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Original Research Paper

Ideal Theory in Sheffer Stroke Hilbert Algebras Using Intuitionistic Fuzzy Points

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Abstract. This paper aims to extend the concept of ideals in Sheffer stroke Hilbert algebras to the framework of intuitionistic fuzzy set theory. We introduce and define the notion of intuitionistic fuzzy ideals using intuitionistic fuzzy points and examine their structural properties. We provide several characterizations of these ideals and identify the necessary and sufficient conditions under which an intuitionistic fuzzy set qualifies as an ideal. Furthermore, we construct the $(0,1)$-set associated with an intuitionistic fuzzy set and investigate the circumstances under which it forms an ideal. The study also explores the roles of intuitionistic level sets and intuitionistic q-sets in the context of ideals, offering a detailed analysis supported by illustrative examples.

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1 Introduction

A fuzzy set, a concept from fuzzy logic, is an extension of classical set theory designed to handle imprecise or uncertain data. While a crisp (or classical) set includes elements that either belong or do not belong to the set (with binary membership values of 0 or 1), a fuzzy set allows each element to possess a degree of membership ranging between 0 and 1, thereby capturing uncertainty or vagueness commonly encountered in real-world situations.

The intuitionistic fuzzy (IF) set, introduced by Krasimir Atanassov in 1986, generalizes fuzzy sets and serves as a more powerful tool for modeling uncertainty and imprecision. Unlike a fuzzy set, which considers only the degree of membership, an IF set accounts for both membership and non-membership degrees simultaneously, along with a degree of hesitation (or uncertainty) associated with each element.

An intuitionistic fuzzy point (see [10]) is an extension of the classical notion of a point in set theory, adapted to the IF set framework. It plays a significant role in the structure and analysis of intuitionistic fuzzy sets.

The Sheffer stroke (also known as the Sheffer operation) is a logical operation in Boolean algebra that yields a false result only when both inputs are true. It is equivalent to the NAND operation and is commonly denoted by |" or, alternatively, by ↑". The formal definition of this operation is presented in Table 1.

Table 1: The truth table for the Sheffer stroke "|"

P	Q	$P\|Q$
F	F	T
F	T	T
T	F	T
T	T	F

The Sheffer stroke has been applied to several algebraic structures, such as Boolean algebras, MV-algebras, BL-algebras, BCK-algebras, and ortholattices, and has also been studied within the context of fuzzy set theory (see [2, 3, 4, 5, 6, 7, 8, 9]).

In 2021, Oner et al. [6] extended this approach by applying the Sheffer stroke to Hilbert algebras. They introduced the concept of Sheffer stroke Hilbert algebras and explored several of their fundamental properties. Subsequently, in [5], Oner et al. introduced the notions of deductive systems and filters for Sheffer stroke Hilbert algebras, and investigated their fuzzification. Furthermore, they proposed the concept of ideals within this framework and examined their structural characteristics [6].

Building on the results introduced in ideals of Sheffer stroke Hilbert algebras based on fuzzy points (see [13]), this paper extends the ideal theory in Sheffer stroke Hilbert algebras by incorporating intuitionistic fuzzy points, providing a more generalized framework for studying fuzzy ideals and deductive systems.

The purpose of this paper is to study intuitionistic fuzzy (IF) versions of ideals in Sheffer stroke Hilbert algebras using intuitionistic fuzzy points. We introduce the concept of IF ideals and investigate their key properties. In particular, we discuss their characterization and determine the necessary and sufficient conditions under which an IF set becomes an IF ideal.

Moreover, we construct the $(0,1)$-set $H_{(0,1)}$ associated with an IF set $A^* := (\hbar_A, \eth_A)$ and analyze the conditions under which it constitutes an ideal. We also establish criteria for the intuitionistic level set and intuitionistic q-set to form ideals within this algebraic structure.

2 Preliminaries

Definition 2.1 ([11]). *Let $\mathcal{A} := (A, |)$ be a groupoid. Then the operation "$|$" is said to be Sheffer stroke or Sheffer operation if it satisfies:*

(s1) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ (\mathfrak{a}|\mathfrak{b} = \mathfrak{b}|\mathfrak{a})$,

(s2) $(\forall \mathfrak{a}, \mathfrak{b} \in A)\ ((\mathfrak{a}|\mathfrak{a})|(\mathfrak{a}|\mathfrak{b}) = \mathfrak{a})$,

(s3) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ (\mathfrak{a}|((\mathfrak{b}|\mathfrak{c})|(\mathfrak{b}|\mathfrak{c})) = ((\mathfrak{a}|\mathfrak{b})|(\mathfrak{a}|\mathfrak{b}))|\mathfrak{c})$,

(s4) $(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in A)\ ((\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b})))|(\mathfrak{a}|((\mathfrak{a}|\mathfrak{a})|(\mathfrak{b}|\mathfrak{b}))) = \mathfrak{a})$.

Let $\mathbb{H} := (H, |)$ be a groupoid. For every element $\mathfrak{a} \in H$, consider the following mapping:

$$f_{\mathfrak{a}} : H \to H,\ \mathfrak{b} \mapsto \mathfrak{a}|(\mathfrak{b}|\mathfrak{b}).$$

Definition 2.2 ([6]). *A Sheffer stroke Hilbert algebra is a groupoid $\mathbb{H} := (H, |)$ with a Sheffer stroke "$|$" that satisfies:*

(sH1) $(\mathfrak{a}|(f_{\mathfrak{b}}(\mathfrak{c})|f_{\mathfrak{b}}(\mathfrak{c})))|((f_{\mathfrak{a}}(\mathfrak{b})|(f_{\mathfrak{a}}(\mathfrak{c})|f_{\mathfrak{a}}(\mathfrak{c})))|(f_{\mathfrak{a}}(\mathfrak{b})|(f_{\mathfrak{a}}(\mathfrak{c})|f_{\mathfrak{a}}(\mathfrak{c})))) = f_{\mathfrak{a}}(\mathfrak{a})$,

(sH2) $f_{\mathfrak{a}}(\mathfrak{b}) = f_{\mathfrak{b}}(\mathfrak{a}) = f_{\mathfrak{a}}(\mathfrak{a}) \ \Rightarrow\ \mathfrak{a} = \mathfrak{b}$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in H$.

