Journal of Mathematical Extension
Vol. 18, No. 10, (2024) (6)1-14
URL: https://doi.org/10.30495/JME.2024.3074
ISSN: 1735-8299
Original Research Paper

Norm of Difference of General Polynomial Weighted Differentiation Composition Operators from Cauchy Transform Spaces into Derivative Hardy Spaces

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Abstract. In this paper, we study the boundedness of the difference of general polynomial weighted differentiation composition operators from the Cauchy transform spaces into the function spaces $S = \{f : \quad f' \in H^1\}$ and $S^2 = \{f : \quad f' \in H^2\}$ with derivative in Hardy spaces. We also derive an exact formula for the norm of these operators. Furthermore, as a corollary, we will prove that there is no composition isometry from the Cauchy transform spaces into the spaces S and S^2.

AMS Subject Classification: 47B38; 30H10; 30H99
Keywords and Phrases: Boundedness, Cauchy transform space, isometry, norm

Received: July 2024; Accepted: February 2025
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1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. For $0 < p < \infty$, a function $f \in H(\mathbb{D})$ is said to belong to the Hardy space H^p if

$$\|f\|_{H^p} = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \Big)^{\frac{1}{p}} < \infty.$$

When $1 \leq p < \infty$, H^p is a Banach space with the norm $\| \cdot \|_p$.

The space S^p consists of all functions $f \in H(\mathbb{D})$ such that $f' \in H^p$. This space is Banach space with the following norm

$$\|f\|_{S^p} = |f(0)| + \|f'\|_{H^p}.$$

The space of Cauchy transform functions can be viewed as a connection between analytic function theory and measure theory. If f is analytic in $\overline{\mathbb{D}}$, then using the Cauchy formula, we have

$$f(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z} d\zeta, \qquad z \in \mathbb{D},$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. The above formula is a special case of the Cauchy transform: A function f is Cauchy transform or $f \in \mathcal{F}$ if it has a representation as

$$f(z) = \int_{\mathbb{T}} \frac{1}{1 - \bar{\zeta} z} d\mu(\zeta), \qquad z \in \mathbb{D},$$

where $\mu \in \mathcal{M}$, $\mathcal{M}$ is the space of all complex valued Borel measures on $\mathbb{T}$ with the total variation norm.

For $\varphi \in \mathcal{S}(\mathbb{D}) = \{\varphi \in H(\mathbb{D}) : \varphi(\mathbb{D}) \subset \mathbb{D}\}$ and $\psi \in H(\mathbb{D})$, the composition operator C_φ, the multiplication operator M_ψ, and the iterated differentiation operator D^n are defined respectively, as

$$C_\varphi f = f \circ \varphi, \quad M_\psi f = \psi f, \quad D^n f = f^{(n)}, \quad f \in H(\mathbb{D}),$$

where $f^{(0)} = f$. These operators and products of these operators and their differences have been studied extensively on spaces of analytic functions in the past four decades, (see, e.g., [1] -[24] and the related references therein).

Stević et. al. came to an idea of investigating the sums of two product-type operators in [19], [20] and [21]. After publishing [21] in 2015, Stević proposed to study the operator

$$T_{\vec{u},\varphi}f(z) = \sum_{k=0}^{n} u_k(z) f^{(k)}(\varphi)(z),\ f \in H(\mathbb{D}),$$

where $\vec{u} = (u_0, ..., u_n)$ such that $\{u_k\}_{k=0}^{n} \subset H(\mathbb{D})$ and $\varphi \in \mathcal{S}(\mathbb{D})$. Investigations of these types of operators as well as some related ones attracted some attention during the last decade (see, for instance, [1, 2, 3, 4, 9, 15, 17, 22, 24] and the related references therein). Recently Stević [16] introduced general polynomial differentiation composition operator which includes the previous operators and is defined as

$$T^n_{\vec{u},\vec{\varphi}}f(z) = \sum_{j=0}^{n} u_j(z) f^{(j)}(\varphi_j(z)) = \sum_{j=0}^{n} (D^j_{u_j,\varphi_j} f)(z), \quad n \in \mathbb{N}_0,$$

where $u_j \in H(\mathbb{D})$ and $\varphi_j \in S(\mathbb{D})$.

