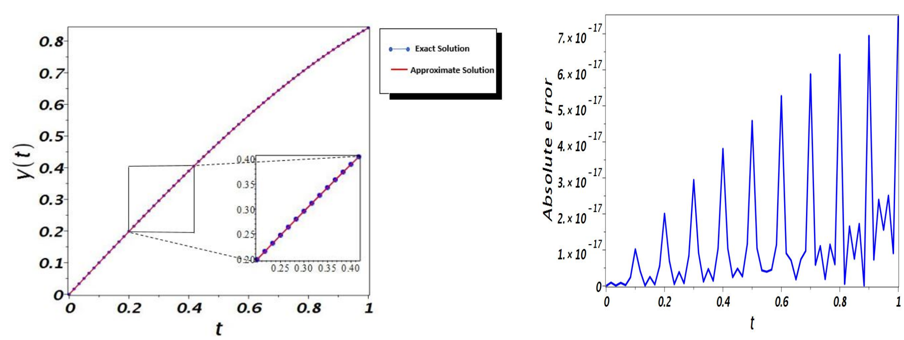
Example 4.2. ([1, 16, 30]) Consider the following problem:

$$y(t) = \frac{3}{2} - \frac{1}{2}e^{-2t} - \int_0^t (y(s) + y^2(s))ds \qquad , 0 \leq t < 1,$$

with analytic solution $y(t) = e^{-t}$. Table 2, shows the comparison of the absolute errors using our method with four- dimensional blocks for $n = 9$ and Newton–Kantorovich and Haar Wavelets Methods [1], hybrid of block-pulse functions and Taylor series [16], hybrid Taylor polynomials and Block-Pulse functions [30], which confirms the proposed method is more accurate than the methods given in [1, 16, 30]. Moreover, the computational time in the last row of Table 2 shows that the presented

Table 1: Comparison of absolute errors for Example 4.1.

t	[23]	[1]	[5]	Proposed Method
0.0	0.000000	8.29E-03	———	0.000000
0.1	3.33E-04	4.40E-04	6.49E-03	6.18E-13
0.2	1.54E-03	2.79E-06	1.49E-02	1.22E-12
0.3	6.65E-03	4.69E-04	2.52E-02	1.78E-12
0.4	1.39E-02	7.69E-04	1.58E-02	2.30E-12
0.5	0.000000	7.31E-04	2.01E-02	2.75E-12
0.6	4.70E-02	7.68E-05	7.02E-02	3.15E-12
0.7	6.90E-02	1.84E-04	9.74E-02	3.48E-12
0.8	9.59E-02	6.48E-04	1.53E-01	3.77E-12
0.9	1.43E-01	6.48E-04	2.06E-01	4.03E-12
1.0	1.85E-01	4.51E-01	———	4.29E-12
time(second)	——	——	——	$0.499''$



(a) (b)

Figure 1: Approximate and exact solutions(a) and Absolute error(b) of Example 4.1 with six-dimensional blocks for $n = 9$.

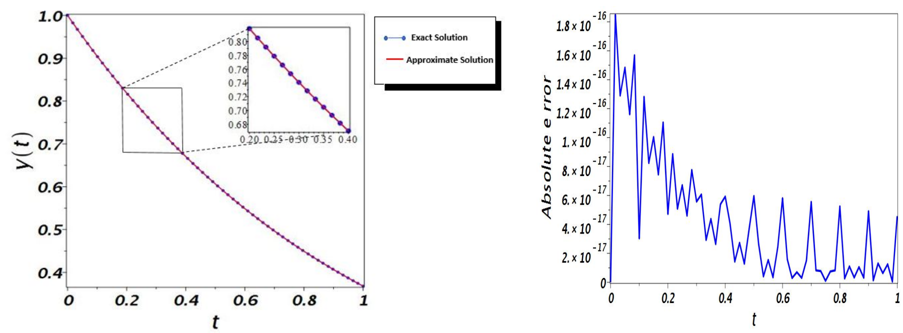
method has fewer calculations. Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 9$ is apparent in Fig 2.

