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Solving Non-Linear Quadratic Optimal Control Problems by Variational Iteration Method

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Abstract. In this article, we give an analytical approximate solution for non-linear quadratic optimal control problems using the variational iteration method (VIM). First by means of the Pontryagins maximum principle the non-linear two-point boundary-value problem (TPBVP), transformed into an initial value problem (IVP), then we construct variational iterations correction functional to find the approximate solution. Finally, an example is given to illustrate the efficiency and applicability of the proposed method.

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1. Introduction

Theory of optimal control has been used with great success not only in traditional areas such as aerospace engineering [5], robotics [22] and chemical engineering but also in areas as diverse as economics to biomedicine [14]. But in general, solving optimal control problems (OCP) by classic

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control theory is difficult. One new strategy which recently is used by some authors is to transform the optimal control problem to another problem, based on Pontryagins maximum principle (PMP), such that implementing PMP, the OCP reduces to a TPBVP. Yousefi et al. [23] used the original or basic Variational Iteration Method (VIM) for linear quadratic OCPs. They transfer the linear TPBVP obtained from PMP to an initial value problem (IVP) and then implement the basic VIM to get a feedback controller. The variational iteration method as an approximation method for solving linear and nonlinear problems has been the centre of attention of many authors. This method was introduced by the Chinese mathematician Ji-Huan He [6] first, by modifying the general Lagrange multiplier method. The main idea in the variational iteration method is to construct an iterative sequence of functions converging to an exact solution [21]. Since the method works without discretization, it is not affected by round of error. The variational iteration method has been applied for solving a wide range of problems successfully, such as partial differential equations [13, 17], fractional differential equations [12, 18], delay differential equations [7], etc. The variational iteration method is used in [3] to solve the Fokker-Planck equation. He in [9] solved large kinds of equations by VIM such that Blasius equation. In [2], the variational iteration method is employed to solve the Burgers and coupled Burgers equations and in [17] it was applied to Helmholtz equation. In current work we solve the nonlinear quadratic OCP by VIM.

2. Non-Linear Quadratic OCPs and Solution Guidelines

Consider the non-linear control system

$$\begin{aligned} &\dot{x}(t) = f(t, x(t)) + g(t, x(t))u(t), \quad t \in [t_0, t_f], \\ &x(t_0) = x_0, \quad x(t_f) = x_f, \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state variable, $u(t) \in \mathbb{R}^m$ the control variable and x_0 and x_f the given initial and final states at t_0 and t_f, respectively. also, $f(t, x(t)) \in \mathbb{R}^n$ and $g(t, x(t)) \in \mathbb{R}^{n\times m}$ are two continuously differentiable functions in all arguments. Our aim is to minimize the quadratic objective functional

$$J[x,u] = \frac{1}{2}\int_{t_0}^{t_f}(x^T(t)Qx(t) + u^T(t)Ru(t))dt, \tag{2}$$

subject to the non-linear system (1), for $Q \in \mathbb{R}^{n\times n}$ and $R \in \mathbb{R}^{m\times m}$, positive semi-definite and positive definite matrices, respectively. Hamiltonian for system (1),(2) define as follows (see [4, 15]):

$$H(x,u,\lambda) = \frac{1}{2}[x^TQx + u^TRu] + \lambda^T[f(t,x) + g(t,x)u]. \tag{3}$$

The following extreme necessary conditions are also sufficient for optimality, because the performance index (2) is convex,

$$\begin{aligned} &u^* = argmin_u H(x,u,\lambda),\\ &\dot{\lambda} = -H_x(x,u^*,\lambda),\\ &\dot{x} = f(t,x) + g(t,x)u^*,\\ &x(t_0) = x_0, \quad x(t_f) = x_f. \end{aligned} \tag{4}$$