Let $\mathbb{H} := (H, |)$ be a Sheffer stroke Hilbert algebra. Then the order relation "$\preceq$" on H is defined as follows:

$$(\forall \mathfrak{a}, \mathfrak{b} \in H)(\mathfrak{a} \preceq \mathfrak{b} \ \Leftrightarrow\ f_{\mathfrak{a}}(\mathfrak{b}) = 1). \tag{1}$$

We observe that the relation "$\preceq$" is a partial order in a Sheffer stroke Hilbert algebra $\mathbb{H} := (H, |)$ (see [6]). Recall that the Sheffer stroke Hilbert algebra $\mathbb{H} := (H, |)$ satisfies the identity $f_{\mathfrak{a}}(\mathfrak{a}) = f_{\mathfrak{b}}(\mathfrak{b})$, which is denoted by 1, for all $\mathfrak{a}, \mathfrak{b} \in H$ (see [6]).

Proposition 2.3 ([6]). *Every Sheffer stroke Hilbert algebra* $\mathbb{H} := (H, |)$ *satisfies:*

$$(\forall \mathfrak{a} \in H)(f_{\mathfrak{a}}(\mathfrak{a}) = 1), \tag{2}$$
$$(\forall \mathfrak{a} \in H)(f_{\mathfrak{a}}(1) = 1), \tag{3}$$
$$(\forall \mathfrak{a} \in H)(f_1(\mathfrak{a}) = \mathfrak{a}), \tag{4}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in H)(\mathfrak{a} \preceq f_{\mathfrak{b}}(\mathfrak{a})), \tag{5}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in H)(f_{\mathfrak{a}}(\mathfrak{b})|(\mathfrak{b}|\mathfrak{b}) = f_{\mathfrak{b}}(\mathfrak{a})|(\mathfrak{a}|\mathfrak{a})), \tag{6}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in H)\left((f_{\mathfrak{a}}(\mathfrak{b})|(\mathfrak{b}|\mathfrak{b}))|(\mathfrak{b}|\mathfrak{b}) = f_{\mathfrak{a}}(\mathfrak{b})\right), \tag{7}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in H)\left(\mathfrak{a}|(f_{\mathfrak{b}}(\mathfrak{c})|f_{\mathfrak{b}}(\mathfrak{c})) = \mathfrak{b}|(f_{\mathfrak{a}}(\mathfrak{c})|f_{\mathfrak{a}}(\mathfrak{c}))\right), \tag{8}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in H)(\mathfrak{a} \preceq \mathfrak{b} \ \Rightarrow\ f_{\mathfrak{c}}(\mathfrak{a}) \preceq f_{\mathfrak{c}}(\mathfrak{b}),\ f_{\mathfrak{b}}(\mathfrak{c}) \preceq f_{\mathfrak{a}}(\mathfrak{c})), \tag{9}$$
$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in H)(\mathfrak{a}|(f_{\mathfrak{b}}(\mathfrak{c})|f_{\mathfrak{b}}(\mathfrak{c})) = (f_{\mathfrak{a}}(\mathfrak{b})|(f_{\mathfrak{a}}(\mathfrak{c})|f_{\mathfrak{a}}(\mathfrak{c}))). \tag{10}$$

By (3), we know that the element 1 is the greatest element in $\mathbb{H} := (H, |)$ with respect to the order $\preceq$.

Proposition 2.4. *[6] Let* $\mathbb{H} := (H, |)$ *be a Sheffer stroke Hilbert algebra with the smallest element* 0. *Then*

$$0|0 = 1,\ \ 1|1 = 0, \tag{11}$$
$$f_1(0) = 0,\ \ f_0(0) = 1. \tag{12}$$

Definition 2.5 ([6]). *Let* $\mathbb{H} := (H, |)$ *be a Sheffer stroke Hilbert algebra with the smallest element* 0. *A subset* D *of* H *is called an ideal of* $\mathbb{H} := (H, |)$ *if it satisfies:*

$$0 \in D, \tag{13}$$
$$(\forall \mathfrak{a}, \mathfrak{b} \in H)(\mathfrak{b} \in D,\ f_{\mathfrak{a}}(\mathfrak{b})|f_{\mathfrak{a}}(\mathfrak{b}) \in D \ \Rightarrow\ \mathfrak{a} \in D). \tag{14}$$

Let H be a set. An *intuitionistic fuzzy set* A^* in H (see [1]) is an object having the form

$$A^* := \{\langle \mathfrak{a}, \hbar_A(\mathfrak{a}), \eth_A(\mathfrak{a})\rangle \mid \hbar_A(\mathfrak{a}) + \eth_A(\mathfrak{a}) \leq 1,\ \mathfrak{a} \in H\},$$

which is simply denoted by $A^* := (\hbar_A, \eth_A)$ where $\hbar_A$ and $\eth_A$ are fuzzy sets in H,

The intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H can be represented as follows:

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1], \mathfrak{a} \mapsto (\hbar_A(\mathfrak{a}), \eth_A(\mathfrak{b}))$$

such that $\hbar_A(\mathfrak{a}) + \eth_A(\mathfrak{b}) \leq 1$.

An intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in a set H of the form

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1], \mathfrak{b} \mapsto \begin{cases} (s,t) \in (0,1] \times [0,1) & \text{if } \mathfrak{b} = \mathfrak{a}, \\ (0,1) & \text{if } \mathfrak{b} \neq \mathfrak{a}, \end{cases}$$

is said to be an *intuitionistic fuzzy point* with support $\mathfrak{a}$ and values (s,t) such that $s + t \leq 1$, and is denoted by $\langle \mathfrak{a}_{(s,t)} \rangle$ (see [12]).

