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Original Research Paper

On $\mathcal{I}^K$-Connectedness and $\mathcal{I}^K$-Compactness

H. S. Behmanush
Mersin University

Abstract. This work aims to provide clear definitions and explore various aspects associated with the concepts of $\mathcal{I}^K$-compactness and $\mathcal{I}^K$-connectedness in a topological space (Ω, τ), which is a generalization of $\mathcal{I}$-connectedness and $\mathcal{I}$-compactness and $\mathcal{I}^*$-compactness.

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1 Introduction

Convergence is a fundamental concept in mathematical analysis that plays a crucial role in comprehending the topological and geometrical properties of the space. In recent years, some new topologies have been produced using different types of convergence in topological spaces. Examining known concepts with these newly produced topologies and identifying differences has become one of the most interesting research areas.

In this study, especially by considering $\mathcal{I}^K$-convergence, which will be defined below, some topological concepts will be re-examined.

By ideal of $\mathbb{N}$, we mean a subfamily of $\mathcal{P}(\mathbb{N})$ which is closed under finite union and has hereditary property. Similarly, by filter, we mean a

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subfamily of $\mathcal{P}(\mathbb{N})$ which is closed under superset and finite intersection. The set of complements of all elements of an ideal $\mathcal{I}$ of $\mathbb{N}$ is a filter known as a filter associated with the ideal and denoted by $\mathcal{F}(\mathcal{I})$. If all singleton subsets of $\mathbb{N}$ belongs to $\mathcal{I}$, then it is called an admissible ideal, and if $\mathcal{I} \neq \phi$ and $\mathbb{N} \notin \mathcal{I}$ then it is called a nontrivial ideal (see more in [17, 21]).

The relatively new notion of $\mathcal{I}^K$-convergence for functions was first introduced by M. Macaj and M. Slezial [12] in the year 2021 as a generalization of $\mathcal{I}^*$-convergence. In a space, this subject was first discussed in [6]. After that, some more results relating to $\mathcal{I}^K$-convergence can be found in the papers [3, 4, 5, 16, 15, 8], etc.

In connection with these concepts, the author S.K. Pal in [20] introduced the $\mathcal{I}$-sequential topological space, and later X. Zhou, L. Liv, and S. Lin in [22] gave more results about the $\mathcal{I}$-sequential topological space in 2020. After these studies, in the paper [9] $\mathcal{I}^*$-sequential topological space, and recently, in [10] the notion of $\mathcal{I}^K$-sequential topological space was introduced.

In the year 2000, A. Blali et al. [1] defined the concept of $\mathcal{I}$-compactness. Later, several results about compactness in relation to ideals were published in [7, 14, 19, 2, 13, 18]. In 2023, M. Singha and R. Sima in [11] defined the $\mathcal{I}$-sequential compact space and $\mathcal{I}^*$-sequential compact space by taking the $\mathcal{I}$-nonthin subsequence of a given sequence by considering an admissible ideal $\mathcal{I}$.

In this work, the ideas considered in the papers [1] and [11] will be generalized, and $\mathcal{I}^K$-connectedness and $\mathcal{I}^K$-compactness are going to be introduced for any ideals $\mathcal{I}$ and $\mathcal{K}$. The first section will present certain established definitions and outcomes. The second and third sections will focus on the concepts of $\mathcal{I}^K$-compactness and $\mathcal{I}^K$-connectedness in a topological space.

Throughout this paper, $(\Omega, \mathcal{T})$ will be referred to as a topological space and instead of using the term "topological space," we shall use the abbreviation "space". The collection of finite subsets of natural numbers is an ideal, which is symbolized as F_{in}.

Definition 1.1. [1] *A sequence $\vec{t} = (t_n)$ in a space $(\Omega, \mathcal{T})$ is said to be $\mathcal{I}$-convergent to a point $t \in \Omega$ if the set $\{n \in \mathbb{N} : t_n \in w\}$ is an element* *of the filter $\mathcal{F}(\mathcal{I})$ for every neighborhood w of t.*

The point t can be described as the ideal limit of the sequence, $\vec{t}$ and it is expressed by $\vec{t} \xrightarrow{\mathcal{I}} t$ (or $\mathcal{I} - \lim t_n = t$).

Definition 1.2. [1] *Consider $F \subset \Omega$. The $\mathcal{I}$-closure of a set F is defined as the set of all $t \in \Omega$ for which there exists a sequence $(t_n) \subset F$ that $\mathcal{I}$-converges to t. F is considered $\mathcal{I}$-closed if its $\mathcal{I}$-closure is identical to itself, while a subset of Ω is considered $\mathcal{I}$-open if its complement is $\mathcal{I}$-closed.*

Definition 1.3. [1] *Consider a space $(\Omega, \mathcal{T})$ with an ideal $\mathcal{I}$. (i) A subset F of Ω is considered $\mathcal{I}$-compact if, for any $\mathcal{I}$-open cover of F, there exists a finite subcover. (ii) A set F is said to be sequentially $\mathcal{I}$-compact if every sequence (t_n) in F has a subsequence (t_{n_k}) that $\mathcal{I}$-converges to a point of F.*

Definition 1.4. [12] *Consider $\mathcal{I}$ and $\mathcal{K}$ as arbitrary ideals, and let $(\Omega, \mathcal{T})$ be a space. A sequence $\vec{t} = (t_n) \subset \Omega$ is $\mathcal{I}^K$-convergent to a point $t \in \Omega$ if there exists $M \in \mathcal{F}(\mathcal{I})$ such that the related sequence (y_n) defined by*

$$y_n := \begin{cases} t_n, & n \in M, \\ t, & n \notin M, \end{cases}$$

is $\mathcal{K}$-convergent to t. The expression $\mathcal{I}^K - \lim(t_n) = t$ or $\vec{t} \xrightarrow{\mathcal{I}^K} t$ represents the limit of the sequence t_n as it approaches the value of t under the context of $\mathcal{I}^K$.