Example 4.3. ([27, 32]) Consider the following problem:

$$y(t) = 1 + (sin(t))^2 - \int_0^t 3sin(t-s)y^2(s)ds \qquad ,0 \le t \le 1,$$

Table 2: Comparison of absolute errors for Example 4.2.

t	[30]	[1]	[16]	Proposed Method
0.0	0.000000	7.90E-03	0.000000	0.000000
0.1	8.33E-04	2.33E-04	1.63E-04	1.90E-12
0.2	3.75E-04	1.28E-05	2.44E-04	2.96E-12
0.3	1.11E-03	4.25E-04	1.27E-04	3.49E-12
0.4	3.51E-04	3.64E-07	2.86E-04	3.71E-12
0.5	5.80E-04	1.19E-04	3.99E-04	3.73E-12
0.6	1.32E-04	7.97E-06	2.25E-04	3.63E-12
0.7	4.95E-04	2.41E-04	3.59E-04	3.47E-12
0.8	1.73E-04	2.74E-05	1.04E-04	3.28E-12
0.9	3.68E-04	2.56E-04	2.97E-04	3.07E-12
time(second)	——	——	——	$0.515''$



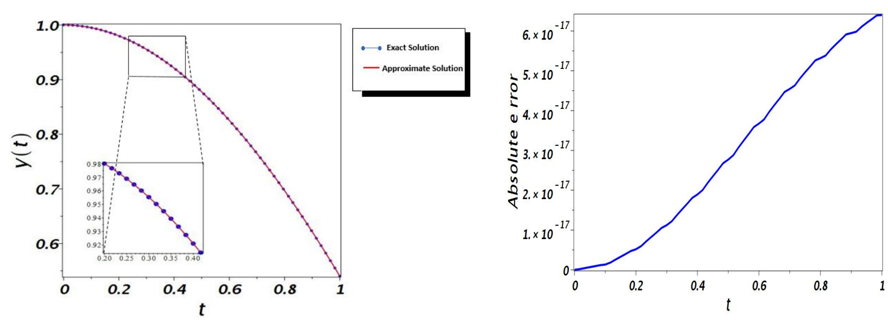
(a) (b)

Figure 2: Approximate and exact solutions(a) and Absolute error(b) of Example 4.2 with six-dimensional blocks for $n = 9$.

with analytic solution $y(t) = cos(t)$. The comparison among the absolute errors of our method with four- dimensional blocks for $n = 9$ with Collocation method [32] and Iterative continuous collocation method [27] are reported in Table 3, which confirms the proposed method is more accurate than the methods given in [27, 32]. Moreover, the computational time in the last row of Table 3 shows that the presented method has fewer calculations . Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 9$ is shown in Fig 3.

Table 3: Comparison of absolute errors for Example 4.3.

t	[32]	[27]	Proposed Method
0.0	0.000000	0.000000	0.000000
0.1	1.59E-06	7.92E-09	7.65E-14
0.2	3.26E-06	5.15E-09	2.99E-13
0.3	4.72E-06	3.87E-09	6.49E-13
0.4	5.87E-06	8.00E-09	1.09E-12
0.5	6.63E-06	8.90E-10	1.60E-12
0.6	6.98E-06	5.90E-09	2.12E-12
0.7	6.92E-06	9.71E-09	2.63E-12
0.8	6.47E-06	3.34E-09	3.07E-12
0.9	5.70E-06	2.07E-09	3.44E-12
1.0	4.71E-06	5.13E-09	3.70E-12
time(second)	——	——	$1.05''$



(a) (b)

Figure 3: Approximate and exact solutions(a) and Absolute error(b) of Example 4.3 with six- dimensional blocks for $n = 9$.

Example 4.4. ([10, 22]) Consider the following problem:

$$y(t) = \frac{1}{4} + \frac{t}{2} + e^t - \frac{e^{2t}}{4} + \int_0^t (t-s)y^2(s)ds \qquad , 0 \le t \le 1,$$

with analytic solution $y(t) = e^t$. The comparison among the absolute errors of our method with four- dimensional blocks for $n = 9$ with modified Laplace Adomian decomposition method [10] and Touchard method[22] are reported in Table 4, which confirms the proposed method is more accurate than the methods given in [10, 22]. Moreover, the computational time in the last row of Table 4 shows that the presented method has fewer calculations . Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 9$ is ploted in Fig 4.

Table 4: Comparison of absolute errors for Example 4.4.

