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General Local Cohomology Modules and Faltings' Local-Global Principles

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Abstract. In this article, we study the Local-global Principles for the Artinianness of ordinary local cohomology modules and the finiteness of general local cohomology modules. Let R be a Noetherian ring, Φ be a system of ideals of R and N be an R-module. Assume that $\mathcal{S}$ is a Serre subcategory of $\mathrm{Mod}(R)$ satisfying the condition C_Φ and the *Residual Fields condition* (briefly $\mathcal{RF}$ condition) and let $\mathcal{S}_\mathcal{A}$ be the class of Artinian R-modules. For $t \in \mathbb{N}_0$, we first show that the general local cohomology modules $H^i_\Phi(N) \in \mathcal{S}$ for every $i < t$ if and only if $H^i_\mathfrak{b}(N) \in \mathcal{S}$ for any $\mathfrak{b} \in \Phi$ and every $i < t$. Then, for a finite R-module N, we conclude that if $H^i_\Phi(N) \in \mathcal{S}_\mathcal{A}$ for every $i < t$, thus $H^i_\Phi(N) \in \mathcal{S}$ for every $i < t$. Consequently, we show that the least non-negative integer i in which $H^i_\Phi(N)$ is not Artinian, is a lower bound for all $\mathcal{S}$-$\mathrm{depth}_\Phi(N)$. Finally, we prove that if $n \in \mathbb{N}_0$ is such that N is in dimension $< n$ and $\left(\mathrm{Ass}_R(H_\Phi^{h^n_\Phi(N)}(N))\right)_{\geq n}$ is a finite set, then $f^n_\Phi(N) = h^n_\Phi(N)$.

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1 Introduction

In this paper, R will always denote a non-trivial commutative Noetherian ring and N will denote an R-module. By a finite R-module, we mean an R-module which is finitely generated. For the set of non-negative integers and the category of all R-modules and R-homomorphisms we shall use $\mathbb{N}_0$ and $\mathrm{Mod}(R)$, respectively.

It is well known that for each $i \geq 0$, for the i-th local cohomology R-module N relative to the ideal $\mathfrak{b}$, there is a natural R-isomorphism as follows:

$$H^i_{\mathfrak{b}}(N) \cong \varinjlim_{n \in \mathbb{N}} \mathrm{Ext}^i_R\left(R/\mathfrak{b}^n, N\right).$$

Let $\Phi \neq \emptyset$ be a set of ideals of R. Recall that Φ is called a system of ideals if the multiplication of any two ideals of Φ, always, contains an ideal of Φ. For such a system and for any R-module N, we consider the following submodule

$$\Gamma_\Phi(N) := \{x \in N |\, \mathfrak{b}\, x = 0 \text{ for some } \mathfrak{b} \in \Phi\}.$$

Then Γ_Φ is a covariant, R-linear and left exact functor from $\mathrm{Mod}(R)$ to $\mathrm{Mod}(R)$. When Φ is taken as the power of an ideal, say $\mathfrak{b}$, then $H^i_\Phi(-)$ is naturally equivalent to the ordinary functor $H^i_{\mathfrak{b}}(-)$. As of now, we refer to $H^i_\Phi(N)$ as the general local cohomology module. The ordinary local cohomology and its generalization on a system of ideals, have been studied in [7, 8, 9, 12].

Also, it is well known that, Faltings' Local-global Principle for the finiteness of local cohomology modules, [11, satz1], states that for any integer $m > 0$, $H^j_{\mathfrak{b}}(N)$ is a finite R-module for every $j < m$ if and only if $H^j_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}})$ is a finite $R_{\mathfrak{p}}$-module for every $j < m$ and any $\mathfrak{p} \in \mathrm{Spec}(R)$. Hence, another formulation of $f_{\mathfrak{b}}(N)$, the finiteness dimension of N with respect to $\mathfrak{b}$, (see [9, Theorem 9.6.2]), is as follows:

$$\begin{aligned} f_{\mathfrak{b}}(N) &= \inf\{j \in \mathbb{N}_0 |\ H^j_{\mathfrak{b}}(N) \text{ is not a finite } R\text{-module}\} \\ &= \inf\{f_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) |\, \mathfrak{p} \text{ is a prime ideal of } R\}. \end{aligned}$$

Then, for any $n \in \mathbb{N}_0$, Bahmanpour et al., in [5], presented the n-th finiteness dimension $f^n_{\mathfrak{b}}(N)$ of N relative to $\mathfrak{b}$ by

$$f^n_{\mathfrak{b}}(N) := \inf\{f_{\mathfrak{b}R_{\mathfrak{p}}}(N_{\mathfrak{p}}) |\, \mathfrak{p} \in \mathrm{Supp}(N/\mathfrak{b}N),\, \dim R/\mathfrak{p} \geq n\}.$$

As some applications of this notion, it is shown that

$$\inf\{i \in \mathbb{N}_0 | H^i_{\mathfrak{b}}(N) \text{ is not minimax (weakly Laskerian)}\},$$

is equal to $f^1_{\mathfrak{b}}(N)$ ($f^2_{\mathfrak{b}}(N)$). An R-module N is called a minimax R-module, if N/N' is Artinian module for some finite submodule N' of N. The class of minimax modules has been studied by Zink [21], Zöschinger [22, 23] and Rudlof [18]. Moreover, an R-module N is called skinny or weakly Laskerian module, if $\mathrm{Ass}_R\, N/N'$ is a finite set, for any submodule N' of N (cf. [10] or [14]).

The class of in dimension $< n$ modules is presented in [4]. The authors generalized Faltings' Local-global Principle on a complete local ring R, for any finite R-module N and ideal $\mathfrak{b}$ of R, as:

$$f_{\mathfrak{b}}^n(N) = h_{\mathfrak{b}}^n(N) := \inf\{i \geq 0 |\, H_{\mathfrak{b}}^i(N) \text{ is not in dimension} < n\}.$$

Then Mehrvarz et al., in[15] showed that the equality $f_{\mathfrak{b}}^n(N) = h_{\mathfrak{b}}^n(N)$ is true on an arbitrary Noetherian (not necessarily complete and local) ring, too.

On the other hand, Tang in [20] proved a similar Local-global Principle for the Artinianness of local cohomology modules. He proved that, for any $m \in \mathbb{N}$, the necessary and sufficient condition for $H_{\mathfrak{b}}^i(N)$ to be Artinian for every $i < m$ is $H_{\mathfrak{b}R_{\mathfrak{p}}}^i(N_{\mathfrak{p}})$ to be Artinian for every $i < m$ and any $\mathfrak{p} \in \operatorname{Spec}(R)$, in which R is an arbitrary Noetherian ring, $\mathfrak{b}$ is an ideal of R and N is an finite R-module (see [20, Theorem 2.2]).