Definition 1.5. [10] *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals and $(\Omega, \mathcal{T})$ be a space. Then, (i) A subset $F \subseteq \Omega$ is considered to be $\mathcal{I}^K$-closed if the $\mathcal{I}^K$-limit point of all sequences F is a point of F. (ii) A subset V of Ω is considered to be $\mathcal{I}^K$-open if its complement V^c is $\mathcal{I}^K$-closed.*

Remark 1.6. *Define $\mathcal{K}$ as the set of all finite subsets of the set of natural numbers. Then, (i) $\mathcal{I}^K$-convergence is transformed into the familiar concept of $\mathcal{I}^*$-convergence.*

(ii) $\mathcal{I}^K$*-open and* $\mathcal{I}^K$*-closed are identical to* $\mathcal{I}^*$*-open and* $\mathcal{I}^*$*-closed, respectively.*

Proof. The proof is provided in [10]. □

Remark 1.7. *According to the reference [12], it is evident that any* $\mathcal{K}$*-converges sequence is also* $\mathcal{I}^K$*-converges. Therefore, it can be demonstrated that any set that is* $\mathcal{I}^K$*-open is also* $\mathcal{K}$*-open.*

Definition 1.8. *Consider two ideals,* $\mathcal{I}$ *and* $\mathcal{K}$*, and a space* $(\Omega, \mathcal{T})$*. Then,* $\mathcal{I}^K$*-closure of* A *is denoted by* $\overline{A}^{\mathcal{I}^K}$ *and defined as*

$$\overline{A}^{\mathcal{I}^K} := \{t \in \Omega : \exists (t_n) \subseteq A,\ t_n \xrightarrow{\mathcal{I}^K} t\}$$

for any subset $A \subseteq \Omega$*.*

Remark 1.9. *The following statements are true: (i) The* $\mathcal{I}^K$*-closure of the empty set and* Ω *are equal to themselves. Additionally, for any* $A \subseteq \Omega$*,* A *is a subset of its* $\mathcal{I}^K$*-closure. (ii) A subset of the space* Ω *is* $\mathcal{I}^K$*-closed if and only if its* $\mathcal{I}^K$*-closure is equal to itself.*

Proof. The proof is provided in [10]. □

Theorem 1.10. *Consider a space* (Ω, τ)*, and let* $\mathcal{I}$ *and* $\mathcal{K}$ *be two ideals of* $\mathbb{N}$*. Then,*

$$\mathcal{T}_{\mathcal{I}^K} := \{A \subset \Omega : cl_{\mathcal{I}^K}(\Omega - A) = \Omega - A\}$$

is a topology over the set Ω*.*

Proof. The proof can be obtained by considering definitions. Therefore, we have excluded it from this discussion. □

Definition 1.11. [10] $(\Omega, \tau_{\mathcal{I}^K})$ *is said to be* $\mathcal{I}^K$*-discrete space if every subset of* Ω *is* $\mathcal{I}^K$*-open set.*

2 $\mathcal{I}^K$-Compactness

For any ideal $\mathcal{I}$ and $\mathcal{K}$, the notions $\mathcal{I}^K$-compactness and $\mathcal{I}^K$-connectedness will be defined.

Definition 2.1. *Consider $\mathcal{I}$ and $\mathcal{K}$ as arbitrary ideals of $\mathbb{N}$, and let (Ω, τ) be a space. A set C that is a subset of Ω is referred to as: (i) $\mathcal{I}^K$-compact, for any collection of $\mathcal{I}^K$-open sets that covers C, there exists a finite subcollection that also covers C.*

(ii) A set C is sequentially $\mathcal{I}^K$-compact if, for any sequence (t_n) in C, there exists a subsequence that is $\mathcal{I}^K$-convergent and $\mathcal{I}^K$-converges to a point in C.

(iii) Locally $\mathcal{I}^K$-compact if for any point $t \in \Omega$, there is a neighborhood that is $\mathcal{I}^K$-compact.

In the case where $\mathcal{K} = Fin$, the concepts of $\mathcal{I}^K$-compactness and sequentially $\mathcal{I}^K$-compactness are identical to $\mathcal{I}^*$-compactness and sequentially $\mathcal{I}^*$-compactness, respectively.

Every space that is $\mathcal{I}^K$-compact is also locally $\mathcal{I}^K$-compact.

Theorem 2.2. *Consider $\mathcal{I}$ and $\mathcal{K}$ as arbitrary ideals of $\mathbb{N}$. Let (Ω, τ) be a space. A subset of an $\mathcal{I}^K$-compact space that is also $\mathcal{I}^K$-closed, is $\mathcal{I}^K$-compact.*

Proof. Let A be an $\mathcal{I}^K$-closed subset of Ω. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be a collection of $\mathcal{I}^K$-open sets that covers the set A, then $\{U_\alpha\}_{\alpha\in\Lambda} \cup A^c$ is $\mathcal{I}^K$-open cover of Ω. Since Ω is $\mathcal{I}^K$-compact, then there exists a finite subcover of Ω. Hence, we have

$$\Omega \subset \bigcup_{\alpha=1}^{n} U_\alpha \cup A^c$$

and

$$A = A \cap \Omega \subset A \cap \left(\bigcup_{\alpha=1}^{n} U_\alpha \cup A^c \right).$$

So, the set A is an $\mathcal{I}^K$-compact set. □

Theorem 2.3. *Consider $\mathcal{I}$ and $\mathcal{K}$ as arbitrary ideals of $\mathbb{N}$. Let (Ω, τ) denote a space. An $\mathcal{I}^K$-closed subset of sequentially $\mathcal{I}^K$-compact space is sequentially $\mathcal{I}^K$-compact.*

Proof. Let Ω be a sequentially $\mathcal{I}^K$-compact space. Let C be an $\mathcal{I}^K$-closed subset of Ω, and $(t_n) \subset C$ be any sequence. Since Ω is sequentially $\mathcal{I}^K$-compact and $C \subset \Omega$, then there exists an $\mathcal{I}^K$-convergent subsequence (t_{n_k}) of (t_n) such that $\mathcal{I}^K$-converges to some point $t \in \Omega$. It is clear that t is the $\mathcal{I}^K$ closure point of C. Since C is $\mathcal{I}^K$-closed, then $t \in C$. Hence, C is sequentially $\mathcal{I}^K$-compact. □

Theorem 2.4. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and let (Ω, τ) be a space. If A_1 and A_2 are $\mathcal{I}^K$-compact subsets, then $A_1 \cup A_2$ is $\mathcal{I}^K$-compact.*