Given an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ and intuitionistic fuzzy point $\langle \mathfrak{a}_{(s,t)} \rangle$ in H, we say

$$\langle \mathfrak{a}_{(s,t)} \rangle \in A^* \text{ if } \hbar_A(\mathfrak{a}) \geq s \text{ and } \eth_A(\mathfrak{a}) \leq t. \tag{15}$$

$$\langle \mathfrak{a}_{(s,t)} \rangle \, q \, A^* \text{ if } \hbar_A(\mathfrak{a}) + s > 1 \text{ and } \eth_A(\mathfrak{a}) + t < 1. \tag{16}$$

$$\langle \mathfrak{a}_{(s,t)} \rangle \in\!\vee q \, A^* \text{ if } \langle \mathfrak{a}_{(s,t)} \rangle \in A^* \text{ or } \langle \mathfrak{a}_{(s,t)} \rangle \, q \, A^*. \tag{17}$$

Given $(s,t) \in (0,1] \times [0,1)$ and an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H, consider the following sets:

$$\begin{aligned}
&(\hbar_A, s)_\in := \{\mathfrak{a} \in H \mid \hbar_A(\mathfrak{a}) \geq s\},\\
&(\eth_A, t)_\in := \{\mathfrak{a} \in H \mid \eth_A(\mathfrak{a}) \leq t\},\\
&(\hbar_A, s)_q := \{\mathfrak{a} \in H \mid \hbar_A(\mathfrak{a}) + s > 1\},\\
&(\eth_A, t)_q := \{\mathfrak{a} \in H \mid \eth_A(\mathfrak{a}) + t < 1\}.\\
&(\hbar_A, s)_{\in\vee q} := \{\mathfrak{a} \in H \mid \hbar_A(\mathfrak{a}) \geq s \text{ or } \hbar_A(\mathfrak{a}) + s > 1\}.\\
&(\eth_A, t)_{\in\vee q} := \{\mathfrak{a} \in H \mid \eth_A(\mathfrak{a}) \leq t \text{ or } \eth_A(\mathfrak{a}) + t < 1\}.
\end{aligned}$$

Also, we consider the sets below.

$$\begin{aligned}
&(A^*, (s,t))_\in := (\hbar_A, s)_\in \cap (\eth_A, t)_\in,\\
&(A^*, (s,t))_q := (\hbar_A, s)_q \cap (\eth_A, t)_q,\\
&(A^*, (s,t))_{\in\vee q} := (\hbar_A, s)_{\in\vee q} \cap (\eth_A, t)_{\in\vee q},
\end{aligned}$$

which are called the *intuitionistic level set*, *intuitionistic q-set* and *intuitionistic* $\in\!\vee q$-*set* of $A^* := (\hbar_A, \eth_A)$, respectively.

3 Intuitionistic Fuzzy Ideals

In what follows, let $\mathbb{H} := (H, |)$ denote a Sheffer stroke Hilbert algebra. Also, (s,t) and (s_i, t_i) are elements of $(0,1] \times [0,1)$ that satisfies $s + t \leq 1$ and $s_i + t_i \leq 1$, respectively, for $i = 1, 2, 3, \cdots$.

First, we introduce a central concept that will be used throughout the paper.

Definition 3.1. *An intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H is called an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ if it satisfies:*

$$(\hbar_A, s)_\in \cap (\eth_A, t)_\in \neq \emptyset \;\Rightarrow\; 0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in, \tag{18}$$

$$\left(\begin{array}{l} x \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in,\ f_y(x)|f_y(x) \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in \\ \;\Rightarrow y \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in \end{array} \right) \tag{19}$$

for all $x, y, z \in H$.

Example 3.2. Consider a set $H = \{1, 2, 3, 0\}$, and define a Sheffer stroke "|" by Table 2.

Table 2: Cayley table for the Sheffer stroke "|"

$\mid$	1	2	3	0
1	0	3	2	1
2	3	3	1	1
3	2	1	2	1
0	1	1	1	1

Then $\mathbb{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [6]). Let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H given by

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$\mathfrak{b} \mapsto \begin{cases} \left(\frac{1}{2n}, \frac{2(n-1)}{2n}\right) & \text{if } \mathfrak{b} = 0, \\ \left(\frac{1}{5n}, \frac{5(n-1)}{5n}\right) & \text{if } \mathfrak{b} = 1, \\ \left(\frac{1}{3n}, \frac{5(n-1)}{5n}\right) & \text{if } \mathfrak{b} = 2, \\ \left(\frac{1}{5n}, \frac{3(n-1)}{3n}\right) & \text{if } \mathfrak{b} = 3, \end{cases}$$

where n is a natural number. Then $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$.

Example 3.3. Consider a set $H = \{0, 1, 2, 3, 4, 5, 6, 7\}$, and define a Sheffer stroke "|" by Table 3.

Table 3: Cayley table for the Sheffer stroke " | "

$\mid$	0	2	3	4	5	6	7	1
0	1	1	1	1	1	1	1	1
2	1	7	1	1	7	7	1	7
3	1	1	6	1	6	1	6	6
4	1	1	1	5	1	5	5	5
5	1	7	6	1	4	7	6	4
6	1	7	1	5	7	3	5	3
7	1	1	6	5	6	5	2	2
1	1	7	6	5	4	3	2	0

Then $\mathbb{H} := (H, |)$ is a Sheffer stroke Hilbert algebra (see [6]).

Let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H defined by:

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$x \mapsto \begin{cases} \left(\frac{1}{2n}, \frac{2(n-1)}{2n}\right) & \text{if } x = 0, \\ \left(\frac{1}{4n}, \frac{3(n-1)}{3n}\right) & \text{if } x = 1, \\ \left(\frac{1}{5n}, \frac{4(n-1)}{4n}\right) & \text{if } x = 2, \\ \left(\frac{1}{6n}, \frac{5(n-1)}{5n}\right) & \text{if } x = 3, \\ \left(\frac{1}{4n}, \frac{2(n-1)}{2n}\right) & \text{if } x = 4, \\ \left(\frac{1}{3n}, \frac{2(n-1)}{2n}\right) & \text{if } x = 5, \\ \left(\frac{1}{5n}, \frac{3(n-1)}{3n}\right) & \text{if } x = 6, \\ \left(\frac{1}{4n}, \frac{4(n-1)}{4n}\right) & \text{if } x = 7, \end{cases}$$

where n is a natural number. Then $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $(H, |)$.