In section 2, we deal with the Serre subcategories which satisfy the conditions C_Φ and $\mathcal{RF}$ (see Definitions 2.7, 2.13) and the situations that the general local cohomology modules belong to these Serre subcategories. Let $\mathcal{S}$ be a subcategory of $\operatorname{Mod}(R)$. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence in $\operatorname{Mod}(R)$. $\mathcal{S}$ is said to be a Serre subcategory, whenever $M \in \mathcal{S}$ if and only if $M', M'' \in \mathcal{S}$. Assume that $\mathcal{S}$ is a Serre subcategory of $\operatorname{Mod}(R)$ satisfying the condition C_Φ and $t \in \mathbb{N}_0$. As an important achievement in this section, in Proposition 2.12, we show that $H_\Phi^i(N)$ belongs to $\mathcal{S}$ for every $i < t$ if and only if $H_{\mathfrak{b}}^i(N)$ belongs to $\mathcal{S}$ for every $\mathfrak{b} \in \Phi$ and every $i < t$.

Section 3, is about Local-global Principles. As another important result about the ordinary local cohomology modules, Theorem 3.1, shows that if N is an R-module, $\mathfrak{b}$ an ideal of R and $t \in \mathbb{N}_0$ are such that $H_{\mathfrak{b}}^i(N)$ is $\mathfrak{b}$-cofinite Artinian R-module for every $i < t$, then $H_{\mathfrak{b}}^i(N) \in \mathcal{S}$ for every $i < t$ and for any Serre subcategory $\mathcal{S}$, which satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$. An application of Theorem 3.1, we present some equivalent conditions to the Local-global Principle for Artinianness (see Proposition 3.2). In sequel, assume that $\mathcal{S}$ is an arbitrary Serre subcategory that satisfies the conditions C_Φ and $\mathcal{RF}$, N is a finite R-module and t is a non-negative integer. Set $\mathcal{S}_{\mathcal{A}}$ as the class of Artinian R-modules. In Corollary 3.3, we show that if $H_\Phi^i(N)$ belongs to $\mathcal{S}_{\mathcal{A}}$ for every $i < t$, then $H_\Phi^i(N)$ belongs to $\mathcal{S}$ for every $i < t$. Then, in Proposition 3.7, we show that $\inf\{i \in \mathbb{N}_0 |\, H_\Phi^i(N) \text{ is not Artinian}\}$ is a lower bound for all $\mathcal{S}\text{-depth}_\Phi(N)$ (see Definition 3.5), i.e.,

$$\text{f-depth}_\Phi(N) \leq \mathcal{S}\text{-depth}_\Phi(N).$$

As the last important result of this paper, we conclude that $f_\Phi^n(N) = h_\Phi^n(N)$, for any $n \in \mathbb{N}_0$ (see Theorem 3.10). This generalizes the main results of [4, Theorem 2.5] for an arbitrary Noetherian (not necessary complete local) ring and [15, Theorem 2.10] for an arbitrary system of ideals of R.

2 General local cohomology modules and conditions C_Φ and $\mathcal{RF}$

We begin this section with one of the results of [19] which is used in most results of this paper. Let F, T be two left exact covariant functors from $\operatorname{Mod}(R)$ to itself. For any $j \geq 1$, we shall use the j-th right derived functors of F, H and the composition FH by F^j, T^j and $(FT)^j$, respectively.

Proposition 2.1. (see[19, Proposition2.2]) *Assume that $\mathcal{S}$ is a Serre subcategory of* $\mathrm{Mod}(R)$*, $\mathfrak{b}$ is an ideal of R and N is an R-module. Let T be a covariant and left exact functor of* $\mathrm{Mod}(R)$ *to itself such that $(0 :_X \mathfrak{b}) = (0 :_{T(X)} \mathfrak{b})$ for any R-module X. Assume that $T(E)$ is an injective R-module for any injective R-module E. Let $n \in \mathbb{N}$ and let $\mathrm{Ext}_R^{t-j}(R/\mathfrak{b}, T^j(N)) \in \mathcal{S}$ for $t = n, n+1$ and every $j < n$. Then* $\mathrm{Ker}\psi \in \mathcal{S}$ *and* $\mathrm{Coker}\psi \in \mathcal{S}$*, in which*

$$\psi : \mathrm{Ext}_R^n(R/\mathfrak{b}, N) \to \mathrm{Hom}_R(R/\mathfrak{b}, T^n(N))$$

is the natural homomorphism. Thus

$$\mathrm{Ext}_R^n(R/\mathfrak{b}, N) \in \mathcal{S} \text{ if and only if } \mathrm{Hom}_R(R/\mathfrak{b}, T^n(N) \in \mathcal{S}.$$

Proof. The assertion follows from [3, Proposition 3.1], by taking $F(-) = \mathrm{Hom}_R(R/\mathfrak{b}, -)$. Because $FT(N) = F(N)$ for any R-module N. □

Corollary 2.2. *Assume that $\mathcal{S}$ is a Serre subcategory of* $\mathrm{Mod}(R)$*, Φ is a system of ideals of R and N is an R-Module. Let $n \in \mathbb{N}_0$ be such that $\mathrm{Ext}_R^j(R/\mathfrak{b}, H_\Phi^i(N)) \in \mathcal{S}$ for every $i, j < n$. Then $\mathrm{Ext}_R^n(R/\mathfrak{b}, N) \in \mathcal{S}$ if and only if $\mathrm{Hom}_R(R/\mathfrak{b}, H_\Phi^n(N)) \in \mathcal{S}$.*

Proof. Apply Proposition 2.1 for $F(N) = \mathrm{Hom}_R(R/\mathfrak{b}, N)$ and $T(N) = \Gamma_\Phi(N)$. □

As it is mentioned in the introduction, the authors in [4], presented the class of R-modules in dimension $< n$ for an arbitrary integer $n \in \mathbb{N}_0$. An R-module N is said to be in dimension$< n$, if $\dim \mathrm{Supp}_R(N/N') < n$, for some finite submodule N' of N (also see the definition of the class of $\mathrm{FD}_{\leq n}$ in [1, Definition 2.1]). It is obvious that, the class of in dimension $< n$ modules is a generalization of the class of finitely generated, Artinian, and minimax modules for some $n \in \mathbb{N}_0$ (see [15, Remark 2.2]). Moreover, it is clear that, the class of in dimension $< n$ modules are a Serre subcategory of $\mathrm{Mod}(R)$.