Proof. Assume that A_1 and A_2 are $\mathcal{I}^K$ compact subsets of the space (Ω, τ). Let $\mathcal{C} = \{U_i : i \in \lambda\}$ be an $\mathcal{I}^K$-open cover of $A_1 \cup A_2$, i.e.,

$$A_1 \cup A_2 \subset \bigcup_{i \in \lambda} U_i.$$

This inclusion implies that the family $\mathcal{C}$ is also $\mathcal{I}^K$-open cover of A_1 and A_2. Because of the $\mathcal{I}^K$-compactness of A_1 and A_2, there exists a finite subcover $\mathcal{C}_1 = \{U_i : i = 1, 2, \ldots, n\}$ and $\mathcal{C}_2 = \{U_j : j = 1, 2, \ldots, m\}$ of $\mathcal{C}$ that covers A_1 and A_2, respectively. Then, the collection $\mathcal{C}_1 \cup \mathcal{C}_2$ is a finite subcover of $A_1 \cup A_2$. □

Theorem 2.5. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and let (Ω, τ) be a space. If A_1 and A_2 are sequentially $\mathcal{I}^K$-compact sets, then $A_1 \cup A_2$ is sequentially $\mathcal{I}^K$-compact.*

Proof. Let $\vec{t} = (t_n) \subset A_1 \cup A_2$ be any arbitrary sequence. Then, at least an infinite number of terms of the sequence $\vec{t}$ are in the set A_1 (or A_2). Since A_1 and A_2 are sequentially $\mathcal{I}^K$-compact, then there exists an $\mathcal{I}^K$-convergence subsequence (t_{n_k}) of $\vec{t}$ in A_1 (or in A_2), which $\mathcal{I}^K$-converges to some point of $A_1 \cup A_2$. □

Corollary 2.6. *The finite union of $\mathcal{I}^K$-compact (sequentially $\mathcal{I}^K$-compact) sets is $\mathcal{I}^K$-compact (sequentially $\mathcal{I}^K$-compact), respectively.*

Theorem 2.7. *Arbitrary union of locally $\mathcal{I}^K$-compact space is locally $\mathcal{I}^K$-compact.*

Proof. The proof is clear from the definition of locally $\mathcal{I}^K$-compactness. $\square$

Theorem 2.8. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and let (Ω, τ) be a space. Then, (i) Every $\mathcal{K}$-compact space is $\mathcal{I}^K$-compact. (ii) Every sequentially $\mathcal{K}$-compact space is sequentially $\mathcal{I}^K$-compact.*

Proof. (i) Let Ω be a $\mathcal{K}$-compact space. Then, for any $\mathcal{K}$-open cover of Ω, there exists a finite subcover. Now, consider an arbitrary family of $\mathcal{I}^K$-open sets $\{U_\alpha\}_{\alpha\in\Lambda}$ as a cover of Ω. Then, by Remark 1.7, it is clear that $\{U_\alpha\}_{\alpha\in\Lambda}$ is a $\mathcal{K}$-open cover, and by the $\mathcal{K}$-compactness of Ω there exists a finite subcover.

(ii) Let Ω be a sequentially $\mathcal{K}$-compact space. Let $\vec{t} = (t_n) \subset \Omega$ be a sequence. Then there exists a subsequence (t_{n_k}) of $\vec{t}$ such that $\mathcal{K}$-converges to somepoint $t \in \Omega$. And by [12] the sequence $\vec{t}$ is $\mathcal{I}^K$-convergent to $t \in \Omega$. $\square$

Remark 2.9. *By (Proposition 2 in [1]), if $\mathcal{I}$ is an admissible ideal, then every open subset of space Ω is $\mathcal{I}$-open. So, every $\mathcal{I}$-compact space is also a compact space.*

The following example shows that every sequentially $\mathcal{I}$-compact space is not sequentially $\mathcal{I}^K$-compact.

Example 2.10. Let $\Omega = [0, 1]$ be equipped with the standard topology inherited from $\mathbb{R}$. Consider the set $\mathbb{N}$, which may be decomposed as the union of infinitely many subsets Δ_j. Each Δ_j is infinite, and the intersection between any two different subsets Δ_i and Δ_j is empty. Let

$$\mathcal{I} = \{A \subset \mathbb{N} : A \cap \Delta_i \text{ is finite, for all but finitely many } i\text{'s}\} \cup Fin$$

and

$$\mathcal{K} = \mathcal{K}_d = \{A \subset \mathbb{R} : \delta(A) = 0\}.$$

Here $\delta(A)$ shows the asymptotic density of A, (see [22]). Clearly, the ideals $\mathcal{I}$ and $\mathcal{K}$ are non-trivial and admissible. We claim that Ω is sequentially $\mathcal{I}$-compact. Let $(t_n) \subset \Omega$ be a sequence. Then, by the Bolzano–Weierstrass theorem, it has a $\mathcal{I}$-convergent subsequence (t_{n_k}) such that $(t_{n_k}) \to t,\ t \in \Omega$. As the ideal $\mathcal{I}$ is admissible by (Proposition 1 in [1]) $t_{n_k} \xrightarrow{\mathcal{I}} t$. Therefore, Ω is sequentially $\mathcal{I}$-compact. But it is not a sequentially $\mathcal{I}^K$-compact space. Now consider the sequence $(t_n) = \left(\frac{1}{n}\right)$. It is clear that in the decomposition of $\mathbb{N}$, each $\Delta_i \in \mathcal{I}$. Let (y_n) be a sequence defined as $y_n = t_j$ when $n \in \Delta_j$. The sequence (y_n) is not $\mathcal{I}^K$-convergent. Assume that (y_n) is $\mathcal{I}^K$-convergent to zero. Then, there exists $M \in \mathcal{F}(\mathcal{I})$ such that the following sequence

$$s_n := \begin{cases} y_n, & n \in M, \\ 0, & n \notin M, \end{cases}$$

is $\mathcal{K}$-convergent to 0. So, for any neighborhood U of zero, $\{n \in \mathbb{N} : t_n \notin U\} \in \mathcal{K}$. This implies that $\{n \in M : t_n \notin U\} \in \mathcal{K}$. But $\delta(\{n \in \mathbb{N} : t_n \notin U\}) \neq 0$ because $M \in \mathcal{F}(\mathcal{I})$ implies that there exists $H \in \mathcal{I}$ such that $M = \mathbb{N} - H$. So there exists $l \in \mathbb{N}$ and assume that $H \subset \Delta_1 \cup \Delta_2 \cup \ldots \cup \Delta_l$, and for all $i > l + 1$, there exist infinitely many terms of y_n that are equal to t_i.