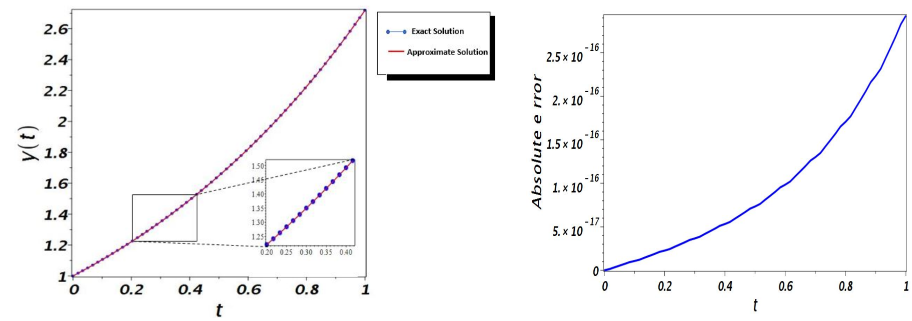
t	[10]	[22]	Proposed Method
0.0	0.000000	1.18E-03	0.000000
0.1	8.01E-04	2.49E-03	6.49E-13
0.2	2.60E-03	3.72E-03	1.38E-12
0.3	5.77E-03	4.41E-03	2.21E-12
0.4	9.89E-03	4.22E-03	3.21E-12
0.5	1.49E-02	2.94E-03	4.41E-12
0.6	2.03E-02	5.07E-04	5.91E-12
0.7	2.51E-02	2.96E-03	7.79E-12
0.8	2.77E-02	7.19E-03	1.02E-11
0.9	2.61E-02	1.67E-02	1.33E-11
1.0	1.69E-02	1.57E-02	1.74E-11
time(second)	——	——	$0.869''$

Example 4.5. ([3, 11, 38]) Consider the following problem:

$$y(t) = Bt + C - A\int_a^t e^{\lambda y(s)} ds \qquad , 0 \le t \le T,$$

with analytic solution as below

$$y(t) = \begin{cases} \dfrac{-1}{\lambda} ln[A\lambda(t-a) + e^{-C\lambda}], & B = 0, \\ \\ \dfrac{-1}{\lambda} ln\left[\dfrac{A}{B} + \left(e^{-\lambda y_0} - \dfrac{A}{B}\right) e^{\lambda B(a-t)}\right], \quad y_0 = aB + C, & B \neq 0. \end{cases}$$



(a) (b)

Figure 4: Approximate and exact solutions(a) and Absolute error(b) of Example 4.4 with six-dimensional blocks for $n = 9$.

We show the superiority of the proposed method with four- dimensional blocks for $n = 9$, $\lambda = \frac{1}{2}$, $A = 4$, $B = 3$, $a = 0$ and $C = \frac{1}{8}$ by comparing its results with the results of the Homotopy perturbation method [11], the Adomian method [3] and the Block by Block method[38] in Table 5. It is clear from these results that our presented method is more accurate than the proposed method in [38] for large values of t, in particular at the end points. Moreover, the computational time in the last row of Table 5 shows that the presented method has a short time to enforce . Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 9$ is drawn in Fig 5.

Example 4.6. ([3, 11, 38]) Consider the following problem:

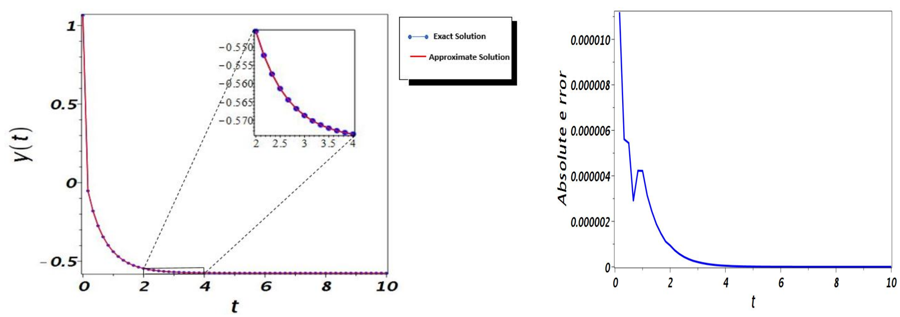
$$y(t) = Bt + C - A\int_a^t y^2(s)ds \qquad ,0 \le t \le T,$$

with analytic solution as below

$$y(t) = \begin{cases} K\dfrac{(K+y_a)e^{2AK(t-a)} + y_a - K}{(K+y_a)e^{2AK(t-a)} - y_a + K}, \; K = \sqrt{\dfrac{B}{A}}, \; y_a = aB + C, \; AB > 0, \\[2ex] \dfrac{C}{AC(t-a)+1}, \qquad\qquad\qquad\qquad\qquad\qquad\qquad AB = 0, \\[2ex] K\tan\left[AK(a-t) + \arctan\dfrac{y_a}{K}\right], \; K = \sqrt{\dfrac{-B}{A}}, \; y_a = aB + C, \; AB < 0. \end{cases}$$

Table 5: Comparison of absolute errors for Example 4.5.