Since the Hamiltonian function $H(x,u,\lambda)$ must choose its maximum with respect to $u(.)$ at $u^*(.)$, so one can find that (see[19] for more details),

$$u^* = -R^{-1}g^T(t,x)\lambda. \tag{5}$$

So equivalently (4) can be written in the following form where $\lambda(t) \in R^n$ is the co-state vector with the ith component $\lambda_i(t)$, $i = 1,2,\cdots,n$ and $g(t,x) = \big(g_1(t,x)\cdots g_n(x,t)\big)^T$ with $g_i(t,x) \in \mathbb{R}^m$, $i = 1,2,\cdots,n$.

$$\begin{aligned} &\dot{\lambda} = -(Qx + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^{n}\lambda_i[-R^{-1}g^T(t,x)\lambda]^T\frac{\partial g_i(t,x)}{\partial x}),\\ &\dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda],\\ &x(t_0) = x_0, \quad x(t_f) = x_f. \end{aligned} \tag{6}$$

Now we deal with such a TPBVP in (6) instead of non-linear OCP in (1),(2). For solving such a TPBVP, first we use a shooting-method-like procedure, so we obtain the following IVP:

$$\begin{aligned}
&\dot{\lambda} = -(Qx + (\frac{\partial f(t,x)}{\partial x})^T\lambda + \sum_{i=1}^{n} \lambda_i[-R^{-1}g^T(t,x)\lambda]^T \frac{\partial g_i(t,x)}{\partial x}) \\
&\dot{x} = f(t,x) + g(t,x)[-R^{-1}g^T(t,x)\lambda] \\
&x(t_0) = x_0, \quad \lambda(t_0) = \alpha. \qquad (7)
\end{aligned}$$

Then we apply VIM to solve the IVP (6). Where $\alpha \in \mathbb{R}$ is an unknown parameter which can be approximated by imposing final condition in (6) as seen in Section 4.

3. Variational Iteration Method

Consider the following general problem:

$$L(u(t)) + N(u(t)) = g(t),$$

where L is a linear operator, N is a nonlinear operator and $g(t)$ is a known analytical function. The variational iteration method constructs an iterative sequence called correction functional as

$$u_{n+1}(t) = u_n(t) + \int_{t_0}^{t} \mu(s)\big(L(u_n(s)) + N(\tilde{u}_n(s)) - g(s)\big)ds, \qquad (8)$$

where μ is the general Lagrange multiplier that can be identified optimally via the variational theory , $\tilde{u}_n(s)$ is considered as the restricted variation, i.e. $\delta\tilde{u}_n(s) = 0$ and the index n denotes the nth iteration(for more details, see [1] and [6]).

4. Suboptimal Control Design

Consider the OCP of the non-linear system (1) with the quadratic cost function (2). Then, the Nth order suboptimal trajectory-control pair is obtained as follows:

$$\begin{cases} x^N(t) = x_n(t), \\ u^N(t) = -R^{-1}g^T(t,x)\lambda_n(t). \end{cases} \qquad (9)$$

Then the following quadratic performance index (QPI) can be calculated as

$$J^{(N)} = \frac{1}{2}\int_{t_0}^{t_f} [(x^{(N)}(t))^T Q x^{(N)}(t) + (u^{(N)}(t))^T R u^{(N)}(t)]dt. \qquad (10)$$

The Nth-order suboptimal trajectory-control pair in (9) has desirable accuracy if for two given positive constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, the following conditions hold jointly:

$$\begin{aligned} &\mid \frac{J^{(N)} - J^{(N-1)}}{J^{(N)}} \mid < \epsilon_1, \\ &\parallel x(t_f) - x_f \parallel < \epsilon_2. \end{aligned} \qquad (11)$$

where $\parallel . \parallel$ is a suitable norm on $\mathbb{R}^n$ and $x(t_f)$ is the value of the corresponding state trajectory at the final time t_f.