Definition 2.11. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and (Ω, τ) and (Ω', ρ) as spaces. A function f from the space Ω to Ω' is said to be $\mathcal{I}^K$-continuous if it provides the inverse image of any $\mathcal{I}^K$-open subset of Ω' is $\mathcal{I}^K$-open in Ω.*

Theorem 2.12. *(i) The image of $\mathcal{I}^K$-compact space under $\mathcal{I}^K$-continuous function is $\mathcal{I}^K$-compact, (ii) The image of sequentially-$\mathcal{I}^K$-compact space under $\mathcal{I}^K$-continuous function is sequentially-$\mathcal{I}^K$-compact.*

Proof. (i) Let us assume that $f : \Omega \to \Omega'$ be an $\mathcal{I}^K$-continuous function and Ω be an $\mathcal{I}^K$-compact space. Take $\{U_\alpha\}_{\alpha\in\Lambda}$ be an $\mathcal{I}^K$-open cover of $f(\Omega)$. Therefore, $f(\Omega) \subset \bigcup_{\alpha\in\Lambda} U_\alpha$ implies that

$$\Omega \subset \bigcup_{\alpha\in\Lambda} f^{-1}(U_\alpha).$$

Since Ω is $\mathcal{I}^K$-compact and f is $\mathcal{I}^K$-continuous function, then we have

$$\Omega \subset \bigcup_{\alpha=1}^{n} f^{-1}(U_\alpha) = f^{-1}\left(\bigcup_{\alpha=1}^{n} U_\alpha\right)$$

and this implies that $f(\Omega) \subset \bigcup_{\alpha=1}^{n} U_\alpha$ holds.

(ii) Let Ω be a sequentially $\mathcal{I}^K$-compact space. Let $(y_n) \subset f(\Omega)$ be an arbitrary sequence. Then, there exists a sequence $(t_n) \subset \Omega$ such that $y_n = f(t_n)$ for all n. Since Ω is sequentially $\mathcal{I}^K$-compact, then there is an $\mathcal{I}^K$-convergence subsequence (t_{n_k}) of (t_n) such that $t_{n_k} \xrightarrow{\mathcal{I}^K} t$. Since every $\mathcal{I}^K$-continuous function is a sequentially $\mathcal{I}^K$-continuous function, then $f(t_{n_k}) \xrightarrow{\mathcal{I}^K} f(t)$ which completes the proof. $\square$

Theorem 2.13. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$. Let (Ω, τ) be a space. Assume that a sequence $(t_n) \subset \Omega$ is $\mathcal{I}^K$-convergent to a point t. Then, following set*

$$A = \{t_n : n \in \mathbb{N}\} \cup \{t\}$$

is sequentially $\mathcal{I}^K$-compact.

Proof. Let $(y_n) \subset A$ be a sequence. Case 1: If there are infinitely many terms of (y_n) that are equal to t, we can take the subsequence $(y_{n_k}) = t$, and it is $\mathcal{I}^K$-convergent to $x \in \Omega$. Case 2: Let finitely many terms of y_n be equal to t. Then infinitely many terms of (y_n) are equal to terms in (t_n). Take $y_{n_k} = y_n$ where $y_n \neq t$, then $(y_{n_k}) = (t_n)$, which is $\mathcal{I}^K$-converges to $x \in \Omega$. And in case (3), if the infinite number of terms of the sequence is equal to t_n and also equal to x, clearly there exists $\mathcal{I}^K$-convergent subsequence which is converging to some point of set A. □

Definition 2.14. *Consider two $\mathcal{I}^K$-sequential spaces denoted as $(\Omega, \tau_{\mathcal{I}^K})$ and $(\Omega', \rho_{\mathcal{I}^K})$. Then, the $\mathcal{I}^K$-product topology on $\Omega \times \Omega'$ is the topology having as basis the collection*

$$\mathcal{B} = \{U \times V : U \in \tau_{\mathcal{I}^K}, V \in \rho_{\mathcal{I}^K}\}.$$

Lemma 2.15. *Consider two sequential spaces denoted by $(\Omega, \tau_{\mathcal{I}^K})$ and $(\Omega', \rho_{\mathcal{I}^K})$. Then the projection maps $\pi_j : \Omega \times \Omega' \to \Omega,\ j = 1, 2$ are $\mathcal{I}^K$-continuous.*

Proof. The proof is evident based on the concept of function $\mathcal{I}^K$-continuity, hence it is omitted. □

Theorem 2.16. *If Ω and Ω' are spaces, then (i) $\Omega \times \Omega'$ is $\mathcal{I}^K$-compact if and only if Ω and Ω' are both $\mathcal{I}^K$-compact. (ii) The Cartesian product of Ω and Ω' is sequentially $\mathcal{I}^K$-compact if and only if Ω and Ω' are both sequentially $\mathcal{I}^K$-compact.*

Proof. (i) Let $\Omega \times \Omega'$ be an $\mathcal{I}^K$-compact space. Since the projection maps π_1 and π_2 are $\mathcal{I}^K$-continuous maps, Ω and Ω' are $\mathcal{I}^K$-compact spaces.

Conversely, let Ω and Ω' be $\mathcal{I}^K$-compact spaces. Consider an arbitrary $\mathcal{I}^K$-open cover $\{U_i \times V_j,\ i, j \in \lambda\}$ of $\Omega \times \Omega'$ such that $\Omega \times \Omega' \subset \bigcup_{i,j\in\lambda} U_i \times V_j$ holds. So, for all $(x, y) \in \Omega \times \Omega'$, there exists $m, n \in \lambda$ such that $x \in U_n$ and $y \in V_m$. Hence, $\Omega \subset \bigcup_{n\in\lambda} U_n$ and $\Omega' \subset \bigcup_{m\in\lambda} V_m$ are satisfied. From the $\mathcal{I}^K$-compactness of Ω and Ω', we have $\Omega \subset \bigcup_{n=1}^{n_0} U_n$ and $\Omega' \subset \bigcup_{m=1}^{m_0} V_m$. So, $\Omega \times \Omega' \subset \bigcup_{n=1}^{n_0} \bigcup_{m=1}^{m_0} U_n \times V_m$ holds.

