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Multivalued Rings

H. Behravesh
University of Urmia

Abstract. V. M. Buchstaber, defined multivalued groups. We define multivalued ring and we show that some similar results in ring theory, hold for multivalued rings.

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1. Introduction

The literature on multivalued groups and their applications is very large and includes titles from 19th century. In 1971, V. M. Buchstaber and S. P. Novikov introduced a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of element is an n-multiset, the set of n points with multiplicities (see [2] for more details). This construction leads to the notion of n-valued groups.

Let $(X)^n$ be the nth symmetric power of a space X, that is, the quotient space of the Cartesian nth power $(X)^n$ modulo the symmetric group of coordinate permutation.

An n-valued multiplication on the set X is a map

$$\mu : X \times X \to (X)^n,$$

defined by $\mu(x, y) = x \star y = [z_1, z_2, \ldots, z_n]$, where $z_k = (x \star y)_k$ for all $1 \leqslant k \leqslant n$. If the n^2-sets

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$$[x \star (y \star z)_1, x \star (y \star z)_2, \ldots, x \star (y \star z)_n],$$

$$[(x \star y)_1 \star z, (x \star y)_2 \star z, \ldots, (x \star y)_n \star z],$$

are equal for all $x, y, z \in X$, then we say μ is associative. If an element $e \in X$ exists such that, for all $x \in X$

$$e \star x = x \star e = [x, x, \ldots, x],$$

we say e is a strong unit element. If a map $\mathrm{inv} : X \to X$ exists such that for all $x \in X$

$$e \in \mathrm{inv}(x) \star x \text{ and } e \in x \star \mathrm{inv}(x),$$

we say x has a strong inverse. Let μ be an n-valued multiplication on X. Then, it is easy to see that, if the strong unit element exists, then it is unique.

An n-valued multiplication μ on X is a group when mapping which is associative together with a strong (two sided) unit e and a weak (two side) inverse $inv : X \to X$.

In [[1], Example 3.2] we showed that Lagrange Theorem does not hold for multivalued groups. Also in [[1], Example 6.6] we showed that First Isomorphism Theorem does not hold for multivalued groups.

In this paper we define an n-valued ring. Then by using the coset group construction (see [3]), we show that n-valued rings exist, and we call them coset rings. We show that maximal ideals for non-zero multivalued ring exists. Then we define principal integral domain and multivalued Neotherian ring and we prove that a multivalued PID ring is a multivalued Neotherian ring.

2. Multivalued Rings

In this section we will define multivalued rings. Also by using coset groups (see [3]), we will show that multivalued rings exist, and we will call them coset rings.

Definition 2.1. *Let X be a set with the following properties*

(1) $(X, +)$ *is an n-valued commutative group;*

(2) $(X, .)$ *is an n-valued associative map;*

(3) 1 *is the strong unit element of* $(X\backslash\{0\}, .)$*;*

(4) *for all $x, y, z \in X$, n^2-sets $x(y+z)$ is a subset of n^3-set $xy+xz$, and $(y+z)x$ is a subset of $yx+zx$;*

(5) *for all $x \in X$, $0.x = x.0 = [0, \ldots, 0]$.*
Then we call $(X, +, .)$ an n-valued ring.

It is easy to see that 0 and 1 are unique and 1-valued ring is a ring (see [[1], Introduction]).

Lemma 2.2. *Let G be a commutative group, $A \leqslant \mathrm{Aut}(G)$ and $|A| = n$. Then the coset group (G, A) is an n-valued commutative group.*

Proof. See [3, Theorem 1]. □

Let R be a ring with the unit element 1. Let $A \leqslant \mathrm{Aut}(R)$ and $|A| = n$. It is easy to see that $A \times R \to R$ is an action of A on R. Now we define

$$X = R/A = \{\mathrm{orbit}(r) : r \in R\}.$$

Let define

$$\pi : R \to X = R/A \text{ by } \pi(r) = \mathrm{orbit}(r), \text{ for all } r \in R.$$

The coset group $(X, +)$ is an n-valued commutative group, where

$$+ : X \times X \to (X)^n,$$

is defined by

$$x + y = [\pi(u + v^\alpha) : \alpha \in A],$$

for $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$.