We discuss characterizations of an intuitionistic fuzzy ideal.

Theorem 3.4. *An intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ if and only if the following assertions are valid.*

$$\hbar_A(0) \geq \hbar_A(x),\ \eth_A(0) \leq \eth_A(x), \tag{20}$$

$$\left(\begin{array}{l} \hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \\ \eth_A(x) \leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\} \end{array} \right) \tag{21}$$

for all $x, y \in H$.

Proof. Assume that $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. If there exists $\mathfrak{a} \in H$ such that $\hbar_A(0) < \hbar_A(\mathfrak{a})$ or $\eth_A(0) > \eth_A(\mathfrak{a})$, then $\mathfrak{a} \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ for $s = \hbar_A(\mathfrak{a})$ and $t = \eth_A(\mathfrak{a})$, and so $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ by (18). Hence $\hbar_A(0) \geq s = \hbar_A(\mathfrak{a})$ and $\eth_A(0) \leq t = \eth_A(\mathfrak{a})$ which is a contradiction. Thus (20) is valid. Let $x, y \in H$ be such that $(s_1, t_1) = (\hbar_A(y), \eth_A(y))$ and $(s_2, t_2) = (\hbar_A(f_x(y)|f_x(y)), \eth_A(f_x(y)|f_x(y)))$. Then $y \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$ and $f_x(y)|f_x(y) \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$. Thus

$$x \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in$$

by (19), and so $\hbar_A(x) \geq \min\{s_1, s_2\} = \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}$ and

$$\eth_A(x) \leq \max\{t_1, t_2\} = \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\}.$$

Conversely, suppose that $A^* := (\hbar_A, \eth_A)$ satisfies (20) and (21). If $(\hbar_A, s)_\in \cap (\eth_A, t)_\in \neq \emptyset$, then there exists $\mathfrak{a} \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ which implies from (20) that

$\hbar_A(0) \geq \hbar_A(\mathfrak{a}) \geq s$ and $\eth_A(0) \leq \eth_A(\mathfrak{a}) \leq t$. Hence $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. Let $x, y \in H$ be such that $x \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$ and $f_y(x)|f_y(x) \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$. Then $\hbar_A(x) \geq s_1$, $\eth_A(x) \leq t_1$, $\hbar_A(f_y(x)|f_y(x)) \geq s_2$ and $\eth_A(f_y(x)|f_y(x)) \leq t_2$. It follows from (21) that

$$\hbar_A(y) \geq \min\{\hbar_A(x), \hbar_A(f_y(x)|f_y(x)\} \geq \min\{s_1, s_2\}$$

and $\eth_A(y) \leq \max\{\eth_A(x), \eth_A(f_y(x)|f_y(x)\} \leq \max\{t_1, t_2\}$. Hence

$$y \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in,$$

and therefore $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. □

Theorem 3.5. *An intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ if and only if the nonempty sets $(\hbar_A, s)_\in$ and $(\eth_A, t)_\in$ are ideals of $\mathbb{H} := (H, |)$.*

Proof. The necessity is clear. Assume that the nonempty sets $(\hbar_A, s)_\in$ and $(\eth_A, t)_\in$ are ideals of $\mathbb{H} := (H, |)$. If $\hbar_A(0) < \hbar_A(\mathfrak{a})$ or $\eth_A(0) > \eth_A(\mathfrak{a})$ for some $\mathfrak{a} \in H$, then $\mathfrak{a} \in (\hbar_A, s)_\in$ and $\mathfrak{a} \in (\eth_A, t)_\in$ where $s = \hbar_A(\mathfrak{a})$ and $t = \eth_A(\mathfrak{a})$. Thus $0 \in (\hbar_A, s)_\in$ and $0 \in (\eth_A, t)_\in$, and so $\hbar_A(0) \geq s = \hbar_A(\mathfrak{a})$ and $\eth_A(0) \leq t = \eth_A(\mathfrak{a})$. This is a contradiction. Thus $\hbar_A(0) \geq \hbar_A(x)$ and $\eth_A(0) \leq \eth_A(x)$ for all $x \in H$. Suppose that $\hbar_A(\mathfrak{a}) < \min\{\hbar_A(\mathfrak{b}), \hbar_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b})\}$ or

$$\eth_A(\mathfrak{a}) > \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b})\}$$

for some $\mathfrak{a} \in H$. If we take $s := \frac{1}{2}\left(\hbar_A(\mathfrak{a}) + \min\{\hbar_A(\mathfrak{b}), \hbar_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\}\right)$ and

$$t := \tfrac{1}{2}\left(\eth_A(\mathfrak{a}) + \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\}\right),$$

then $\mathfrak{b} \in (\hbar_A, s)_\in$, $f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\hbar_A, s)_\in$, $\mathfrak{a} \notin (\hbar_A, s)_\in$, $\mathfrak{b} \in (\eth_A, t)_\in$, $f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\eth_A, t)_\in$, and $\mathfrak{a} \notin (\eth_A, t)_\in$. This is a contradiction. Hence (21) is valid, and therefore $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ by Theorem 3.4. □

Corollary 3.6. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$, then its nonempty intuitionistic level set $(A^*, (s, t))_\in$ is an ideal of $\mathbb{H} := (H, |)$.*

Proposition 3.7. *Every intuitionistic fuzzy ideal $A^* := (\hbar_A, \eth_A)$ of $\mathbb{H} := (H, |)$ satisfies:*

$$x \preceq y,\ y \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in \ \Rightarrow\ x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in \tag{22}$$

for all $x, y \in H$.

Proof. Let $x, y \in H$ be such that $x \preceq y$ and $y \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. Then $f_x(y) = 1$, and thus $f_x(y)|f_x(y) = 1|1 = 0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ by (11) and (18). It follows from (19) that $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. □

In the following example, we can confirm that the converse of Proposition 3.7 is not true in general.