(ii) Let Ω and Ω' be sequential $\mathcal{I}^K$-compact spaces. Let $\vec{x} = ((t_n, y_n)) \subset \Omega \times \Omega'$ be a sequence, then $\pi_1(\vec{x}) = (t_n)$ and $\pi_2(\vec{x}) = (y_n)$ are in Ω and Ω', respectively. Since Ω and Ω' are sequentially $\mathcal{I}^K$-compact spaces, there exists $\mathcal{I}^K$-convergent sub-sequences (t_{n_k}) of (t_n) and (y_{m_j}) of (y_n), which are $\mathcal{I}^K$-converge to some point $t \in \Omega$ and $y \in \Omega'$. So, there exists $M_1, M_2 \in \mathcal{F}(\mathcal{I})$ such that the sequences

$$y_n^1 = \begin{cases} t_{n_k}, & n_k \in M_1, \\ t, & n_k \notin M_1, \end{cases}$$

is $\mathcal{K}$-converges to t and the sequence

$$y_n^2 = \begin{cases} y_{m_j}, & m_j \in M_2, \\ y, & m_j \notin M_2, \end{cases}$$

is $\mathcal{K}$-converges to y. So, for any neighborhood U^1 of t and for any neighborhood U^2 of y

$$\{n_k \in M_1 : t_{n_k} \in U^1\} \in \mathcal{F}(\mathcal{K}) \quad \text{and} \quad \{m_j \in M_2 : y_{m_j} \in U^2\} \in \mathcal{F}(\mathcal{K}).$$

Let $M = M_1 \cap M_2$. Define a sequence

$$t_n = \begin{cases} (t_{n_k}, y_{m_k}), & n_k \in M, \\ (t, y), & n_k \notin M. \end{cases}$$

Let U be any $\mathcal{I}^K$-open set that contains (t, y). So, there exists $B_1 \times B_2 \in \mathcal{B}$ such that $B_1 \times B_2 \subset U$ and $(t, y) \in B_1 \times B_2$. Hence,

$$\{(n_k) \in M : (t_{n_k}, y_{m_k}) \notin U\} \subset \{(n_k, m_j) \in M : t_{n_k}, y_{m_j} \notin B_1 \times B_2\} \subset \{(n_k \in M_1 : t_{n_k} \notin B_1) \cup (m_j \in M$$

is satisfied. Therefore, the sequence $\mathcal{K}$-converges to (t, y). This implies that $\Omega \times \Omega'$ is sequentially $\mathcal{I}^K$-compact.

The converse of the proof is clear by the $\mathcal{I}^K$-continuity of the projection mappings. □

Definition 2.17. *Consider $\mathcal{I}$ and $\mathcal{K}$ as arbitrary ideals of $\mathbb{N}$. Let $(\Omega_j, \tau^j_{\mathcal{I}^K})_{j \in \Lambda}$ be a family of $\mathcal{I}^K$-sequential spaces. Let $\Omega = \prod_{j \in \Lambda} \Omega_j$. The $\mathcal{I}^K$-product space is defined as the product set Ω equipped with a topology $\tau_{\mathcal{I}^K}$, having as its basis the family*

$$\mathcal{B} = \left\{ \prod_{j\in\Lambda} O_j : O_j \in \tau^j_{\mathcal{I}^K} \text{ and } O_j = \Omega_j \text{ for all but finite numbers of } j \right\}.$$

Proposition 2.18. *Let $(\Omega_j, \tau^j_{\mathcal{I}^K})_{j\in\Lambda}$ be a family of $\mathcal{I}^K$-sequential spaces. Then, the projection maps $\pi_i : \prod_{j\in\Lambda} \Omega_j \to \Omega_i$ are $\mathcal{I}^K$-continuous maps.*

Proof. The proof is self-evident based on the definition of $\mathcal{I}^K$-continuous maps and projection maps, so it is omitted. □

Theorem 2.19. *Let $(\Omega_j, \tau^j_{\mathcal{I}^K})_{j\in\Lambda}$ be a family of $\mathcal{I}^K$-sequential spaces. Then, the $\mathcal{I}^K$-product space Ω is $\mathcal{I}^K$-compacted iff the set Ω_i is compacted for all $i \in \Lambda$.*

Proof. We will prove the theorem by the fact that a sequential space Ω is $\mathcal{I}^K$-compact iff every family $\mathcal{E}$ of $\mathcal{I}^K$-closed subset of Ω with the finite intersection property (F.I.P.) satisfy $\bigcap_{E\in\mathcal{E}} E \neq \varnothing$.

Let Ω_i be an $\mathcal{I}^K$-compact space for each i. Let $\mathcal{E}$ be a family of $\mathcal{I}^K$-closed subsets of Ω with the F.I.P.; we prove that $\bigcap_{E\in\mathcal{E}} E \neq \varnothing$.

By using Zorn's Lemma, it can be shown that there exists a maximal family $\mathcal{G}$ subsets of Ω (not necessary $\mathcal{I}^K$-closed) that contain $\mathcal{E}$ and have the F.I.P. We will demonstrate that $\bigcap_{G\in\mathcal{G}} G \neq \varnothing$. This implies the desired result, as each $E \in \mathcal{E}$ is closed.

If $G_1, G_2, \ldots, G_n \in \mathcal{G}$, for any $n \in \mathbb{N}$, then the set $G' = G_1 \cap G_2 \cap \ldots \cap G_n \in \mathcal{G}$. Assuming that this is not true, we can conclude that the set $G' = G \cup \{G'\}$ properly contains $\mathcal{G}$, has the F.I.P., and contains $\mathcal{E}$. This contradicts the maximality of $\mathcal{G}$.