Now we define an associative map

$$. : X \times X \to (X)^n,$$

by

$$x.y = [\pi(uv^{\alpha}) : \alpha \in A],$$

where $u \in \pi^{-1}(x)$, $v \in \pi^{-1}(y)$. We claim that $(X, +, .)$ is an n-valued ring with unit element $\{0\}$ for sum and $\{1\}$ for product. We call this n-valued ring, coset ring. □

Example 2.3. Let $R = \{z = x+iy : x, y \in \mathbb{Z}, i^2 = -1\}$ and $A = \{I_R, \phi\}$, where $\phi(z) = \bar{z}$. Then $X = R/A = \{\{z, \bar{z}\} : z \in R\}$.

Now

$$\{z_1, \bar{z_1}\} + \{z_2, \bar{z_2}\} = [\{z_1 + z_2, \bar{z_1} + \bar{z_2}\}, \{z_1 + \bar{z_2}, \bar{z_1} + z_2\}],$$

and

$$\{z_1, \bar{z_1}\}.\{z_2, \bar{z_2}\} = [\{z_1 z_2, \bar{z_1}\bar{z_2}\}, \{z_1\bar{z_2}, \bar{z_1}z_2\}],$$

also $\{0\}, \{1\} \in X$.

Definition 2.4. *Let Y be a subset of X, where X is an n-valued ring. If $(Y, +, .)$ is an n-valued ring, then we call Y an n-valued subring of X. An n-valued ring X is commutative if $x.y = y.x$ for all $x, y \in X$.*

We can prove that the intersection of two n-valued subrings, is an n-valued subring.

Definition 2.5. *Let Y be a subset of X, where X is an n-valued ring. Suppose $(Y, +)$ be an n-valued group and for all $x \in X$, $y \in Y$, we have $xy \in (Y)^n$. Then we call Y an ideal of X.*

We can prove that the intersection of two ideals of an n-valued ring, is an ideal.

Lemma 2.6. *Let R be a ring, J an ideal of R and $A \leqslant \mathrm{Aut}(R)$. Then $\{\mathrm{orbit}(j) : j \in J\}$ is an ideal of $X = R/A$.*

Definition 2.7. *Let R and S be an n-valued rings. We call $f : R \to S$ a homomorphism if*

(1) $f(0) = 0$;

(2) $f(1) = 1$;

(3) *$f(-x) = -f(x)$, for all $x \in (R, +)$;*

(4) *$f(x+y) = f(x) + f(y)$, for all $x, y \in R$;*

(5) *$f(x^{-1}) = (f(x))^{-1}$, for all $x \in (R \backslash \{0\}, .)$;*

(6) *$f(xy) = f(x)f(y)$, for all $x, y \in R$.*

We define $\ker(f) = \{x \in R : f(x) = 0\}$. We can prove that $(\ker(f), +)$ is an n-valued group (see [[1], Lemma 6.2]). Also let $x \in \ker(f)$ and $r \in R$. Then $f(r.x) = f(r).f(x) = f(r).0 = [0, \ldots, 0]$. Also $f(x.r) = [0, \ldots, 0]$. This shows that $\ker(f)$ is an ideal of R.

If the inverse of elements of $(R, +)$ is unique, then we can prove that: $\ker(f) = \{0\}$ if and only if f is one to one (see [[1], Lemma 6.5]).

Let define $\mathrm{Im}(f) = \{f(r) : r \in R\}$. We can prove that $(\mathrm{Im}(f), +, .)$ is an n-valued subring of S (see [[1], Lemma 6.3]).

Definition 2.8. *We will call an n-valued ring $(X, +, .)$, an n-valued field, if $(X \backslash \{0\}, .)$ is an n-valued commutative group.*

Example 2.9. Let $X = \{0, 1\}$ and define "+" and "." by the following tables:

Table 1

+	0	1
0	$[0,0]$	$[1,1]$
1	$[1,1]$	$[0,0]$

and

.	0	1
0	$[0,0]$	$[0,0]$
1	$[0,0]$	$[1,1]$

Table 2

+	0	1
0	$[0,0]$	$[1,1]$
1	$[1,1]$	$[0,1]$

and

.	0	1
0	$[0,0]$	$[0,0]$
1	$[0,0]$	$[1,1]$

Clearly Table 1 and Table 2 are not isomorphic. So finite fields of the same order not necessarily isomorphic. Also if we define an n-valued prime field as the intersection of the n-valued subfields, then n-valued prime fields are not necessarily isomorphic.