Corollary 2.3. *Assume that Φ is a system of ideals of R and $s, n \in \mathbb{N}_0$ are such that the R-modules N and $H_\Phi^i(N)$ are in dimension $< n$ for every $i < s$. Thus, the R-module $\mathrm{Hom}_R(R/\mathfrak{b}, H_\Phi^s(N)/N')$ is in dimension $< n$ for any in dimension $< n$ submodule N' of $H_\Phi^s(N)$ and all $\mathfrak{b} \in \Phi$. Consequently, $\big(\mathrm{Ass}_R(H_\Phi^s(N)/N') \cap \mathrm{V}(\mathfrak{b})\big)_{\geq n}$ is a finite set.*

Proof. As N is in dimension $< n$, one can see that the R-module $\mathrm{Ext}_R^i(R/\mathfrak{b}, N)$ is in dimension $< n$ for every $i \geq 0$. Now, for any in dimension $< n$ submodule N' of $H_\Phi^s(N)$, use Corollary 2.2 and the short exact sequence $0 \to N' \to H_\Phi^s(N) \to H_\Phi^s(N)/N' \to 0$. □

We shall use Proposition 2.4 for some further results.

Proposition 2.4. *For a Serre subcategory $\mathcal{S}$ of* $\mathrm{Mod}(R)$*, an ideal $\mathfrak{b}$ of R and an R-Module X, we have the following:*

(i) $\mathfrak{b}X \in \mathcal{S}$ *if and only if* $X/(0:_X \mathfrak{b}) \in \mathcal{S}$.

(ii) $X \in \mathcal{S}$ *if and only if there is* $k \in \mathbb{N}_0$ *such that* $(0:_X \mathfrak{b}^k) \in \mathcal{S}$ *and* $\mathfrak{b}^k X \in \mathcal{S}$.

Proof. (i) Let $\mathfrak{b} = \sum_{i=1}^{n} Rb_i$ where $b_i \in R$ and $n \in \mathbb{N}$. Suppose that $\mathfrak{b}X \in \mathcal{S}$. Consider the homomorphism $g : X \to (\mathfrak{b}X)^n$ by $g(x) = (\mathfrak{b}_i x)_{i=1}^n$ for every $x \in X$. Thus, the R-module $X/(0:_X \mathfrak{b})$ is isomorphic to a submodule of $(\mathfrak{b}X)^n$. Conversely, consider the homomorphism $f : X^n \to \mathfrak{b}X$ given by $f((x_i)_{i=1}^n) = \sum_{i=1}^{n} b_i x_i$. Then f is surjective and $(0:_X \mathfrak{b})^n \subseteq \operatorname{Ker} f$, so $\mathfrak{b}X$ is a homomorphic image of $\big(X/(0:_X \mathfrak{b})\big)^n$. Now, the assertion follows from $\big(X/(0:_X \mathfrak{b})\big)^n \in \mathcal{S}$.

(ii) This part is immediately followed by part (i) and the short exact sequence

$$0 \to (0:_X \mathfrak{b}^k) \to X \to X/(0:_X \mathfrak{b}^k) \to 0.$$

□

Proposition 2.5. *Assume that* $\mathcal{S}$ *is a Serre subcategory of* $\operatorname{Mod}(R)$, Φ *is a system of ideals of* R *and* N *is an* R*-module. Suppose that* $\mathfrak{b} \in \Phi$ *and* $k, t \in \mathbb{N}_\circ$ *are such that* $\mathfrak{b}^k H^i_\Phi(N) \in \mathcal{S}$ *for every* $i < t$. *Thus* $H^i_\Phi(N) \in \mathcal{S}$ *for every* $i < t$, *if one of the following statements satisfies:*

(i) $\Gamma_\Phi(N) \in \mathcal{S}$ *and* $\operatorname{Ext}^i_R(R/\mathfrak{b}^k, N) \in \mathcal{S}$ *for every* $i < t$.

(ii) $N \in \mathcal{S}$.

(iii) $\Gamma_\Phi(N) \in \mathcal{S}$ *and there exists* $n \in \mathbb{N}_0$ *such that* $\mathfrak{b}^n N \in \mathcal{S}$.

Proof. First, suppose that condition (i) holds and use induction on t. For $t = 0, 1$, we do not have anything to prove. Now, assume that $t > 1$ and the assertion is valid for every $i \leq t-2$ i.e., the R-modules $H^0_\Phi(N), H^1_\Phi(N), \cdots, H^{t-2}_\Phi(N)$ belong to $\mathcal{S}$. We show that $H^{t-1}_\Phi(N) \in \mathcal{S}$. Since $\operatorname{Ext}^{t-1}_R(R/\mathfrak{b}^k, N) \in \mathcal{S}$ and $H^i_\Phi(N) \in \mathcal{S}$, for every $i < t-1$, we have $(0:_{H^{t-1}_\Phi(N)} \mathfrak{b}^k) \in \mathcal{S}$, by Corollary 2.2. On the other hand, by the assumption, $\mathfrak{b}^k H^{t-1}_\Phi(N) \in \mathcal{S}$. Now, the result holds by Proposition 2.4 (ii). Under each condition (ii) and (iii), use part (i) and Proposition 2.4 (ii). □

As a corollary of Proposition 2.5, we achieve the following result that generalizes [9, Proposition 9.1.2].

Corollary 2.6. *Assume that* $\mathcal{S}$ *is a Serre subcategory of* $\operatorname{Mod}(R)$, N *is an* R*-module such that* $N \in \mathcal{S}$ *and* $t \in \mathbb{N}_0$. *Let* $\mathfrak{b}$ *be an ideal of* R *such that* $\mathfrak{b} \subseteq \sqrt{0 : H^i_\mathfrak{b}(N)}$ *for every* $i < t$. *Then* $H^i_\mathfrak{b}(N)$ *belongs to* $\mathcal{S}$ *for every* $i < t$.