Example 3.8. Consider the Sheffer stroke Hilbert algebra $\mathbb{H} := (H, |)$ in Example 3.2 and let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H given by

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$\mathfrak{b} \mapsto \begin{cases} (0.87, 0.19) & \text{if } \mathfrak{b} = 0, \\ (0.43, 0.52) & \text{if } \mathfrak{b} = 1, \\ (0.63, 0.33) & \text{if } \mathfrak{b} \in \{2, 3\}. \end{cases}$$

Then $A^* := (\hbar_A, \eth_A)$ satisfies (22). If we take $(s_1, t_1) = (0.48, 0.41)$ and $(s_2, t_2) = (0.56, 0.35)$, then $f_1(2)|f_1(2) = 3 \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$ and $2 \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$, but $1 \notin (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in$. Hence $A^* := (\hbar_A, \eth_A)$ is not an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$.

Example 3.9. Consider the Sheffer stroke Hilbert algebra $(H, |)$ in Example 3.3 and let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H defined by

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$x \mapsto \begin{cases} (0.72, 0.24) & \text{if } x = 0, \\ (0.35, 0.60) & \text{if } x = 1, \\ (0.45, 0.42) & \text{if } x = 2, \\ (0.58, 0.33) & \text{if } x = 3, \\ (0.62, 0.29) & \text{if } x = 4, \\ (0.39, 0.55) & \text{if } x = 5, \\ (0.53, 0.37) & \text{if } x = 6, \\ (0.60, 0.31) & \text{if } x = 7, \end{cases}$$

Then $A^* := (\hbar_A, \eth_A)$ satisfies (22). If we take $(s_1, t_1) = (0.40, 0.48)$ and $(s_2, t_2) = (0.55, 0.38)$, then $x = 2$ and $y = 3$ satisfy $x \mid y = 1 \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$ and $x = 2 \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$, but $y = 3 \notin (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in$. Hence $A^* := (\hbar_A, \eth_A)$ is **not** an intuitionistic fuzzy ideal of $(H, |)$.

We explore the conditions under which an intuitionistic fuzzy set can be an intuitionistic fuzzy ideal.

Lemma 3.10 ([6])**.** *In a Sheffer stroke Hilbert algebra $\mathbb{H} := (H, |)$, the set $\{x, y\}$ has the least upper bound $f_x(y)|(y|y)$ for all $x, y \in H$.*

Theorem 3.11. *An intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ if and only if it satisfies (22) and*

$$\begin{aligned} & x \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in,\ y \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in \\ & \Rightarrow\ f_x(y)|(y|y) \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in \end{aligned} \tag{23}$$

for all $x, y \in H$.

Proof. Assume that $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. Then it satisfies (22) by Proposition 3.7. Let $x \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$ and $y \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$. We can observe that

$$\begin{aligned}
&((((f_x(y)|(y|y))|(y|y))|((f_x(y)|(y|y))|(y|y)))|(x|x))| \\
&\quad ((((f_x(y)|(y|y))|(y|y))|((f_x(y)|(y|y))|(y|y)))|(x|x)) \\
&= ((f_x(y)|f_x(y))|(x|x))|((f_x(y)|f_x(y))|(x|x)) \\
&= ((y|y)|(f_x(x)|f_x(x)))|((y|y)|(f_x(x)|f_x(x))) \\
&= ((y|y)|(1|1))|((y|y)|(1|1)) \\
&= 1|1 = 0 \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in.
\end{aligned}$$

Since $x \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$, we have

$$\begin{aligned}
&(((f_x(y)|(y|y))|(y|y))|((f_x(y)|(y|y))|(y|y))) \\
&\in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in
\end{aligned}$$

by (19). Also, since $y \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$, it follows from (19) that

$$f_x(y)|(y|y) \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in.$$

Conversely, suppose that $A^* := (\hbar_A, \eth_A)$ satisfies (22) and (23). Clearly $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ by (22). Let $x, y \in H$ be such that $y \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$ and $f_x(y)|f_x(y) \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$. Using (S2), (S3) and (23), we have

$$\begin{aligned}
f_x(y)|(y|y) &= (x|(f_{y|y}(y)|f_{y|y}(y)))|(y|y) \\
&= ((f_x(y)|f_x(y))|(y|y))|(y|y) \\
&\in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in.
\end{aligned}$$

Since $x \preceq f_x(y)|(y|y)$ by Lemma 3.10, it follows from (22) that

$$x \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in.$$

Therefore $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. □

Proposition 3.12. *Let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H that satisfies:*

$$\left(\begin{array}{l} (\hbar_A, s)_\in \neq \emptyset \ \Rightarrow 0 \in (\hbar_A, s)_\in \ \textit{or}\ 0 \in (\hbar_A, s)_q \\ (\eth_A, t)_\in \neq \emptyset \ \Rightarrow 0 \in (\eth_A, t)_\in \ \textit{or}\ 0 \in (\eth_A, t)_q \end{array} \right), \tag{24}$$

$$\left(\begin{array}{l} y \in (\hbar_A, s_1)_\in, f_x(y)|f_x(y) \in (\hbar_A, s_2)_\in \\ \ \Rightarrow \left\{ \begin{array}{l} x \in (\hbar_A, \min\{s_1, s_2\})_\in \ \textit{or} \\ x \in (\hbar_A, \min\{s_1, s_2\})_q \end{array} \right\} \\ \mathfrak{b} \in (\eth_A, t_1)_\in, f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\eth_A, t_2)_\in \\ \ \Rightarrow \left\{ \begin{array}{l} \mathfrak{a} \in (\eth_A, \max\{t_1, t_2\})_\in \ \textit{or} \\ \mathfrak{a} \in (\eth_A, \max\{t_1, t_2\})_q \end{array} \right\} \end{array} \right), \tag{25}$$

for all $x, y, \mathfrak{a}, \mathfrak{b} \in H$. *Then* (24) *and* (25) *are equivalent to the following arguments, respectively:*

$$\hbar_A(0) \geq \min\{\hbar_A(x), 0.5\},\ \eth_A(0) \leq \max\{\eth_A(x), 0.5\}, \tag{26}$$

$$\left(\begin{array}{l} \hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y)), 0.5\} \\ \eth_A(\mathfrak{a}) \leq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b})), 0.5\} \end{array} \right) \tag{27}$$

for all $x, y, \mathfrak{a}, \mathfrak{b} \in H$.