Lemma 2.10. *If we assume $(R,+,.)$ is a field, $A \leqslant \mathrm{Aut}(R)$ and $|A| = n$, then $X = R/A$ is an n-valued field.*

Example 2.11. Let $R = \mathbb{Q}(\sqrt{2}) = \{z = a+b\sqrt{2} : a, b \in \mathbb{Q}\}$, $\bar{z} = a-b\sqrt{2}$ and $A = \{I_R, \phi\}$, where $\phi(z) = \bar{z}$, for all $z \in R$. Then $X = R/A = \{\{z, \bar{z}\} : z \in R\}$.
Now

$$\{z_1, \bar{z_1}\} + \{z_2, \bar{z_2}\} = [\{z_1 + z_2, \bar{z_1} + \bar{z_2}\}, \{z_1 + \bar{z_2}, \bar{z_1} + z_2\}],$$

and

$$\{z_1, \bar{z_1}\}.\{z_2, \bar{z_2}\} = [\{z_1 z_2, \bar{z_1}\bar{z_2}\}, \{z_1\bar{z_2}, \bar{z_1}z_2\}],$$

also $\{0\}, \{1\} \in X$ and the inverse of $\{z, \bar{z}\}$ in $(X\backslash\{0\}, .)$ is equal to $\{1/z, 1/\bar{z}\}$.

Definition 2.12. *Let X be an n-valued commutative ring and $1 \neq 0$. We will call X an integral domain if for all $x, y \in X$, we have $xy = [0, \ldots, 0]$, then $x = 0$ or $y = 0$.*

Lemma 2.13. *Let X be an n-valued field. Then X is an integral domain.*

Proof. Let $x, y \in X$ and $x \neq 0$. Also let $x.y = [0, \ldots, 0]$. Then $[0, \ldots, 0] = x^{-1}.(x.y) = (x^{-1}.x).y$. Since $1 \in x^{-1}.x$, so $y = 0$ and the result follows. $\square$

Example 2.14. Let $R = \{z = a + ib : a, b \in \mathbb{Z}_n\}$, where n is an integer. Also let $A = \{I_R, \phi\}$, where $\phi(z) = \bar{z}$. If $n = 2$, then $X = R/A$ is not an integral domain, as $(1 + i)^2 = 0$. But if $n = 3$, then R is an integral domain.

Lemma 2.15. *Let R be an integral domain and let $A \leqslant \mathrm{Aut}(R)$. Then $X = R/A$ is an integral domain.*

Example 2.16. Let Y be an ideal of an n-valued ring. Also let $1 \in Y$. Then $1.x = x.1 = [x, \ldots, x] \in (Y)^n$. So for all $x \in X$ we have $x \in Y$. Hence $Y = X$.

Example 2.17. Let X be an n-valued field and Y an ideal of X. Also let $x \neq 0$ and $x \in Y$. Then $x \in X$. So $x^{-1} \in X$. Therefore $1 \in x^{-1}.x =$

$x.x^{-1}$ and $x^{-1}.x = x.x^{-1} \in (Y)^n$. So $1 \in Y$ and $Y = X$. Hence we have only two ideal $\{0\}$ and X.

Conversely let $\{0\}$ and X be the only ideals of an n-valued commutative ring X. Also let $x \neq 0$ and $x \in X$. Define Y to be the ideal generated by x. So $Y = X$. Since $1 \in X$, therefore there is an element $y \in X$, such that, $1 \in y.x = x.y$. This shows that X is an n-valued field.