Consider S, a subset of Ω, which has intersection with every element in $\mathcal{G}$. We assert that the set $\mathcal{G} \cup \{S\}$ possesses the F.I.P. To see this, let $G'_1, G'_2, \ldots, G'_m$ be members of $\mathcal{G}$. By previous paragraph, $G'_1 \cap G'_2 \cap \ldots \cap G'_m \in \mathcal{G}$, and by assumption, $S \cap (G'_1 \cap G'_2 \cap \ldots \cap G'_m) \neq \varnothing$, hence $\mathcal{G} \cup \{S\}$ has the F.I.P. and contains $\mathcal{E}$. By utilizing the properties of $\mathcal{G}$ being maximum, having a F.I.P., and including $\mathcal{E}$, it becomes evident that S belongs to $\mathcal{G}$. Fix i and let $P_i : \Omega \to \Omega_i$ be the projection maps, then the family $\{P_i(G) : G \in \mathcal{G}\}$ has the F.I.P. As (Ω_i, τ_i) is $\mathcal{I}^K$-compact, $\bigcap_{G\in\mathcal{G}} P_i(G) \neq \varnothing$. Select $x_i \in \bigcap_{G\in\mathcal{G}} P_i(G)$, then for each i we can find a point $x_i \in \bigcap_{G\in\mathcal{G}} P_i(G)$, put $x = \prod_i x_i \in \Omega$.

We shall prove that $x \in \bigcap_{G\in\mathcal{G}} G$. Let O be any $\mathcal{I}^K$-open set containing x. Then O contains a basic $\mathcal{I}^K$-open set about x of the form $\bigcap_{i\in J} P_i^{-1}(W_i)$, where $W_i \in \tau_i$, $x_i \in W_i$, and J is a finite subset of Λ. As $x_i \in P_i(G)$, $W_i \cap P_i(G) \neq \varnothing$, for all $G \in \mathcal{G}$, thus $P_i^{-1}(W_i) \cap G \neq \varnothing$ for all $G \in \mathcal{G}$. By observation above, this implies that $P_i^{-1}(W_i) \in \mathcal{G}$, $\forall i \in J$. As $\mathcal{G}$ has F.I.P., $\bigcap_{i\in J} P_i^{-1}(W_i) \cap H \neq \varnothing$ for all $G \in \mathcal{G}$, so $O \cap G \neq \varnothing$ for all $G \in \mathcal{G}$, hence $x \in \bigcap_{G\in\mathcal{G}} G$, as required.

Conversely, let the product space Ω be $\mathcal{I}^K$-compact. Since the projection maps p_i are $\mathcal{I}^K$-continuous maps and they map $\mathcal{I}^K$-compact space into $\mathcal{I}^K$-compact spaces, so Ω_i is $\mathcal{I}^K$-compact for all $i \in \Lambda$. □

3 $\mathcal{I}^K$-Connectedness

In this section, the idea of connectedness of a space has been generalized to $\mathcal{I}^K$-connectedness.

Definition 3.1. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and let $(\Omega, \mathcal{T})$ be a space. The subsets U and V of Ω are said to be $\mathcal{I}^K$-semi separated if*

$$cl_{\mathcal{I}^K}(U) \cap V = \varnothing = U \cap cl_{\mathcal{I}^K}(V).$$

Lemma 3.2. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$. Let $(\Omega, \mathcal{T})$ be a space and Ω' be a subspace of Ω. Then, any subsets $U, V \subset \Omega'$ are $\mathcal{I}^K$-semi separated in Ω iff they are $\mathcal{I}^K$-semi separated in Ω'.*

Proof. Let U and V be $\mathcal{I}^K$-semi separated in Ω. Then,

$$cl_{\mathcal{I}^K}(U) \cap V = \varnothing = U \cap cl_{\mathcal{I}^K}(V)$$

holds. So, $cl_{\mathcal{I}^K}(U) \cap V = cl_{\mathcal{I}^K}(U) \cap \Omega' \cap V = \varnothing$. Similarly, we have

$$cl_{\mathcal{I}^K}(V) \cap U = cl_{\mathcal{I}^K}(V) \cap \Omega' \cap U = \varnothing.$$

Conversely, let U and V be $\mathcal{I}^K$-semi separated in Ω'. Then,

$$cl_{\mathcal{I}^K}(U) \cap V = cl_{\mathcal{I}^K}(U) \cap U.$$

Therefore, $cl_{\mathcal{I}^K}(U) \cap \Omega' \cap V = \varnothing$. Since $\Omega' \neq \varnothing$, then $cl_{\mathcal{I}^K}(U) \cap V = \varnothing$. Similarly, we can show that $cl_{\mathcal{I}^K}(V) \cap U = \varnothing$. □

Lemma 3.3. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and $(\Omega, \mathcal{T})$ and $(\Omega', \mathcal{T}')$ as spaces. Let f be a function from X to Y that is $\mathcal{I}^K$-continuous. If A and B are $\mathcal{I}^K$-semi separated subsets of Ω', then the preimage of A and B under f are $\mathcal{I}^K$-semi separated subsets of Ω.*

Proof. Let $A, B \subset \Omega'$ be $\mathcal{I}^K$-semi separated. Then, we have

$$cl_{\mathcal{I}^K}(A) \cap B = \varnothing = A \cap cl_{\mathcal{I}^K}(B)$$

and we are going to prove,

$$f^{-1}(A) \cap cl_{\mathcal{I}^K}(f^{-1}(B)) = \varnothing = f^{-1}(B) \cap cl_{\mathcal{I}^K}(f^{-1}(A))$$

holds. Let $x \in f^{-1}(A) \cap cl_{\mathcal{I}^K}(f^{-1}(B))$. Then, $x \in f^{-1}(A)$ and $x \in cl_{\mathcal{I}^K}(f^{-1}(B))$. So, $f(x) \in A$ and there exists a sequence $(t_n) \subset f^{-1}(B)$ such that $\mathcal{I}^K$-converges to x. Since f is an $\mathcal{I}^K$-continuous function, the sequence $(f(t_n))$ $\mathcal{I}^K$ converges to $f(x)$ in Ω'. Because of $(t_n) \subset B$, $f(x) \in cl_{\mathcal{I}^K}(B)$. Therefore, $f(x) \in A \cap cl_{\mathcal{I}^K}(B)$. But A and B are $\mathcal{I}^K$-semi separated subsets of Ω, which is a contradiction. Hence $f^{-1}(A) \cap cl_{\mathcal{I}^K}(f^{-1}(B)) = \varnothing$. Similarly, we can prove that $f^{-1}(B) \cap cl_{\mathcal{I}^K}(f^{-1}(A)) = \varnothing$. □