Let $i, n \in \mathbb{N}_0$, $u_i, v_i \in H(\mathbb{D})$ and $\varphi_i, \psi_i \in S(\mathbb{D})$. We set

$$L = \sum_{j=0}^{n} (D^j_{u_j,\varphi_j} - D^j_{v_j,\psi_j}) = T^n_{\vec{u},\vec{\varphi}} - T^n_{\vec{v},\vec{\psi}}.$$

The purpose of this paper is to characterize differences of general polynomial differentiation composition operator L from the Cauchy transform spaces into the spaces S and S^2. Also we will obtain an exact formula for the norm of these operators. Furthermore, as a corollary, we will prove that there is no composition isometry from the Cauchy transform spaces into the spaces $S = \{f : \quad f' \in H\}$ and $S^2 = \{f : \quad f' \in H^2\}$.

2 Main Results

In the following theorem, we consider the boundedness of the operator $L : \mathcal{F} \to S$ and we find the exact formula for the norm of this operator.

Theorem 2.1. *Let $0 \le j \le n$, $\varphi_j, \psi_j \in S(\mathbb{D})$ and $u_j, v_j \in H(\mathbb{D})$. Then the operator $L : \mathcal{F} \to S$ is bounded if and only if*

$$\sup_{\xi \in \mathbb{T}} \left(Q_1(\xi) + Q_2(\xi) \right) < \infty, \tag{1}$$

where

$$Q_1(\xi) = \left| \sum_{j=0}^{n} j! \bar{\xi}^j \left(\frac{u_j(0)}{(1 - \bar{\xi}\varphi_j(0))^{j+1}} - \frac{-v_j(0)}{(1 - \bar{\xi}\psi_j(0))^{j+1}} \right) \right|,$$

$$Q_2(\xi) = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)! \bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1 - \bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} - \frac{E^t_{v_j,\varphi_j}(re^{i\theta})}{(1 - \bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \Big) \Big| d\theta,$$

$E^0_{u_j,\varphi_j}(re^{i\theta}) = u'_j(re^{i\theta})$ *and* $E^1_{u_j,\varphi_j}(re^{i\theta}) = u_j(re^{i\theta})\varphi'(re^{i\theta})$.
Moreover, in this case,

$$\|L\| = \sup_{\xi \in \mathbb{T}} \left(Q_1(\xi) + Q_2(\xi) \right).$$

Proof. For any $\xi \in \mathbb{T}$, the function

$$f_\xi(z) = \frac{1}{1 - \bar{\xi}z}, \qquad z \in \mathbb{D}$$

belong to $\mathcal{F}$ with $\|f_\xi\|_{\mathcal{F}} = 1$ (see[23]) also for any $j \in \mathbb{N}_0$, $f_\xi^{(j)}(z) = \frac{j!\bar{\xi}^j}{(1-\bar{\xi}z)^{j+1}}$. Therefore, for any $\xi \in \mathbb{T}$, we have

$$\begin{aligned} |(Lf_\xi)(0)| &= \left| \sum_{j=0}^{n} (u_j(0) f_\xi^{(j)}(\varphi_j(0)) - v_j(0) f_\xi^{(j)}(\psi_j(0)) \right| \\ &= \left| \sum_{j=0}^{n} j! \bar{\xi}^j \left(\frac{u_j(0)}{(1 - \bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1 - \bar{\xi}\psi_j(0))^{j+1}} \right) \right| \\ &= Q_1(\xi) \end{aligned} \tag{2}$$

and

$$\begin{aligned}\|(Lf_\xi)'\|_{H^1} &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}|(Lf_\xi)'(re^{i\theta})|d\theta\\ &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{j=0}^{n}\sum_{t=0}^{1}E^t_{u_j,\varphi_j}(re^{i\theta})f_\xi^{(j+t)}(\varphi_j(re^{i\theta}))\\ &\quad - E^t_{v_j,\psi_j}(re^{i\theta})f_\xi^{(j+t)}(\psi_j(re^{i\theta}))\Big|d\theta\\ &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\ &\quad - \frac{E^t_{v_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big|d\theta\\ &= Q_2(\xi).\end{aligned}$$

Let $L:\mathcal{F}\to S$ be bounded. Then for any $\xi\in\mathbb{T}$, we get

$$\|L\|\ge\|Lf_\xi\|_{S^1}\ge|(Lf_\xi)(0)|+\|(Lf_\xi)'\|_{H^1}=Q_1(\xi)+Q_2(\xi).$$