Proof. Assume that $A^* := (\hbar_A, \eth_A)$ satisfies (24). If (26) is not valid, then $\hbar_A(0) < s \leq \min\{\hbar_A(x), 0.5\}$ or $\eth_A(0) > t \geq \max\{\eth_A(y), 0.5\}$ for some $x, y \in H$ and $(s, t) \in (0, 0.5) \times (0.5, 1)$. Hence $x \in (\hbar_A, s)_\in$ and $0 \notin (\hbar_A, s)_\in$, or $y \in (\eth_A, t)_\in$ and $0 \notin (\eth_A, t)_\in$. Also, $\hbar_A(0) + s < 1$ and $\eth_A(0) + t > 1$, i.e., $0 \notin (\hbar_A, s)_q \cap (\eth_A, t)_q$. This is a contradiction, and hence (26) is valid. Suppose that $A^* := (\hbar_A, \eth_A)$ satisfies (25). Let $x, y, \mathfrak{a}, \mathfrak{b} \in H$. If

$$\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} < 0.5 \text{ and } \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} > 0.5,$$

then $\hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}$ and

$$\eth_A(\mathfrak{a}) \leq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\},$$

respectively. In fact, if not then there exists $(s, t) \in (0, 0.5) \times (0.5, 1)$ such that $\hbar_A(x) < s \leq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}$ and

$$\eth_A(\mathfrak{a}) > t \geq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\},$$

respectively. Thus $y \in (\hbar_A, s)_\in$, $f_x(y)|f_x(y) \in (\hbar_A, s)_\in$, $\mathfrak{b} \in (\eth_A, t)_\in$, $f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\eth_A, t)_\in$, but $x \notin (\hbar_A, s)_\in \cap (\hbar_A, s)_q$ and $\mathfrak{a} \notin (\eth_A, t)_\in \cap (\eth_A, t)_q$. This is a contradiction, and so $\hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}$ and

$$\eth_A(\mathfrak{a}) \leq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\}$$

whenever $\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} < 0.5$ and

$$\max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} > 0.5,$$

respectively. If $\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \geq 0.5$ and

$$\max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} \leq 0.5,$$

then $y \in (\hbar_A, 0.5)_\in$, $f_x(y)|f_x(y) \in (\hbar_A, 0.5)_\in$, $\mathfrak{b} \in (\eth_A, 0.5)_\in$, $f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\eth_A, 0.5)_\in$, which imply from (25) that $x \in (\hbar_A, 0.5)_\in$ or $x \in (\hbar_A, 0.5)_q$, and $\mathfrak{a} \in (\eth_A, 0.5)_\in$ or $\mathfrak{a} \in (\eth_A, 0.5)_q$. Hence $\hbar_A(x) \geq 0.5$ and $\eth_A(\mathfrak{a}) \leq 0.5$ because if $\hbar_A(x) < 0.5$ and $\eth_A(\mathfrak{a}) > 0.5$, respectively, then $\hbar_A(x) + 0.5 < 0.5 + 0.5 = 1$ and $\eth_A(\mathfrak{a}) + 0.5 > 0.5 + 0.5 = 1$, i.e., $x \notin (\hbar_A, 0.5)_q$ and $\mathfrak{a} \notin (\eth_A, 0.5)_q$, a contradiction. Therefore $\hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y)), 0.5\}$ and

$$\eth_A(\mathfrak{a}) \leq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b})), 0.5\}$$

for all $x, y, \mathfrak{a}, \mathfrak{b} \in H$.

Conversely, (26) and (27) are valid. Suppose that $(\hbar_A, s)_\in \neq \emptyset \neq (\eth_A, t)_\in$. Then $x \in (\hbar_A, s)_\in$ and $\mathfrak{a} \in (\eth_A, t)_\in$ for some $x, \mathfrak{a} \in H$. Assume that $0 \notin (\hbar_A, s)_\in$ and $0 \notin (\eth_A, t)_\in$, respectively. Then $\hbar_A(x) \geq 0.5$ and $\eth_A(\mathfrak{a}) \leq 0.5$ because if $\hbar_A(x) < 0.5$ and $\eth_A(\mathfrak{a}) > 0.5$, respectively, then

$$\hbar_A(0) \geq \min\{\hbar_A(x), 0.5\} = \hbar_A(x) \geq s$$

and $\eth_A(0) \leq \max\{\eth_A(\mathfrak{a}), 0.5\} = \eth_A(\mathfrak{a}) \leq t$, respectively. This is a contradiction. Hence $\hbar_A(0) + s > 2\hbar_A(0) \geq 2\min\{\hbar_A(x), 0.5\} = 1$ and $\eth_A(0) + t < 2\eth_A(0) \leq 2\max\{\eth_A(\mathfrak{a}), 0.5\} = 1$, that is, $0 \in (\hbar_A, s)_q$ and $0 \in (\eth_A, t)_q$. This shows that $A^* := (\hbar_A, \eth_A)$ satisfies (24). Now let $y \in (\hbar_A, s_1)_\in$, $f_x(y)|f_x(y) \in (\hbar_A, s_2)_\in$, $\mathfrak{b} \in (\eth_A, t_1)_\in$, and $f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}) \in (\eth_A, t_2)_\in$. Suppose that $\hbar_A(x) < \min\{s_1, s_2\}$ and $\eth_A(\mathfrak{a}) > \max\{t_1, t_2\}$, respectively. If $\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} < 0.5$ and

$$\max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} > 0.5,$$

respectively, then

$$\begin{aligned}\hbar_A(x) &\geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y)), 0.5\}\\ &= \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}\\ &\geq \min\{s_1, s_2\}\end{aligned}$$

and

$$\begin{aligned}\eth_A(\mathfrak{a}) &\leq \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b})), 0.5\}\\ &= \max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\}\\ &\leq \max\{t_1, t_2\}\end{aligned}$$

which is a contradiction. Hence $\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \geq 0.5$ and

$$\max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} \leq 0.5,$$

respectively. It follows that

$$\hbar_A(x) + \min\{s_1, s_2\} > 2\hbar_A(x) \geq 2\min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y)), 0.5\} = 1$$

and

$$\eth_A(\mathfrak{a}) + \max\{t_1, t_2\} < 2\eth_A(\mathfrak{a}) \leq 2\max\{\eth_A(\mathfrak{b}), \eth_A(f_\mathfrak{a}(\mathfrak{b})|f_\mathfrak{a}(\mathfrak{b}))\} = 1,$$

that is, $x \in (\hbar_A, \min\{s_1, s_2\})_q$ and $\mathfrak{a} \in (\eth_A, \max\{t_1, t_2\})_q$. Therefore $A^* := (\hbar_A, \eth_A)$ satisfies (25). □