Definition 2.18. *An ideal M of a commutative multivalued ring X is said to be maximal precisely when M is a maximal member, with respect to inclusion to the set of ideals of X.*
In other words, the ideal M of X is maximal if and only if

(a) $M \subset X$,

(b) *there does not exist an ideal K of X, with $M \subset K \subset X$.*

Theorem 2.19. *Let X be a non trivial commutative n-valued ring. Then X has at least one maximal ideal.*

Proof. Since X is a non trivial commutative n-valued ring, so 0 is a proper ideal of X. Let Ω denote the set of all proper ideals of X. Since $0 \in \Omega$, so $\Omega \neq \emptyset$. The relation of inclusion, $\subseteq$, is a partially ordered set $(\Omega, \subseteq)$.

Let Δ be a non-empty totally ordered subset of Ω. Set $J = \bigcup_{I \in \Delta} I$. Let $x \in X$ and $a \in J$. Then $a \in I$ for some $I \in \Delta$. So $ra \in (I)^n$. Hence $ra \in (J)^n$.

Now let $a, b \in J$. Then $a \in I_1$ and $b \in I_2$ for some $I_1, I_2 \in J$. Since either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$, so a and b belong to I_1 or I_2. Therefore $a + b$ belongs to either $(I_1)^n$ or $(I_2)^n$. Hence J is an ideal of X.
We claim J is a proper ideal of X. If $J = X$, then $1 \in J$. So $1 \in I$ for some $I \in \Delta$. Hence $I = X$. This is a contradiction as $I \neq X$. □

Thus we have shown that $J \in \Omega$ and J is an upper bound for Δ in Ω. Since the hypotheses of Zorn's Lemma are satisfied, it follows that the partially ordered set $(\Omega, \subseteq)$ has a maximal element, and X has a maximal ideal.

Definition 2.20. *Let X be an n-valued commutative ring and $a \in X$. The ideal generated by a is denoted by aX and is called the principal ideal of X.*
An n-valued commutative ring X, such that every ideal of X is principal, is called a principal ideal ring. If X is also an integral domain, it is called principal integral domain, often written PID *for short.*

Note: $0X$ is equal to 0 ideal and $1X$ is equal to X.

Example 2.21. Let $X = \{0, 1, 2, \dots, n-1\}$. Define $x + y = [x_1, x_2]$, where $x_1 \equiv x + y \bmod n$, $x_2 \equiv |x - y| \bmod n$ and $x_1, x_2 \in X$. It is easy to see that X is a 2-valued group. Let define $x.y = [x_1, x_2]$, where $x_1 = x_2 = xy \bmod n$. Now it is easy to see that $(X, +, .)$ is a 2-valued ring.

Now let $n = 5$. Then 0 or X are the only proper subgroups of $(X, +)$. Also these are the only ideals of X. Hence X is a PID when $n = 5$.

Definition 2.22. *Let X be an n-valued commutative ring. We say X is Neotherian if and only if every ascending chain*

$$I_1 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \dots,$$

of ideals of X is eventually stationary, and this is the case if and only if every non-empty set of ideals of X has a maximal member with respect to inclusion.

Theorem 2.23. *Let X be an n-valued* PID*. Then X is an n-valued Neotherian ring.*

Proof. Let

$$I_1 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \dots,$$

be an ascending chain of ideals of X. It is easy to prove that

$$J = \bigcup_{n \in \mathbb{N}} I_i,$$

is an ideal of X. Since X is a PID, there exits $a \in X$ such that $J = aX$. So by definition of J, there exists $k \in \mathbb{N}$ such that $a \in I_k$. But then we have

$$J = aX \subseteq I_k \subseteq I_{k+i} \subseteq J,$$

for all $i \in \mathbb{N}$. Thus the result follows. $\square$

References

[1] H. Behravesh and A. Borovik, A note on multivalued groups, *Ricerche mat.,* 61 (2) (2012), 245-253.

[2] V. M. Buchstaber and S. P. Novikov, Formal groups, power systems and Adams oprators, *Russian Mat. Sb.* (N. S.), 84 (126) (1971), 81-118.

[3] V. M. Buchstaber, n-valued groups: theory and applications, *Mosc. Math. J.,* 6 (1) (2006), 57-84.

Houshang Behravesh
Department of Mathematics
Professor of Mathematics
University of Urmia
Urmia, Iran
E-mail: h.behravesh@urmia.ac.ir