Assume that $\mathfrak{b}$ is an ideal of R and $\mathcal{S}$ is a Serre subcategory of $\operatorname{Mod}(R)$. The authors in [2], studied the Serre subcategories which satisfy the condition $C_\mathfrak{b}$. It is said that $\mathcal{S}$ satisfies the condition $C_\mathfrak{b}$, if for any $\mathfrak{b}$-torsion R-module X, the condition $(0:_X \mathfrak{b}) \in \mathcal{S}$ implies that $X \in \mathcal{S}$ (see [2, Definition 2.1]). The Examples 2.4 and 2.5 of [2], show that all the classes of zero R-modules, Artinian modules, modules with finite support, $\mathfrak{b}$-cofinite Artinian modules and the class of all R-modules M with $\dim M \leq s$, where $s \in \mathbb{N}_0$, satisfy the condition $C_{\mathfrak{b}}$. Recall that an R-module X is said to be $\mathfrak{b}$-cofinite whenever $\mathrm{Supp}_R(X) \subseteq V(\mathfrak{b})$ and $\mathrm{Ext}^i_R(R/\mathfrak{b}, X)$ is a finite R-module for every $i \in \mathbb{N}_0$ (see [13]).

Now, in this position, for an arbitrary system of ideals Φ of R, we introduce the condition C_Φ and then we shall conclude some results on the general local cohomology modules. An R-module X is said to be a Φ-torsion module if $\Gamma_\Phi(X) = X$ and it is said to be a Φ-torsion-free module if $\Gamma_\Phi(X) = 0$.

Definition 2.7. Assume that $\mathcal{S}$ is a Serre subcategory of $\mathrm{Mod}(R)$ and Φ is a system of ideals of R. We say that $\mathcal{S}$ satisfies the condition C_Φ, precisely when for every Φ-torsion R-module X, the condition $(0 :_X \mathfrak{b}) \in \mathcal{S}$ for some $\mathfrak{b} \in \Phi$ implies that $X \in \mathcal{S}$.

Remark 2.8. By Proposition 2.4 (ii), it can be said that, $\mathcal{S}$ satisfies the condition C_Φ, whenever for any Φ-torsion R-module N, the condition $(0 :_N \mathfrak{a}^k) \in \mathcal{S}$ for some $\mathfrak{a} \in \Phi$ and $k \in \mathbb{N}_0$, implies that $\mathfrak{a}^k N \in \mathcal{S}$. In addition, it is easy to see that if $\mathcal{S}$ satisfies the condition C_Φ, then $\mathcal{S}$ satisfies the condition $C_{\mathfrak{b}}$, for any $\mathfrak{b} \in \Phi$.

Example 2.9. Let R be a Noetherian ring of finite dimension d. Let Φ be an arbitrary system of ideals of R and $\mathcal{S} = \{M \in \mathrm{Mod}(R) | \dim M \leq d\}$. Then, it is clear that $\mathcal{S}$ satisfies the condition C_Φ.

The next examples show that the class of Artinian R-modules does not satisfy the condition C_Φ, for some system of ideals Φ.

Example 2.10. ([6, Example 3.7]) Let R be a Gorenstein ring of finite dimension d such that has infinite maximal ideal $\mathfrak{m}$ with $\mathrm{ht}\,\mathfrak{m} = d$. Let

$$\Psi = \{\mathfrak{m} \in \mathrm{Max}(R) | \ \mathrm{ht}\,\mathfrak{m} = d\} \text{ and } \Phi = \{\mathfrak{a} | \ \mathfrak{a} \text{ is an ideal of } R \text{ and } \dim R/\mathfrak{a} \leq 0\}.$$

Then, it is easy to see that Φ is a system of ideals of R and $\Psi \subseteq \Phi$. By [6, Example 3.7], we have

$$H^i_\Phi(R) = \begin{cases} \bigoplus_{\mathfrak{m} \in \Psi} E(R/\mathfrak{m}) & if\ i = d \\ 0 & if\ i \neq d. \end{cases}$$

Thus, $H^d_\Phi(R)$ is not Artinian R-module. On the other hand, since $\mathrm{Ext}^d_R(R/\mathfrak{a}, R)$ is an Artinian R-module for all $\mathfrak{a} \in \Phi$, so $(0 :_{H^d_\Phi(R)} \mathfrak{a}) = \mathrm{Hom}_R(R/\mathfrak{a}, H^d_\Phi(R))$ is Artinian R-module for all $\mathfrak{a} \in \Phi$, by Corollary 2.2. Hence, the class of Artinian R-modules does not satisfy the condition C_Φ.

Example 2.11. Let $\mathbb{Z}$ be the ring of integers and let p be a prime number of $\mathbb{Z}$. Set $\Phi = \{(0), p\mathbb{Z}\}$. Then, Φ is a system of ideals of $\mathbb{Z}$. It is clear that $\Gamma_\Phi(\mathbb{Z}) = \mathbb{Z}$ and so $H^i_\Phi(\mathbb{Z}) = 0$ for all $i \geq 1$, by [12, Lemma 2.4]. Since $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, \mathbb{Z})$ is Artinian $\mathbb{Z}$-module and $\mathbb{Z}$ is not Artinian, thus the class of Artinian $\mathbb{Z}$-modules does not satisfy the condition C_Φ.

As a generalization of [2, Theorem 2.9] and [6, Corollary 2.14], we present the following proposition which is one of the useful results in this article.

Proposition 2.12. *Assume that Φ is a system of ideals of R and $t \in \mathbb{N}_0$. For any Serre subcategory $\mathcal{S}$ of $\mathrm{Mod}(R)$ which satisfies the condition C_Φ and any R-module N the following conditions are equivalent:*

(i) $H^i_\Phi(N) \in \mathcal{S}$ *for every* $i < t$;

(ii) $\mathrm{Ext}^j_R(R/\mathfrak{b}, H^i_\Phi(N)) \in \mathcal{S}$ *for any* $\mathfrak{b} \in \Phi$ *and every* $i, j < t$.

(iii) $\mathrm{Ext}^i_R(R/\mathfrak{b}, N) \in \mathcal{S}$ *for any* $\mathfrak{b} \in \Phi$ *and every* $i < t$;

(iv) $H^i_\mathfrak{b}(N) \in \mathcal{S}$ *for any* $\mathfrak{b} \in \Phi$ *and every* $i < t$.

Proof. (i) $\Rightarrow$ (ii) Let $\mathfrak{b} \in \Phi$ and consider the following resolution of $R/\mathfrak{b}$

$$\mathcal{A}_\bullet : \cdots \longrightarrow A_s \longrightarrow A_{s-1} \longrightarrow \cdots \longrightarrow A_1 \longrightarrow A_0 \longrightarrow 0,$$

in which A_i are free R-modules with finite ranks for every $i \geq 0$.