5. A Numerical Example

Consider the following non-linear OCP (see [4, 15]):

$$\begin{aligned} &min J = \int_0^1 u^2(t)dt \\ &s.t \quad \dot{x(t)} = \frac{1}{2}x^2(t) sin x(t) + u(t) \quad , \quad t \in [0,1] \\ &x(0) = 0 \quad , \quad x(1) = 0.5. \end{aligned} \qquad (12)$$

According to (1) and (2) we have $f(t, x(t)) = \frac{1}{2}x^2(t) sin x(t)$, $g(t, x(t)) = 1$, $Q = 0$, $R = 1$, $t_0 = 0$ and $t_f = 1$. As mentioned in Section 2, we solve the following IVP:

$$\begin{aligned} &\dot{x}(t) = \frac{1}{2}x^2(t) sin x(t) - \lambda(t), \\ &\dot{\lambda} = -\lambda(t)x(t) sin x(t) - \frac{1}{2}\lambda(t)x^2(t) cos x(t) \quad , \quad t \in [0,1] \\ &x(0) = 0 \quad , \quad \lambda(0) = \alpha. \end{aligned} \qquad (13)$$

where $\alpha \in \mathbb{R}$ is an unknown parameter. Also the optimal control law is given by

$$u^*(t) = -\lambda(t).$$

Now we construct the connection functional (8) for system (13) as follows:

$$\begin{cases} x_{n+1}(t) = x_n(t) + \int_0^t \mu_1\big(\dot{x}_n(s) - \frac{1}{2}x_n^2(s)\sin x_n(s) + \lambda_n(s)\big)ds, \\ \lambda_{n+1}(t) = \lambda_n(t) + \int_0^t \mu_2\big(\dot{\lambda}_n(s) + \lambda_n(s)x_n(s)\sin x_n(s) \\ +\frac{1}{2}\lambda_n(s)x_n^2(s)\cos x_n(s)\big)ds. \end{cases} \tag{14}$$

By taking variation with respect to independent variables $x_n(t)$ and $\lambda_n(t)$, we get

$$\begin{cases} \delta x_{n+1} = \delta x_n + \delta\int_0^t \mu_1\big(\dot{x}_n - \frac{1}{2}x_n^2\sin x_n + \lambda_n\big)ds = \delta x_n + \delta\mu_1 x_n \\ -\int_0^t \delta x_n \dot{\mu}_1 ds = 0 \\ \delta\lambda_{n+1} = \delta\lambda_n + \delta\int_0^t \mu_2\big(\dot{\lambda}_n(s) + \lambda_n x_n \sin x_n + \frac{1}{2}\lambda_n x_n^2\cos x_n\big)ds \\ = \delta\lambda_n + \delta\mu_2\lambda_n - \int_0^t \delta\lambda_n\dot{\mu}_2 ds = 0. \end{cases} \tag{15}$$

Note that $(-\frac{1}{2}x_n^2\sin x_n+\lambda_n)$ in first equation and $(\lambda_n x_n\sin x_n+\frac{1}{2}\lambda_n x_n^2\cos x_n)$ in the second equation are considered as restricted variations , i.e. $\delta(-\frac{1}{2}x_n^2\sin x_n + \lambda_n) = 0$ and $\delta(\lambda_n x_n\sin x_n + \frac{1}{2}\lambda_n x_n^2\cos x_n) = 0$, so making the above correction functionals stationary implies the following stationary conditions

$$\begin{cases} \delta x_n : 1 - \mu_1(s)\big|_{s=t} = 0 \\ \delta x_n : \dot{\mu}_1(s) = 0 \\ \delta\lambda_n : 1 - \mu_2(s)\big|_{s=t} = 0 \\ \delta\lambda_n : \dot{\mu}_2(s) = 0 \end{cases} \tag{16}$$

The general Lagrange multipliers, therefore, can be readily identified

$$\mu_1 = -1,$$
$$\mu_2 = -1.$$

As a result, we obtain the following iteration formulas:
$x_{n+1}(t) = x_n(t) - \int_0^t \big(\dot{x}_n(s) - \frac{1}{2}x_n^2(s)\sin x_n(s) + \lambda_n(s)\big)ds,$

$$\lambda_{n+1}(t) = \lambda_n(t) - \int_0^t \big(\dot{\lambda}_n(s) + \lambda_n(t)x_n(s)\sin x_n(s) + \tfrac{1}{2}\lambda_n(s)x_n^2(s)\cos x_n(s)\big)ds. \tag{17}$$