Therefore, by taking supremum in the above inequality, we obtain

$$\sup_{\xi\in\mathbb{T}}(Q_1(\xi)+Q_2(\xi))\le\|L\|. \tag{3}$$

Now we assume that the condition of (1) holds. For any $f\in\mathcal{F}$ there exists $\mu\in\mathcal{M}$ such that

$$f(z)=\int_{\mathbb{T}}\frac{d\mu(\xi)}{1-\bar{\xi}z},\qquad z\in\mathbb{D},$$

and $\|f\|_{\mathcal{F}}=\|\mu\|$. Therefore, for each $k\in\mathbb{N}_0$, we have

$$f^{(k)}(z)=k!\int_{\mathbb{T}}\frac{\bar{\xi}^k d\mu(\xi)}{(1-\bar{\xi}z)^{k+1}},\qquad z\in\mathbb{D}.$$

Hence,

$$\begin{aligned}|(Lf)(0)| &= \left|\sum_{j=0}^{n} u_j(0)f^{(j)}(\varphi_j(0)) - v_j(0)f^{(j)}(\psi_j(0))\right| \qquad (4)\\ &= \left|\sum_{j=0}^{n} j!\int_{\mathbb{T}} \bar{\xi}^j\Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big)d\mu(\xi)\right|\\ &\le \int_{\mathbb{T}} \left|\sum_{j=0}^{n} j!\bar{\xi}^j\Big(\frac{u_j(0)}{(1-\bar{\xi}\varphi_j(0))^{j+1}} - \frac{v_j(0)}{(1-\bar{\xi}\psi_j(0))^{j+1}}\Big)\right| d|\mu|(\xi)\\ &= \int_{\mathbb{T}} Q_1(\xi)d|\mu|(\xi),\end{aligned}$$

and by using Fubini's Theorem, we get

$$\begin{aligned}\|(Lf)'\|_{H^1} &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\int_{\mathbb{T}}\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\ &\qquad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)d\mu(\xi)\Big|d\theta\\ &\le \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{T}}\Big|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\ &\qquad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big|d|\mu|(\xi)d\theta\\ &\le \int_{\mathbb{T}}\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\Big|\sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}}\\ &\qquad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big|d\theta d|\mu|(\xi)\\ &\le \int_{\mathbb{T}} Q_2(\xi)d|\mu|(\xi).\end{aligned}$$

By applying the last inequalities, we obtain

$$\begin{aligned}|(Lf)(0)| + \|(Lf)'\|_{H^1} &\leq \int_{\mathbb{T}} (Q_1(\xi) + Q_2(\xi)) d|\mu|(\xi) \\ &\leq \sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_2(\xi)) \int_{\mathbb{T}} d|\mu|(\xi) \\ &\leq \sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_2(\xi)) \|\mu\| \\ &\leq \sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_2(\xi)) \|f\|_{\mathcal{F}}.\end{aligned}$$

Therefore,

$$\|L\| \leq \sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_2(\xi)).$$

By using (3) and the preceding inequality, we have

$$\|L\| = \sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_2(\xi)).$$

The proof is completed. □

In the following theorem, we examine the boundedness of the operator $L : \mathcal{F} \to S^2$ and provide an approximation for the norm of this operator.

Theorem 2.2. *Let $0 \leq j \leq n$, $\varphi_j, \psi_j \in S(\mathbb{D})$ and $u_j, v_j \in H(\mathbb{D})$. Then the operator $L : \mathcal{F} \to S^2$ is bounded if and only if*

$$\sup_{\xi \in \mathbb{T}} \Big(Q_1(\xi) + Q_3(\xi) \Big) < \infty, \tag{5}$$

where

$$\begin{aligned}Q_3(\xi) = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{j=0}^{n} \sum_{0}^{1} (j+t)! \bar{\xi}^{j+t} \big(&\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\ &- \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \big) \Big|^2 d\theta \Big)^{\frac{1}{2}}\end{aligned}$$

and $Q_1(\xi)$, $E^t_{u_j,\varphi_j}$ are defined in Theorem 2.1. Moreover, in this case,

$$\sup_{\xi \in \mathbb{T}} \Big(Q_1(\xi) + Q_3(\xi) \Big) \leq \|L\| \leq \sup_{\xi \in \mathbb{T}} Q_1(\xi) + \sup_{\xi \in \mathbb{T}} Q_3(\xi).$$