Then $\mathrm{Ext}^j_R\big(R/\mathfrak{b}, H^i_\Phi(N)\big) = H^j\big(\mathrm{Hom}_R(\mathcal{A}_\bullet, H^i_\Phi(N)\big)$ is a subquotient of direct sum of finite copies of $H^i_\Phi(N)$ for every i, j. Now, the assertion holds by the fact that $\mathcal{S}$ is a Serre subcategory.

(ii) $\Rightarrow$ (iii) Use Corollary 2.2.

(iii) $\Rightarrow$ (iv) As $\mathcal{S}$ satisfies the condition $C_\mathfrak{b}$ for every $\mathfrak{b} \in \Phi$, thus this implication is true, by Remark 2.8 and [2, Theorem 2.9].

(iv) $\Rightarrow$ (i) We show that $H^i_\Phi(N) \in \mathcal{S}$ by induction on t. For $t = 0$, we do not have anything to show. Suppose that $t = 1$. By hypothesis, $\Gamma_\mathfrak{b}(N) = H^0_\mathfrak{b}(N) \in \mathcal{S}$, and so $(0 :_{\Gamma_\Phi(N)} \mathfrak{b}) = (0 :_N \mathfrak{b}) = (0 :_{\Gamma_\mathfrak{b}(N)} \mathfrak{b}) \in \mathcal{S}$. Therefore, the assertion follows as $\mathcal{S}$ satisfies the condition C_Φ. Now, let $t > 1$ and assume that the assertion is settled for every $i \leq t-2$. We prove that $H^{t-1}_\Phi(N) \in \mathcal{S}$. Since, by hypothesis, $H^i_\mathfrak{b}(N) \in \mathcal{S}$ for any ideal $\mathfrak{b}$ in Φ and every $i \leq t-1$, $\mathrm{Ext}^i_R(R/\mathfrak{b}, N) \in \mathcal{S}$ for every $i \leq t-1$, by [2, Theorem 2.9]. By using inductive hypothesis, $H^i_\Phi(N) \in \mathcal{S}$ for every $i \leq t-2$. Therefore $(0 :_{H^{t-1}_\Phi(N)} \mathfrak{b}) \in \mathcal{S}$, by Corollary 2.2. Now, the assertion holds as $\mathcal{S}$ satisfies the condition C_Φ. $\square$

Definition 2.13. Let R be a Noetherian ring. A Serre subcategory $\mathcal{S}$ of $\mathrm{Mod}(R)$ is said to satisfy the *Residual Fields condition* (briefly $\mathcal{RF}$ condition) if $R/\mathfrak{n} \in \mathcal{S}$ for any $\mathfrak{n} \in \mathrm{Max}(R)$.

Example 2.14. (i) It is obvious that all of the classes of Noetherian R-modules, Artinian R-modules, R-modules with finite support and the class of all R-modules X with $\dim_R X \leq n$, where $n \in \mathbb{N}_0$, satisfy the $\mathcal{RF}$ condition.

(ii) Assume that R is a non-local Noetherian ring and $\mathfrak{b}$ is an ideal of R. Let $\mathfrak{n} \in \mathrm{Max}(R)$ be such that $\mathfrak{n} \notin \mathrm{V}(\mathfrak{b})$. Set

$$\mathcal{S} := \{M \in \mathrm{Mod}(R) \mid M \text{ is an } \mathfrak{b}\text{-torsion } R\text{-module}\}.$$

Clearly, $\mathcal{S}$ does not satisfy the $\mathcal{RF}$ condition. However, the next lemma, (part iii), shows that when $(R, \mathfrak{n})$ is a local ring, any non-zero Serre subcategory $\mathcal{S}$ of $\operatorname{Mod}(R)$ satisfies the $\mathcal{RF}$ condition.

Lemma 2.15. *Assume that $\mathcal{S}$ is an arbitrary Serre subcategory and that $\mathcal{S}_{\mathcal{FL}}$ is the class of the finite length R-modules. Then*

(i) *$\mathcal{S}$ is non-zero if and only if there is $\mathfrak{n} \in \operatorname{Max}(R)$ such that $R/\mathfrak{n} \in \mathcal{S}$.*

(ii) *$\mathcal{S}_{\mathcal{FL}} \subseteq \mathcal{S}$ if and only if $\mathcal{S}$ satisfies the condition $\mathcal{RF}$.*

(iii) *If R is a local ring and $\mathcal{S}$ is non-zero, then $\mathcal{S}$ satisfies the condition $\mathcal{RF}$.*

Proof. (i)($\Rightarrow$) Let $0 \neq M \in \mathcal{S}$ and $0 \neq x \in M$. Thus, there exists $\mathfrak{n} \in \operatorname{Max}(R)$ such that $(0 :_R Rx) \subseteq \mathfrak{n}$. Now, as $Rx \in \mathcal{S}$, considering the natural epimorphism

$$Rx \cong R/(0 :_R Rx) \to R/\mathfrak{n},$$

we get $R/\mathfrak{n} \in \mathcal{S}$.
($\Leftarrow$) It is obvious.

(ii) Assume that $\mathcal{S}_{\mathcal{FL}} \subseteq \mathcal{S}$ and $\mathfrak{n}$ is an arbitrary maximal ideal of R. Since $R/\mathfrak{n}$ has finite length, $R/\mathfrak{n}$ belongs to $\mathcal{S}$. For inverse, let $L \in \mathcal{S}_{\mathcal{FL}}$ have the length l. Thus, there exists a chain

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_l = L$$

of R-submodules of N in which L_j/L_{j-1} is isomorphic to $R/\mathfrak{n}$ for some $\mathfrak{n} \in \operatorname{Max}(R)$ and all $1 \leq j \leq l$. Therefore, the proof is completed by induction on l.

(iii) The assertion follows from (i) and (ii). □

3 Faltings' Local-global Principles

We begin this section with the following theorem, as an important result of the article. This is applied in the proof of Proposition 3.2.