Theorem 3.13. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies (24), (25) and $\hbar_A(x) < 0.5 < \eth_A(x)$ for all $x \in H$, then it is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$.*

Proof. Let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H that satisfies (24), (25) and $\hbar_A(x) < 0.5 < \eth_A(x)$ for all $x \in H$. Assume that $(\hbar_A, s)_\in \cap (\eth_A, t)_\in \neq \emptyset$, and say $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. Then $\hbar_A(x) \geq s$ and $\eth_A(x) \leq t$, which imply from Proposition 3.12 that $\hbar_A(0) \geq \min\{\hbar_A(x), 0.5\} = \hbar_A(x) \geq s$ and $\eth_A(0) \leq \max\{\eth_A(x), 0.5\} = \eth_A(x) \leq t$. Hence $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. Let $y \in (\hbar_A, s_1)_\in \cap (\eth_A, t_1)_\in$ and $f_x(y)|f_x(y) \in (\hbar_A, s_2)_\in \cap (\eth_A, t_2)_\in$. Using Proposition 3.12, we can induce the following:

$$\begin{aligned}\hbar_A(x) &\geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y)), 0.5\}\\ &= \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\}\\ &\geq \min\{s_1, s_2\}\end{aligned}$$

and

$$\begin{aligned}\eth_A(x) &\leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y)), 0.5\}\\ &= \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\}\\ &\leq \max\{t_1, t_2\}.\end{aligned}$$

Hence $x \in (\hbar_A, \min\{s_1, s_2\})_\in \cap (\eth_A, \max\{t_1, t_2\})_\in$, and therefore $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. □

Theorem 3.14. *For every nonempty subset D of H, consider an intuitionistic fuzzy set $A^*_D := (\hbar^D_A, \eth^D_A)$ in H which is defined by*

$$A^*_D := (\hbar^D_A, \eth^D_A) : H \to [0,1] \times [0,1],\ x \mapsto \begin{cases} (s_1, t_1) & \text{if } x \in D, \\ (s_2, t_2) & \text{otherwise} \end{cases}$$

where $s_1 > s_2$ and $t_1 < t_2$ in $[0,1]$. Then $A^*_D := (\hbar^D_A, \eth^D_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ if and only if D is an ideal of $\mathbb{H} := (H, |)$.

Proof. Assume that $A^*_D := (\hbar^D_A, \eth^D_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. Using (20), we have $\hbar^D_A(0) = s_1$ and $\eth^D_A(0) = t_1$. Hence $0 \in D$. Let $x, y \in H$ be such that $y \in D$ and $f_x(y)|f_x(y) \in D$. Then $\hbar^D_A(y) = s_1 = \hbar^D_A(f_x(y)|f_x(y))$ and $\eth^D_A(y) = t_1 = \eth^D_A(f_x(y)|f_x(y))$. Using (21) leads to

$$\hbar^D_A(x) \geq \min\{\hbar^D_A(y), \hbar^D_A(f_x(y)|f_x(y))\} = s_1$$

and

$$\eth^D_A(x) \leq \max\{\eth^D_A(y), \eth^D_A(f_x(y)|f_x(y))\} = t_1.$$

Hence $\hbar^D_A(x) = s_1$ and $\eth^D_A(x) = t_1$, and thus $x \in D$. Therefore D is an ideal of $\mathbb{H} := (H, |)$.

Conversely, suppose that D is an ideal of $\mathbb{H} := (H, |)$. Since $0 \in D$, we have $\hbar^D_A(0) = s_1 \geq \hbar^D_A(x)$ and $\eth^D_A(0) = t_1 \leq \eth^D_A(x)$ for all $x \in H$. Let $x, y \in H$. If $y \in D$ and $f_x(y)|f_x(y) \in D$, then $x \in D$ and thus

$$\hbar^D_A(x) = s_1 = \min\{\hbar^D_A(y), \hbar^D_A(f_x(y)|f_x(y))\}$$

and $\eth^D_A(x) = t_1 = \max\{\eth^D_A(y), \hbar^D_A(f_x(y)|f_x(y))\}$.

If $y \notin D$ or $f_x(y)|f_x(y) \notin D$, then $\hbar^D_A(y) = s_2$ and $\eth^D_A(y) = t_2$, or

$$\hbar_A^D(f_x(y)|f_x(y)) = s_2 \text{ and } \eth_A^D(f_x(y)|f_x(y)) = t_2.$$

Hence $\hbar_A^D(x) \geq s_2 = \min\{\hbar_A^D(y), \hbar_A^D(f_x(y)|f_x(y))\}$ and

$$\eth_A^D(x) \leq t_2 = \max\{\eth_A^D(y), \eth_A^D(f_x(y)|f_x(y))\}.$$

Therefore $A_D^* := (\hbar_A^D, \eth_A^D)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ by Theorem 3.4. □

Given an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H, we consider the next set called the $(0,1)$-*set* of $A^* := (\hbar_A, \eth_A)$.

$$H_{(0,1)} := \{x \in H \mid \hbar_A(x) \neq 0, \eth_A(x) \neq 1\}.$$

Theorem 3.15. *If $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$, then its nonempty $(0,1)$-set is an ideal of $\mathbb{H} := (H, |)$.*

Proof. Let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$. Suppose $H_{(0,1)} \neq \emptyset$, say $x \in H_{(0,1)}$. Then $\hbar_A(0) \geq \hbar_A(x) \neq 0$ and $\eth_A(0) \leq \eth_A(x) \neq 1$ by (20). Thus $0 \in H_{(0,1)}$. Let $x, y \in H$ be such that $y \in H_{(0,1)}$ and $f_x(y)|f_x(y) \in H_{(0,1)}$. Then $\hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \neq 0$ and $\eth_A(x) \leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\} \neq 1$ by (21). Hence $x \in H_{(0,1)}$, and therefore $H_{(0,1)}$ is an ideal of $\mathbb{H} := (H, |)$. □

In the following example, we can see that the converse of Theorem 3.15 is not true in general.