As first iteration with initial approximations $x_0(t) = x(0) = 0$ and $\lambda_0(t) = \lambda(0) = \alpha$, we have

$$\begin{cases} x_1(t) = x_0(t) - \int_0^t \big(\dot{x}_0(s) - \frac{1}{2}x_0^2(s)\sin x_0(s) + \lambda_0(s)\big)ds = -\alpha t \\ \lambda_1(t) = \lambda_0(t) - \int_0^t \big(\dot{\lambda}_0(s) + \lambda_0(s)x_0(s)\sin x_0(s) \\ +\frac{1}{2}\lambda_0(s)x_0^2(s)\cos x_0(s)ds = \alpha. \end{cases} \tag{18}$$

By imposing final state condition we have

$$\begin{aligned} 0.5 = x(1) \approx x_1(1) = -\alpha, \\ \alpha \approx -0.5, \end{aligned}$$

thus

$$\begin{matrix} x_1(t) = \frac{1}{2}t \\ \lambda_1(t) = -\frac{1}{2} \end{matrix}.$$

If we suppose $\epsilon_1 = 7 \times 10^{-2}$ and $\epsilon_2 = 2 \times 10^{-2}$ as tolerance error bounds, convergence is achieved after two iterations, i.e. $\mid \frac{J^{(2)} - J^{(1)}}{J^{(2)}} \mid = 6.25 \times 10^{-2} < \epsilon_1$ and $\parallel x(1) - 0.5 \parallel = 1.52 \times 10^{-2} < \epsilon_2$. So we have

$$x(t) \approx x_2(t)$$

$$u(t) = -\lambda(t) \approx -\lambda_2(t).$$

Simulation curves of $x(t)$ and $u(t)$ got from second step of variational iteration method are shown in Fig 1. Also, as you see in Fig 1, we compared the results of VIM with the solutions obtained using the collocation method [1], modal series [15] and homotopy perturbation method [4]. Our results are very close to all three of them.

Problem (12) has also been solved by Rubio [19] via the measure theory in which to find an acceptable solution, a linear programming problem with 1000 variables and 20 constraints should be solved.

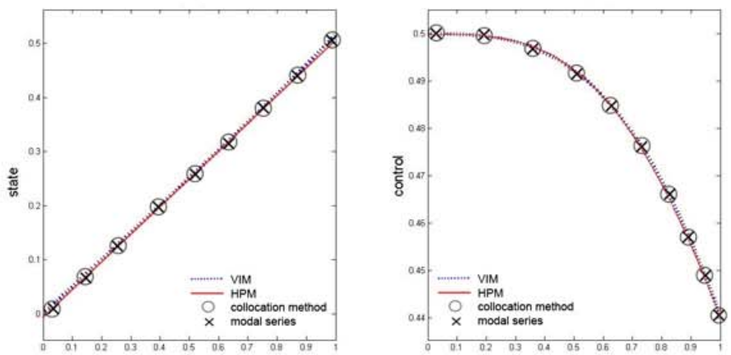

Figure 1. Simulation curves of $x(.)$ and $u(.)$ got from second step of variational iteration method.

Table1 represents simulation results of the example for two steps of VIM. In Table 2 rapidity of VIM is compared with modal series method [15] and homotopy perturbation method [4].

Table 1: Simulation results of the example for two step of VIM

i	$J(i)$	$\left\lvert\frac{J(i)-J(i-1)}{J(i)}\right\rvert$	$\lVert x(t_f)-x_f \rVert$
0	0.25	-	-
1	0.25	0	-
2	0.2353	6.25×10^{-2}	1.52×10^{-2}

Table 2: Result of the VIM and two other methods

Method	Number of steps (n)	Performance index value
HPM	5	0.2353
Modal series	5	0.2353
VIM	2	0.2353

6. Conclusion

In this article, we successfully applayed VIM to solve non-linear quadratic OCPs. We saw the optimal control law and the optimal state trajectory were determined rapidly and easily with few iterations.

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