Definition 3.4. *Consider $\mathcal{I}$ and $\mathcal{K}$ as ideals of $\mathbb{N}$, and let $(\Omega, \mathcal{T})$ be a space. (i) A set $C \subset \Omega$ is considered $\mathcal{I}^K$-connected if it cannot be expressed as the combination of two $\mathcal{I}^K$-semi-separated sets. (ii) The space Ω is considered $\mathcal{I}^K$-connected if there are no subsets U and V of Ω that are semi-separated and satisfy $\Omega = U \cup V$.*

Definition 3.5. *Consider a space $(\Omega, \mathcal{T})$. The component $C_{\mathcal{I}^K}(t)$ of t in Ω is the union of all $\mathcal{I}^K$-connected subsets of Ω containing t.*

Definition 3.6. *A space $(\Omega, \mathcal{T})$ is totally $\mathcal{I}^K$-disconnected if the component $C_{\mathcal{I}^K}(t)$ consists just of the element $\{t\}$ for all $t \in \Omega$.*

Theorem 3.7. *Consider a space $(\Omega, \mathcal{T})$. A subset $Y \subset \Omega$ is $\mathcal{I}^K$-connected if and only if for every subset A of Y,*

$$cl_{\mathcal{I}^K}(A) \cap (Y - A) \ \cup \ cl_{\mathcal{I}^K}(Y - A) \cap A \neq \varnothing$$

Proof. Let $Y \subseteq \Omega$ be an $\mathcal{I}^K$-connected set. Let

$$[cl_{\mathcal{I}^K}(A) \cap (Y - A)] \cup [cl_{\mathcal{I}^K}(Y - A) \cap A] = \varnothing$$

then

$$cl_{\mathcal{I}^K}(A) \cap (Y - A) = \varnothing \quad \wedge \quad cl_{\mathcal{I}^K}(Y - A) \cap A = \varnothing$$

Theorem 3.8. *Consider a space $(\Omega, \mathcal{T})$. The following assertions are equivalent: (i) Ω is $\mathcal{I}^K$-connected. (ii) The only subsets of Ω that are simultaneously $\mathcal{I}^K$-open and $\mathcal{I}^K$-closed are Ω itself and the empty set. (iii) If the $\mathcal{I}^K$-discrete space Ω' has more than one point, there isn't a non-constant $\mathcal{I}^K$-continuous function that maps Ω to Ω'.*

Proof. $(i) \Rightarrow (ii)$: Consider a space Ω that is $\mathcal{I}^K$-connected. Consider a valid subset D of Ω that is simultaneously $\mathcal{I}^K$-open and $\mathcal{I}^K$-closed. There are two possibilities: (1) D is $\mathcal{I}^K$-open. Then, $\Omega - D$ is $\mathcal{I}^K$-closed and $\Omega - D = cl_{\mathcal{I}^K}(\Omega - D)$. Then $D = cl_{\mathcal{I}^K}(D)$. (2) D is $\mathcal{I}^K$-closed. Then $D = cl_{\mathcal{I}^K}(D)$.

Given that both instance (1) and (2) occur simultaneously. So D and $\Omega - D$ are $\mathcal{I}^K$-semi separated subsets of Ω such that their union is Ω. So Ω is $\mathcal{I}^K$-disconnected, which is a contradiction.

$(ii) \Rightarrow (iii)$: Consider a set Ω'' that is an $\mathcal{I}^K$-discrete space with more than one element. Let $f : \Omega \to \Omega''$ be a function that is $\mathcal{I}^K$-continuous. Let's assume that f is not constant. There is a subset D of Ω such that $f(D)$ is equal to a in Ω'', and $f(D^c)$ is equal to b in Ω''. Given that Ω'' is an $\mathcal{I}^K$-discrete space, it follows that both $\{a\}$ and $\{b\}$

are $\mathcal{I}^K$-open

$$cl_{\mathcal{I}^K}(D) = cl_{\mathcal{I}^K} f^{-1}(\{a\}) = f^{-1}(cl_{\mathcal{I}^K}(\{a\})) = f^{-1}(\{a\}) = D.$$

This indicates that D is also $\mathcal{I}^K$-closed under the operation in Ω. Which is a contradiction.

$(iii) \Rightarrow (i)$: Let Ω'' be $\mathcal{I}^K$-discreet space consisting of more than one element. Suppose that Ω is an $\mathcal{I}^K$-disconnected space. Then, there exists $\mathcal{I}^K$-semi-separated subsets D and E of Ω such that $\Omega = D \cup E$. If we consider $E = \Omega - D$, we have $cl_{\mathcal{I}^K}(\Omega - D) \cap D = \varnothing$ and also $\Omega - D \cap cl_{\mathcal{I}^K}(D) = \varnothing$. So $cl_{\mathcal{I}^K}(D) \subset D$ and $cl_{\mathcal{I}^K}(\Omega - D) \subset \Omega - D$. Therefore, $cl_{\mathcal{I}^K}(D) = D$ and $cl_{\mathcal{I}^K}(\Omega - D) = \Omega - D$. Without loss of generality, take $\Omega'' = \{0, 1\}$ with $cl_{\mathcal{I}^K}(\varnothing) = \varnothing$, $cl_{\mathcal{I}^K}(\Omega'') = \Omega''$, $cl_{\mathcal{I}^K}(\{0\}) = \{0\}$, and $cl_{\mathcal{I}^K}(\{1\}) = \{1\}$. Define a function $f : \Omega \to \Omega''$ as $f(D) = \{0\}$ and $f(E) = \{1\}$.