Proof. Let the operator $L : \mathcal{F} \to S^2$ be bounded, therefore for any $\xi \in \mathbb{D}$, we have

$$\begin{aligned}
\|(Lf_\xi)'\|_{H^2}^2 &= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \left|(Lf_\xi)'(re^{i\theta})\right|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \Big|\sum_{j=0}^{n} \sum_{t=0}^{1} (j+t)!\bar{\xi}^{j+t} \Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}} \Big) \Big|^2 d\theta \\
&= Q_3(\xi)^2.
\end{aligned}$$

Applying (2) and preceding equation, for any $\xi \in \mathbb{T}$, we obtain

$$\begin{aligned}
Q_1(\xi) + Q_3(\xi) &= |(Lf_\xi)(0)| + \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|(Lf_\xi)'(re^{i\theta})\right|^2 d\theta \right)^{\frac{1}{2}} \\
&\leq \|Lf_\xi\|_{S^2} \\
&\leq \|L\| \|f_\xi\|_{\mathcal{F}} = \|L\|.
\end{aligned}$$

So,

$$\sup_{\xi \in \mathbb{T}} (Q_1(\xi) + Q_3(\xi)) \leq \|L\|.$$

Now, we suppose that (5) holds. Thus, for any $f \in \mathcal{F}$, there exists $\mu \in \mathcal{M}$ such that

$$f(z) = \int_{\mathbb{T}} \frac{d\mu(\xi)}{1 - \bar{\xi}z}, \qquad z \in \mathbb{D},$$

and $\|f\|_{\mathcal{F}} = \|\mu\|$. Therefore, for any $f \in \mathcal{F}$, by applying a similar calculation as in equation (4), we get

$$|(Lf)(0)| \leq \int_{\mathbb{T}} Q_1(\xi) d|\mu|(\xi) \leq \|\mu\| \sup_{\xi \in \mathbb{T}} Q_1(\xi).$$

By using Jensen's inequality and Fubini's Theorem, we have

$$
\begin{aligned}
\|(Lf)'\|_{H^2}^2 &= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \left|(Lf)'(re^{i\theta})\right|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \Big| \sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)! \int_{\mathbb{T}} \bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big) d\mu(\xi)\Big|^2 d\theta \\
&= \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \|\mu\|^2 \Big| \sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)! \int_{\mathbb{T}} \bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big) \frac{d\mu(\xi)}{\|\mu\|}\Big|^2 d\theta \\
&\le \int_{\mathbb{T}} \|\mu\|^2 \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \Big| \sum_{j=0}^{n}\sum_{t=0}^{1}(j+t)!\bar{\xi}^{j+t}\Big(\frac{E^t_{u_j,\varphi_j}(re^{i\theta})}{(1-\bar{\xi}\varphi_j(re^{i\theta}))^{j+t+1}} \\
&\quad - \frac{E^t_{v_j,\psi_j}(re^{i\theta})}{(1-\bar{\xi}\psi_j(re^{i\theta}))^{j+t+1}}\Big)\Big|^2 d\theta \frac{d|\mu|(\xi)}{\|\mu\|} \\
&\le \int_{\mathbb{T}} \|\mu\| Q_3^2(\xi) d|\mu|(\xi) \\
&\le \|\mu\|^2 (\sup_{\xi\in\mathbb{T}} Q_3(\xi))^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|(Lf)(0)| + \sup_{0<r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}\left|(Lf)'(re^{i\theta})\right|^2 d\theta\right)^{\frac{1}{2}} &\le \\
\left(\sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi)\right)\|\mu\| &\le \\
\left(\sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi)\right)\|f\|_{\mathcal{F}}&.
\end{aligned}
$$

Hence,

$$
\|L\| \le \sup_{\xi\in\mathbb{T}} Q_1(\xi) + \sup_{\xi\in\mathbb{T}} Q_3(\xi).
$$

The proof is completed. □

By setting appropriate parameters in Theorems 2.1 and 2.2, we derive the following two corollaries.