Theorem 3.1. *Assume that $\mathfrak{b}$ is an ideal of R, N is an R-module and $t \in \mathbb{N}_0$. Thus a necessary and sufficient condition for $H^i_{\mathfrak{b}}(N)$ to be a $\mathfrak{b}$-cofinite Artinian R-module is that $H^i_{\mathfrak{b}}(N) \in \mathcal{S}$ for every $i < t$ and any Serre subcategory $\mathcal{S}$ satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$.*

Proof. ($\Rightarrow$) Suppose that $H^i_{\mathfrak{b}}(N)$ is $\mathfrak{b}$-cofinite Artinian R-module for every $i < t$ and $\mathcal{S}$ is an arbitrary Serre subcategory such that satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$. Using induction on t, we prove that $H^i_{\mathfrak{b}}(N) \in \mathcal{S}$ for every $i < t$. When t is zero, we do not have anything to prove. Let $t = 1$. Since $\ell_R(0 :_{\Gamma_{\mathfrak{b}}(N)} \mathfrak{b}) < \infty$, $(0 :_{\Gamma_{\mathfrak{b}}(N)} \mathfrak{b}) \in \mathcal{S}$, by Lemma 2.15 (ii). Thus, $\Gamma_{\mathfrak{b}}(N) \in \mathcal{S}$ as $\mathcal{S}$ satisfies the condition $C_{\mathfrak{b}}$.

Now, assume that $t > 1$ and the result is true for $i = 0, \cdots, t-2$. Using Proposition 2.12 and Corollary 2.2 for the category of $\mathfrak{b}$-cofinite Artinian R-modules and $\Phi = \{\mathfrak{b}^j \,|\, j \geq 0\}$, since $H^i_{\mathfrak{b}}(N)$ is $\mathfrak{b}$-cofinite Artinian R-module for every $i \leq t-2$, $\ell_R(0 :_{H^{t-1}_{\mathfrak{b}}(N)} \mathfrak{b}) < \infty$ and so similar to the argument of the case $t = 1$, we get $(0 :_{H^{t-1}_{\mathfrak{b}}(N)} \mathfrak{b}) \in \mathcal{S}$.

($\Leftarrow$) Choose $\mathcal{S}$ as the category of $\mathfrak{b}$-cofinite Artinian R-modules. □

The following result presents more equivalent conditions to the Local-global Principle for the Artinianness of ordinary local cohomology modules than those have been proven in [20, Theorem 2.2].

Proposition 3.2. *Assume that $\mathfrak{b}$ is an ideal, N is an R-module and $t \in \mathbb{N}_0$ is such that the R-module $\operatorname{Ext}^i_R(R/\mathfrak{b}, N)$ is finite for every $i < t$. Then the following conditions are equivalent:*

(i) *$H^i_{\mathfrak{b}}(N)$ is Artinian for every $i < t$;*

(ii) *$H^i_{\mathfrak{b}}(N)$ is $\mathfrak{b}$-cofinite Artinian for every $i < t$;*

(iii) *$(H^i_{\mathfrak{b}}(N))_{\mathfrak{p}}$ is Artinian for every $i < t$ and any prime ideal $\mathfrak{p}$ of R;*

(iv) *$\operatorname{Supp}(H^i_{\mathfrak{b}}(N)) \subseteq \operatorname{Max}(R)$ for every $i < t$;*

(v) *$H^i_{\mathfrak{b}}(N)$ belongs to $\mathcal{S}$ for every $i < t$ and any Serre subcategory $\mathcal{S}$, which satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$;*

(vi) *$\operatorname{Ext}^i_R(R/\mathfrak{b}, N)$ belongs to $\mathcal{S}$ for every $i < t$ and any Serre subcategory $\mathcal{S}$ which satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$.*

Proof. The conditions (i) to (iv) are equivalent by [6, Theorem 3.2]. Also, conditions (ii) and (v) are equivalent by Theorem 3.1. Finally, (v) and (vi) are equivalent by Proposition 2.12, considering $\Phi = \{\mathfrak{b}^i \,|\, i \geq 0\}$.
□

Corollary 3.3. *Assume that Φ is a system of ideals of R, N is a finite R-module and $\mathcal{S}_{\mathcal{A}}$ is the subcategory of Artinian R-modules of $\operatorname{Mod}(R)$. Let $t \in \mathbb{N}_0$ be such that $H^i_{\Phi}(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$. Then $H^i_{\Phi}(N) \in \mathcal{S}$ for every $i < t$ and any Serre subcategory $\mathcal{S}$ of $\operatorname{Mod}(R)$, which satisfies the conditions C_{Φ} and $\mathcal{RF}$. Specially, for $\Phi = \{\mathfrak{b}^i | i \geq 0\}$, where $\mathfrak{b}$ is an arbitrary ideal of R.*

Proof. Since $H^i_{\Phi}(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$, $H^i_{\mathfrak{b}}(N) \in \mathcal{S}_{\mathcal{A}}$ for every $i < t$ and any $\mathfrak{b} \in \Phi$, by applying [6, Corollary 2.7] for the class of Artinian R-modules. Therefore, Propositions 3.2 (part i $\Rightarrow$ v) implies that $H^i_{\mathfrak{b}}(N) \in \mathcal{S}$ for every $i < t$ and any $\mathfrak{b} \in \Phi$. Now, the assertion follows from Propositions 2.12 (part iv $\Rightarrow$ i). □

Corollary 3.4. *Assume that $\mathfrak{b}$ is an ideal of R such that $\dim R/\mathfrak{b} = 0$ and N is a finite R-module. Then $H^i_{\mathfrak{b}}(N) \in \mathcal{S}$ for any Serre subcategory $\mathcal{S}$ satisfies the conditions $C_{\mathfrak{b}}$ and $\mathcal{RF}$ and every $i \in \mathbb{N}_0$.*

Proof. As $\operatorname{Supp}_R(H^i_{\mathfrak{b}}(N)) \subseteq \operatorname{Max}(R)$ for every $i \geq 0$, the result holds easily by Proposition 3.2. □

Let M be an R-module. As an interesting generalization of various regular sequences on M to a Serre subcategory $\mathcal{S}$ of $\mathrm{Mod}(R)$, the authors in [2, Definition 2.6], introduced the concept of $\mathcal{S}$-sequences on M. Then, in [2, Lemma 2.14 and Definition 2.15], for an ideal $\mathfrak{a}$ of R and a Serre subcategory $\mathcal{S}$ satisfying the condition $C_{\mathfrak{a}}$ with $M/\mathfrak{a}M \notin \mathcal{S}$, they showed that the lengths of all maximal $\mathcal{S}$-sequences on M in $\mathfrak{a}$, are equal. They denoted this number by $\mathcal{S}\text{-depth}_{\mathfrak{a}}(M)$ and in [2, Theorem 2.18], they proved that

$$\mathcal{S}\text{-depth}_{\mathfrak{a}}(M) = \min\{i \geq 0 |\, H^i_{\mathfrak{a}}(M) \notin \mathcal{S}\}.$$