We argue that f is an $\mathcal{I}^K$-continuous function. Let $\phi \neq C \subset \Omega''$. In order to observe this, we have three cases as follows: (i) If $C = \Omega''$, then $f^{-1}(C) = \Omega$ holds. So, $cl_{\mathcal{I}^K}(f^{-1}(C)) = cl_{\mathcal{I}^K}(\Omega) = \Omega = f^{-1}(cl_{\mathcal{I}^K}(C))$. (ii) If $C = \{0\}$, then $f^{-1}(C) = A$ holds. So, $cl_{\mathcal{I}^K}(f^{-1}(\{0\})) = cl_{\mathcal{I}^K}(D) = D = f^{-1}(cl_{\mathcal{I}^K}(\{0\}))$. (iii) If $C = \{1\}$, then $f^{-1}(C) = E$ holds. So, $cl_{\mathcal{I}^K}(f^{-1}(\{1\})) = cl_{\mathcal{I}^K}(E) = D = f^{-1}(cl_{\mathcal{I}^K}(\{1\}))$.

Thus, f is an $\mathcal{I}^K$-continuous function but not constant. Hence, Ω is an $\mathcal{I}^K$-connected space. □

Theorem 3.9. *If M is an $\mathcal{I}^K$-connected subset of a space Ω and S and W are $\mathcal{I}^K$-semi-separated subsets of Ω. Then, either $M \subset S$ or $M \subset W$.*

Proof. The proof is evident based on the concept of $\mathcal{I}^K$-connectedness and the definition of $\mathcal{I}^K$-semi-separated sets. Therefore, it is excluded from discussion. □

Theorem 3.10. *Let M be an $\mathcal{I}^K$-connected subset of a space $(\Omega, \mathcal{T})$ and $M \subset N \subset cl_{\mathcal{I}^K}(M)$. Then, N is $\mathcal{I}^K$-connected.*

Proof. Assume that N is not $\mathcal{I}^K$-connected. Then there exists $\mathcal{I}^K$-semi-separated subsets S and W of N such that $N = S \cup W$. Since

M is $\mathcal{I}^K$-connected, then either $M \subset S$ or $M \subset W$. Suppose that $M \subset S$. Then $cl_{\mathcal{I}^K}(M) \subset S$ and $W \cap cl_{\mathcal{I}^K}(M) = \varnothing$. But by hypothesis, $W \subset N \subset cl_{\mathcal{I}^K}(M)$, and $W = cl_{\mathcal{I}^K}(M) \cap W = \varnothing$, which is a contradiction since $W \neq \varnothing$. □

Corollary 3.11. *If M is an $\mathcal{I}^K$-connected subset of a space $(\Omega, \mathcal{T})$, then $cl_{\mathcal{I}^K}(M)$ is $\mathcal{I}^K$-connected.*

Theorem 3.12. *Let M and N are subsets of an $\mathcal{I}^K$-connected space $(\Omega, \mathcal{T})$. If M and N are $\mathcal{I}^K$-connected but not $\mathcal{I}^K$-semi-separated, then $M \cup N$ is $\mathcal{I}^K$-connected.*

Proof. Assume that $E = M \cup N$ is not $\mathcal{I}^K$-connected. Then there exists $\mathcal{I}^K$-semi-separated subsets C and D of Ω such that $E = C \cup D$. $M \subset C \cup D$, so by Theorem 3.9 either $M \subset C$ or $M \subset D$. Similarly, we can say that either $N \subset C$ or $N \subset D$. If $M \subset C$ and also $N \subset C$, then $M \cup N \subset C$ and $D = \varnothing$, which is not the case. Thus

$$(M \subset C \ \wedge \ N \subset D) \quad \text{or} \quad (M \subset D \ \wedge \ N \subset C).$$

In the first case,

$$cl_{\mathcal{I}^K}(M) \cap N \subset cl_{\mathcal{I}^K}(C) \cap D = \varnothing$$

and

$$cl_{\mathcal{I}^K}(N) \cap M \subset cl_{\mathcal{I}^K}(D) \cap C = \varnothing$$

Likewise, in the second situation, we can have the identical outcome. Thus, it follows that M and N are $\mathcal{I}^K$-semi-separated in Ω, which contradicts the initial statement. □

Theorem 3.13. *If $\{M_i : i \in \lambda\}$ be a non-empty family of $\mathcal{I}^K$-connected subsets of space Ω, such that $\bigcap_{i\in\lambda} M_i \neq \varnothing$. Then, $\bigcup_{i\in\lambda} M_i$ is $\mathcal{I}^K$-connected.*

Proof. Let $T = \bigcup_{i\in\lambda} M_i$. Assume that T is an $\mathcal{I}^K$-disconnected subset of Ω. Then $T = S \cup W$, where S and W are $\mathcal{I}^K$-semi-separated sets in

Ω. Since $\bigcap_{i\in\lambda} M_i \neq \varnothing$, we choose a point $x \in \bigcap_{i\in\lambda} M_i$. Then $x \in T$. So either $x \in S$ or $x \in W$. Suppose that $x \in S$. Since $x \in M_i$, for each $i \in \lambda$ and $(M_i \cup S \neq \varnothing, \forall i \in \lambda)$, so M_i must be either in S or in W. Since S and W are disjoint, so $M_i \subset S$ for all $i \in \lambda$. Hence, $T \subset S$. This means that $W = \varnothing$, which is a contradiction. □

Theorem 3.14. *$\mathcal{I}^K$-continuous image of $\mathcal{I}^K$-connected space is $\mathcal{I}^K$-connected.*

Proof. Let Ω be an $\mathcal{I}^K$-connected space and Ω''' be an $\mathcal{I}^K$-discreet space that has more than one element. Then, any $\mathcal{I}^K$-continuous function from Ω to Ω''' is constant.

Let $g : f(\Omega) \to \Omega'''$ be an $\mathcal{I}^K$-continuous function. Since f and g are both $\mathcal{I}^K$-continuous functions, then $g \circ f : \Omega \to \Omega'''$ is an $\mathcal{I}^K$-continuous function, and by the $\mathcal{I}^K$-connectedness of Ω, $g \circ f$ is constant. This implies that g is constant. Hence, $f(\Omega)$ is $\mathcal{I}^K$-connected. □

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Hassina Sabor Behmanush
Ph.D. Scholar Department of Mathematics
Faculty of Science, Mersin University,
Mersin, Turkiye
E-mail: h.s.behmsnush1989@gmail.com