Corollary 2.3. *Let $\varphi \in S(\mathbb{D})$. Then*

$$\|C_\varphi\|_{\mathcal{F}\to S} = \sup_{\xi\in\mathbb{T}} \left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1-\bar{\xi}\varphi(z)|^2} d\theta \right).$$

Corollary 2.4. *Ler $\varphi \in S(\mathbb{D})$. Then*

$$\sup_{\xi\in\mathbb{T}} \left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|^2}{|1-\bar{\xi}\varphi(z)|^4} d\theta \right)^{\frac{1}{2}} \right) \le \|C_\varphi\|_{\mathcal{F}\to S^2} \le$$

$$\sup_{\xi\in\mathbb{T}} \frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{\xi\in\mathbb{T}} \sup_{0<r<1} \left(\frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|^2}{|1-\bar{\xi}\varphi(z)|^4} d\theta \right)^{\frac{1}{2}}.$$

Corollary 2.5. *Let $\varphi \in S(\mathbb{D})$ such that $\varphi(0) \neq 0$. Then*

$$\max\{\|C_\varphi\|_{\mathcal{F}\to S}, \|C_\varphi\|_{\mathcal{F}\to S^2}\} > 1.$$

Proof. Since $\varphi(0) \neq 0$, so there exists $0 \le \theta_0 < 2\pi$ such that $\varphi(0) = |\varphi(0)|e^{i\theta_0}$, so

$$\sup_{\xi\in\mathbb{T}} \frac{1}{|1-\bar{\xi}\varphi(0)|} \ge \frac{1}{|1-\overline{e^{i\theta_0}}|\varphi(0)|e^{i\theta_0}|} = \frac{1}{1-|\varphi(0)|} > 1.$$

Hence,

$$\begin{aligned} \|C_\varphi\|_{\mathcal{F}\to S} &= \sup_{\xi\in\mathbb{T}} \left(\frac{1}{|1-\bar{\xi}\varphi(0)|} + \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1-\bar{\xi}\varphi(z)|^2} d\theta \right) \\ &\ge \frac{1}{|1-\overline{e^{i\theta_0}}\varphi(0)|} + \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1-\overline{e^{i\theta_0}}\varphi(z)|^2} d\theta \\ &> 1 + \sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1-\overline{e^{i\theta_0}}\varphi(z)|^2} d\theta. \end{aligned}$$

With the same calculation, we have $\|C_\varphi\|_{\mathcal{F}\to S^2} > 1$. □

Corollary 2.6. *Let $\varphi \in S(\mathbb{D})$ such that $\varphi \neq 0$. If $\varphi(0) = 0$ then*

$$\max\{\|C_\varphi\|_{\mathcal{F}\to S}, \|C_\varphi\|_{\mathcal{F}\to S^2}\} > 1.$$

Proof. It is clear that for any $\xi \in \mathbb{T}$, we have

$$|1 - \overline{\xi}\varphi(z)|^2 \leq (1 + |\overline{\xi}\varphi(z)|)^2 \leq 2^2 = 4.$$

Let $\varphi(0) = 0$, so we have

$$\begin{aligned} \|C_\varphi\|_{\mathcal{F}\to S} &= \sup_{\xi\in\mathbb{T}} \left(\frac{1}{|1 - \overline{\xi}\varphi(0)|} + \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1 - \overline{\xi}\varphi(z)|^2} d\theta \right) \\ &= 1 + \sup_{\xi\in\mathbb{T}} \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{|\varphi'(re^{i\theta})|}{|1 - \overline{\xi}\varphi(z)|^2} d\theta \\ &\geq 1 + \frac{1}{8\pi} \sup_{0<r<1} \int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta. \end{aligned}$$

If $\|C_\varphi\|_{\mathcal{F}\to S} = 1$ then $\sup_{0<r<1} \int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta = 0$. This implies that φ must be constant, resulting in $\varphi = \varphi(0) = 0$, which contradicts the initial assumption. Similarly, we get $\|C_\varphi\|_{\mathcal{F}\to S^2} > 1$. □

Recall that a linear operator T between the normed linear spaces X and Y with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ is called an isometry, if for any $x \in X$

$$\|Tx\|_Y = \|x\|_X.$$

Theorem 2.7. *Let $\varphi \in S(\mathbb{D})$. There is no composition isometry from the Cauchy transform spaces into the space S or S^2.*

Proof. If $\varphi \equiv 0$ then the operator $C_\varphi : \mathcal{F} \to S (C_\varphi : \mathcal{F} \to S^2)$ is not injective, so it cannot be an isometry. When $\varphi \neq 0$ the proof is clear by using corollaries 2.5 and 2.6. □

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