In this stage, we first introduce the concept of $\mathcal{S}\text{-depth}_{\Phi}(N)$, in which N is an R-module, Φ is a system of ideals of R and $\mathcal{S}$ is a Serre subcategory of $\mathrm{Mod}(R)$ satisfies the condition C_{Φ}. Then, in Proposition 3.7, we show that for any Serre subcategory $\mathcal{S}$ satisfying the conditions C_{Φ} and $\mathcal{RF}$, the following inequality holds:

$$\inf\{j \in \mathbb{N}_0 |\, H^j_{\Phi}(N) \text{ is not Artinian } \} \leq \mathcal{S}\text{-depth}_{\Phi}(N).$$

Definition 3.5. Assume that Φ is a system of ideals of R, N is an R-module and $\mathcal{S}$ is a Serre subcategory of $\mathrm{Mod}(R)$ satisfying the condition C_{Φ}. We define $\mathcal{S}\text{-depth}_{\Phi}(N)$ as:

$$\mathcal{S}\text{-depth}_{\Phi}(N) = \inf\{j \geq 0 |\, H^j_{\Phi}(N) \notin \mathcal{S}\}$$

if the infimum exists, ∞ otherwise.

Remark 3.6. Let $\mathcal{S}$ be mentioned in Definition 3.5. According to Proposition 2.12 and the paragraph before Definition 3.5, it is obvious that if N is a finite R-module and $\mathfrak{b} \in \Phi$ is such that $N/\mathfrak{b}N$ does not belong to $\mathcal{S}$, then $\mathcal{S}\text{-depth}_{\Phi}(N) \in \mathbb{N}_0$ and

$$\mathcal{S}\text{-depth}_{\Phi}(N) = \min\{\mathcal{S}\text{-depth}_{\mathfrak{b}}(N) |\, \mathfrak{b} \in \Phi\}.$$

In addition, for an arbitrary ideal $\mathfrak{b}$ of R, with choosing suitable Serre subcategories $\mathcal{S}$ which satisfy the condition $C_{\mathfrak{b}}$ and $\Phi = \{\mathfrak{b}^i | i \geq 0\}$ in Definition 3.5, we can obtain the concepts of $\mathrm{depth}_{\mathfrak{b}}(N)$, $\text{f-depth}_{\mathfrak{b}}(N)$ and $\text{g-depth}_{\mathfrak{b}}(N)$. Recall that, by [20, Theorem 3.4], [16, Theorem 3.1] and [17, Proposition 5.2], we have

$$\text{f-depth}_{\mathfrak{b}}(N) = \inf\{i \geq 0 |\, H^i_{\mathfrak{b}}(N) \text{ is not Artinian}\},$$

and

$$\text{g-depth}_{\mathfrak{b}}(N) = \inf\{i \geq 0 |\, \mathrm{Supp}_R H^i_{\mathfrak{b}}(N) \text{ is not a finite set}\}.$$

The next result indicates that $\text{f-depth}_{\Phi}(N)$, i.e., the least integer $j \in \mathbb{N}_0$ that $H^j_{\Phi}(N)$ is not Artinian, is a lower bound for $\mathcal{S}\text{-depth}_{\Phi}(N)$, for all Serre subcategory $\mathcal{S}$ of $\mathrm{Mod}(R)$ satisfying the conditions C_{Φ} and $\mathcal{RF}$.

Proposition 3.7. *Assume that Φ is a system of ideals of R and N is a finite R-module. Let $\mathcal{S}_{\mathcal{A}}$ be the category of Artinian R-modules and $\mathcal{S}$ be a Serre subcategory of* $\mathrm{Mod}(R)$ *satisfying the conditions C_{Φ} and $\mathcal{RF}$. Then*

$$\text{f-depth}_{\Phi}(N) = \inf\{j \geq 0 |\, H^j_{\Phi}(N) \notin \mathcal{S}_{\mathcal{A}}\} \leq \mathcal{S}\text{-depth}_{\Phi}(N).$$

Specially, for any ideal $\mathfrak{b}$ of R that $N/\mathfrak{b}N \notin \mathcal{S}$,

$$\text{f-depth}_{\mathfrak{b}}(N) = \inf\{j \in \mathbb{N}_0 |\, \operatorname{Ext}^j_R(R/\mathfrak{b}, N) \notin \mathcal{S}_{\mathcal{A}}\} \leq \mathcal{S}\text{-depth}_{\mathfrak{b}}(N).$$

Proof. The assertions are obtained from Corollary 3.3 and Proposition 3.2. □

Corollary 3.8. *Suppose that $(R, \mathfrak{n})$ is a local ring. Let N be a finite R-module and $\mathfrak{b}$ be an ideal of R. Then for any non-zero Serre subcategory $\mathcal{S}$ satisfying the condition $C_{\mathfrak{b}}$ and $N/\mathfrak{b}N \notin \mathcal{S}$, we get*

$$\text{f-depth}_{\mathfrak{b}}(N) \leq \mathcal{S}\text{-depth}_{\mathfrak{b}}(N).$$

Specially,

$$\operatorname{depth}_{\mathfrak{b}}(N) \leq \text{f-depth}_{\mathfrak{b}}(N) \leq \text{g-depth}_{\mathfrak{b}}(N).$$

Proof. The assertions will be obtained from Proposition 3.7 and Lemma 2.15 (i). □

As it is mentioned in the introduction, the most important result of [15] is its Theorem 2.10, which shows that the equality $f^n_{\mathfrak{b}}(N) = h^n_{\mathfrak{b}}(N)$ holds for any finite R-module N and any ideal $\mathfrak{b}$ on an arbitrary Noetherian ring R. Following, Theorem 3.10, as the last important result of this article, generalizes the main results of [4, Theorem 2.5], [15, Theorem 2.10] and [19, Theorem 2.17]. To do this, the following definition is needed.