t	[11]	[3]	[38]	Proposed Method
1	7.56E+00	6.81E -01	1.22E-04	1.08E-04
2	4.88E+01	1.16E+01	3.72E-05	2.66E-05
3	2.44E+01	8.09E+01	1.47E-05	6.34E-06
4	1.13E+03	4.05E+04	7.49E-06	1.50E-06
5	5.12E+03	1.87E+03	1.15E-05	3.54E-07
6	2.30E+04	8.44E+03	1.84E-05	8.33E-08
7	1.03E+05	3.79E+04	8.72E-05	1.95E-08
8	4.62E+05	1.70E+05	7.93E-04	4.57E-09
9	2.08E+06	7.62E+05	3.60E-04	1.07E-09
10	9.28E+06	3.41E+06	3.60E-04	2.49E-10
time(second)	——	——	$1.078''$	$0.640''$



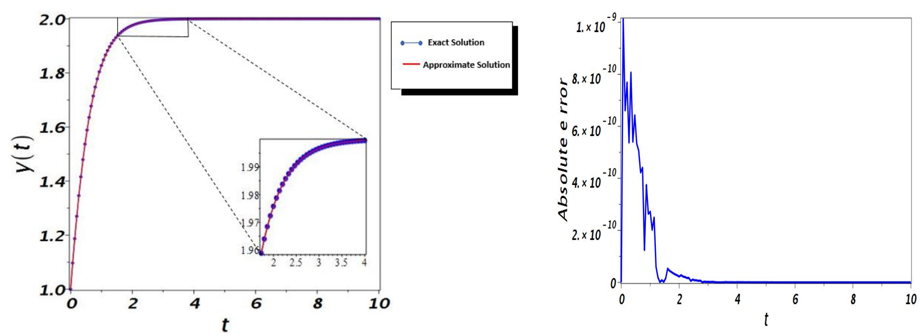
(a) (b)

Figure 5: Approximate and exact solutions(a) and Absolute error(b) of Example 4.5 with six-dimensional blocks for $n = 9$, $\lambda = \frac{1}{2}$, $A = 4$, $B = 3$, $a = 0$ and $C = \frac{1}{8}$.

The comparison between the absolute error of the proposed method with four- dimensional blocks for $n = 24$, $A = \frac{1}{2}$, $B = 2$, $a = 0$ and $C = 1$ with Homotopy perturbation method[11], Adomian method [3] and Block by Block scheme via Romberg quadrature[38] in some points of the interval $[0, 10]$ are reported in Table 6, which confirms our method is more accurate than the proposed method in [38] for large values of t, in particular at the end points. Moreover, the computational time in the last row of Table 6 shows that the presented method has a short time to enforce. Also, the superiority of the proposed scheme by increasing the dimension of blocks and $n = 24$ is depicted in Fig 6.

Table 6: Comparison of absolute errors for Example 4.6.

t	[11]	[3]	[38]	Proposed Method
0.5	1.66E- 01	3.00E-09	3.52E-08	3.67E-08
1.0	1.48E+00	1.05E-05	4.98E-08	1.10E-08
1.5	8.02E+00	1.43E-03	1.49E-07	6.93E-08
2.5	1.12E+70	2.82E+67	8.36E-08	3.75E-10
5.0	3.07E+340	1.00E+320	2.16E-06	2.61E-10
7.5	1.28E+507	2.30E+491	1.79E-06	3.17E-12
8.5	3.87E+559	5.03E+551	1.16E-06	5.46E-13
9.0	3.07E+583	3.50E+576	1.04E-06	2.23E-13
9.5	1.91E+606	1.07E+600	8.60E-07	8.31E-14
10	9.20E+627	2.02E+662	5.35E-06	3.70E-14
time(second)	——	——	0.626″	0.582″



(a) (b)

Figure 6: Approximate and exact solutions(a) and Absolute error(b) of Example 4.6 with six-dimensional blocks for $n = 24$, $A = \frac{1}{2}$, $B = 2$, $a = 0$ and $C = 1$.

5 Conclusions

This paper proposed a multistage technique called Block by Block technique for solving nonlinear Volterra integral equation. Our proposed method utilizing the Gauss-Lobatto quadrature rule compared to the method expressed in [38] shows high accuracy at all distances, especially at the end points of long distances, and when the dimension of the blocks increases to 6, our accuracy also improves. Also, this proposed method can automatically calculates several values of unknown functions simultaneously. Another advantage of the present method is that it takes a relatively short time to perform calculations. Furthermore, using Gronwall inequality, we proved that the degree of the convergence is at least eight. Illustrated examples confirm the high ability and accuracy of the proposed method. For a future studies prescription, the method proposed in this paper can be expanded for such problems in high dimensions.

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