Definition 3.9. Assume that Φ is a system of ideals of R and N is an R-module. Let $n \in \mathbb{N}_0$ and $\Phi_{\mathfrak{p}} := \{\mathfrak{b}R_{\mathfrak{p}} |\, \mathfrak{b} \in \Phi\}$ for any $\mathfrak{p} \in \operatorname{Spec}(R)$. We define

$$h^n_{\Phi}(N) := \inf\{j \geq 0 |\, H^j_{\Phi}(N) \text{ is not in dimension} < n\}$$

and

$$f^n_{\Phi}(N) := \inf\{f_{\Phi_{\mathfrak{p}}}(N_{\mathfrak{p}}) |\, \mathfrak{p} \in \operatorname{Supp}_R(N) \text{ and } \dim R/\mathfrak{p} \geq n\}.$$

Theorem 3.10. *Assume that R is a Noetherian ring and that Φ is a system of ideals of R. Let $n \in \mathbb{N}_0$ be such that the R-module N is in dimension $< n$. Suppose that the set $\left(\operatorname{Ass}_R H^{h^n_{\Phi}(N)}_{\Phi}(N)\right)_{\geq n}$ is finite. Then $h^n_{\Phi}(N) = f^n_{\Phi}(N)$.*

Proof. Put $s := h^n_{\Phi}(N)$. For every $i < s$, $\dim \operatorname{Supp}(H^i_{\Phi}(N)/N') < n$ for some finite submodule N' of $H^i_{\Phi}(N)$. So for any $\mathfrak{q} \in (\operatorname{Spec}(R))_{\geq n}$, we obtain $(H^i_{\Phi}(N)/N')_{\mathfrak{q}}$ is zero. Consequently, $H^i_{\Phi_{\mathfrak{q}}}(N_{\mathfrak{q}})$ is a finite $R_{\mathfrak{q}}$-module. Hence $s \leq f^n_{\Phi}(N)$. Now, we prove that $s = f^n_{\Phi}(N)$. On the contrary, suppose that $s < f^n_{\Phi}(N)$. Assume that

$$(\operatorname{Ass}_R H^s_{\Phi}(N))_{\geq n} = \{\mathfrak{q}_1, \mathfrak{q}_2, ..., \mathfrak{q}_r\}.$$

For any $1 \leq j \leq r$, $\mathfrak{q}_j \in (\mathrm{Spec}(R))_{\geq n}$. Thus, as $s < f^n_\Phi(N)$, $(H^s_\Phi(N))_{\mathfrak{q}_j}$ is a finite $R_{\mathfrak{q}_j}$-module. So that for every $1 \leq j \leq r$, there exists a finite R-submodule M_j of $H^s_\Phi(N)$ such that $(H^s_\Phi(N))_{\mathfrak{q}_j} \cong (M_j)_{\mathfrak{q}_j}$. Put $L_1 := M_1 + M_2 + ... + M_r$. Then L_1 is a finite R-submodule (and Φ-torsion) of $H^s_\Phi(N)$ and we get

$$(\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n} \cap (\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n} = \emptyset.$$

Next, we prove that the set $(\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n}$ is finite, too. According to [7, Lemma 2.1] and since the set $(\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n}$ is finite, there are ideals $\mathfrak{b}_1, \mathfrak{b}_2, ..., \mathfrak{b}_r \in \Phi$ such that

$$(\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n} \subseteq \bigcup_{j=1}^{r} (\mathrm{Ass}_R\, H^s_{\mathfrak{b}_j}(N))_{\geq n} \subseteq \bigcup_{j=1}^{r} (\mathrm{V}(\mathfrak{b}_j))_{\geq n}.$$

As Φ is a system of ideals of R, there is $\mathfrak{b} \in \Phi$ such that $\mathfrak{b} \subseteq \prod_{j=1}^{r} \mathfrak{b}_j$ and thus

$$(\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n} \subseteq \bigcup_{j=1}^{r} (\mathrm{V}(\mathfrak{b}_j))_{\geq n} \subseteq (\mathrm{V}(\mathfrak{b}))_{\geq n}.$$

We claim that $(\mathrm{Supp}_R\, H^s_\Phi(N))_{\geq n} \subseteq (\mathrm{V}(\mathfrak{b}))_{\geq n}$. For this purpose, let $\mathfrak{q}$ be an arbitrary element of $(\mathrm{Supp}_R\, H^s_\Phi(N))_{\geq n}$. Then, there is $\mathfrak{p} \in (\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n}$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. So $\mathfrak{p} \in (\mathrm{V}(\mathfrak{b}))_{\geq n}$. Since $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{q}$ and $\dim R/\mathfrak{q} \geq n$, we get $\mathfrak{q} \in (\mathrm{V}(\mathfrak{b}))_{\geq n}$ as required. Therefore

$$(\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n} \subseteq (\mathrm{Supp}_R\, H^s_\Phi(N)/L_1)_{\geq n} \subseteq (\mathrm{Supp}_R\, H^s_\Phi(N))_{\geq n} \subseteq (\mathrm{V}(\mathfrak{b}))_{\geq n}.$$

Also, as for every $i < s$, $H^i_\Phi(N)$ is an R-module in dimension $< n$, by Corollary 2.3, $(\mathrm{Ass}_R(H^s_\Phi(N)/L_1) \cap \mathrm{V}(\mathfrak{b}))_{\geq n}$ is a finite set and so, $(\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n}$ is a finite set as well. Now, as

$$(\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n} \cap (\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n} = \emptyset,$$

from [15, Lemma 2.5], we conclude that

$$\textstyle\bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n}} \mathfrak{q} \subsetneqq \bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n}} \mathfrak{q}.$$

Using a similar argument, there exists a submodule L_2/L_1 of $H^s_\Phi(N)/L_1$ such that

$$(\mathrm{Ass}_R\, H^s_\Phi(N)/L_2)_{\geq n} \cap (\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n} = \emptyset,$$

and so

$$\textstyle\bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n}} \mathfrak{q} \subsetneqq \bigcap_{\mathfrak{q} \in (Ass_R H^s_\Phi(N)/L_2)_{\geq n}} \mathfrak{q}.$$

Proceeding in the same way, we can find a strictly chain of ideals of R as follows:

$$\textstyle\bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N))_{\geq n}} \mathfrak{q} \subsetneqq \bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N)/L_1)_{\geq n}} \mathfrak{q} \subsetneqq \bigcap_{\mathfrak{q} \in (\mathrm{Ass}_R\, H^s_\Phi(N)/L_2)_{\geq n}} \mathfrak{q} \subsetneqq \cdots.$$

which is not stable and this is a contradiction. Therefore $s = f^n_\Phi(N)$. $\square$

Corollary 3.11. *Assume that R is a Noetherian ring and that $\mathfrak{b}$ is an ideal of R. Let N be a finite R-module. Then for every $n \in \mathbb{N}_0$, $f^n_\mathfrak{a}(N) = h^n_\mathfrak{a}(N)$.*

Proof. Apply $\Phi = \{\mathfrak{b}^j | j > 0\}$ and $s = h^n_\mathfrak{b}(M)$ in Theorem 3.10 and Corollary 2.3. $\square$

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