Example 3.16. Consider the Sheffer stroke Hilbert algebra $\mathbb{H} := (H, |)$ in Example 3.2 and let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H given by

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$\mathfrak{b} \mapsto \begin{cases} \left(\frac{0.44}{k}, \frac{0.33}{k}\right) & \text{if } \mathfrak{b} = 0, \\ \left(\frac{0.65}{k}, \frac{0.22}{k}\right) & \text{if } \mathfrak{b} = 3, \\ (0.00, 1.00) & \text{if } \mathfrak{b} \in \{1, 2\} \end{cases}$$

where k is a natural number. Then $H_{(0,1)} = \{0, 3\}$ and it is an ideal of $\mathbb{H} := (H, |)$. We can observe that $\hbar_A(0) = \frac{0.44}{k} < \frac{0.65}{k} = \hbar_A(3)$ and/or $\eth_A(0) = \frac{0.33}{k} > \frac{0.22}{k} = \eth_A(3)$, that is, (20) is not valid. Hence $A^* := (\hbar_A, \eth_A)$ is not an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$ by Theorem 3.4.

Example 3.17. Consider the Sheffer stroke Hilbert algebra $(H, |)$ in Example 3.3 and let $A^* := (\hbar_A, \eth_A)$ be an intuitionistic fuzzy set in H given by

$$A^* := (\hbar_A, \eth_A) : H \to [0,1] \times [0,1],$$
$$x \mapsto \begin{cases} \left(\frac{0.52}{k}, \frac{0.36}{k}\right) & \text{if } x = 2, \\ \left(\frac{0.69}{k}, \frac{0.24}{k}\right) & \text{if } x = 5, \\ (0.00, 1.00) & \text{if } x \in H \setminus \{2, 5\} \end{cases}$$

where k is a natural number. Then $H_{(0,1)} = \{2, 5\}$ and it is an ideal of $(H, |)$.

We can observe that $\hbar_A(2) = \frac{0.52}{k} < \frac{0.69}{k} = \hbar_A(5)$ and/or $\eth_A(2) = \frac{0.36}{k} > \frac{0.24}{k} = \eth_A(5)$, that is, (20) is not valid.

Hence, $A^* := (\hbar_A, \eth_A)$ is not an intuitionistic fuzzy ideal of $(H, |)$ by Theorem 3.4.

We explore the conditions under which the nonempty $(0,1)$-set is an ideal.

Theorem 3.18. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies:*

$$(\hbar_A, s)_\in \cap (\eth_A, t)_\in \neq \emptyset \;\Rightarrow\; 0 \in (\hbar_A, s)_q \cap (\eth_A, t)_q, \tag{28}$$

$$\left(\begin{array}{l} x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in,\ f_y(x)|f_y(x) \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in \\ \Rightarrow y \in (\hbar_A, s)_q \cap (\eth_A, t)_q \end{array} \right) \tag{29}$$

for all $x, y \in H$, then the nonempty $(0,1)$-set is an ideal of $\mathbb{H} := (H, |)$.

Proof. Let $x \in H_{(0,1)}$. Then $\hbar_A(x) \neq 0$ and $\eth_A(x) \neq 1$. Since $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ for $s := \hbar_A(x)$ and $t := \eth_A(x)$, we have $0 \in (\hbar_A, s)_q \cap (\eth_A, t)_q$ by (28). If $0 \notin H_{(0,1)}$, then $\hbar_A(0) = 0$ or $\eth_A(0) = 1$. Hence $\hbar_A(0) + s = s \not> 1$ or $\eth_A(0) + t = 1 + t \not< 1$, i.e., $0 \in (\hbar_A, s)_q$ or $0 \in (\eth_A, t)_q$ which is a contradiction. Thus $0 \in H_{(0,1)}$. Let $x \in H_{(0,1)}$ and $f_y(x)|f_y(x) \in H_{(0,1)}$. If we take

$$s := \min\{\hbar_A(x), \hbar_A(f_y(x)|f_y(x))\} \text{ and } t := \max\{\eth_A(x), \eth_A(f_y(x)|f_y(x))\},$$

then $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ and $f_y(x)|f_y(x) \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. It follows from (29) that $y \in (\hbar_A, s)_q \cap (\eth_A, t)_q$. Hence $\hbar_A(y) + s > 1$ and $\eth_A(y) + t < 1$ which shows that $\hbar_A(y) \neq 0$ and $\eth_A(y) \neq 1$. Thus $y \in H_{(0,1)}$, and therefore $H_{(0,1)}$ is an ideal of $\mathbb{H} := (H, |)$. □

Theorem 3.19. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies:*

$$(\hbar_A, s)_q \cap (\eth_A, t)_q \neq \emptyset \;\Rightarrow\; 0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in, \tag{30}$$

$$\left(\begin{array}{l} x \in (\hbar_A, s)_q \cap (\eth_A, t)_q,\ f_y(x)|f_y(x) \in (\hbar_A, s)_q \cap (\eth_A, t)_q \\ \Rightarrow y \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in \end{array} \right) \tag{31}$$

for all $x, y \in H$, then the nonempty $(0,1)$-set is an ideal of $\mathbb{H} := (H, |)$.

Proof. Let $x \in H_{(0,1)}$. Then $\hbar_A(x) \neq 0$ and $\eth_A(x) \neq 1$, and so $\hbar_A(x) + 1 > 1$ and $\eth_A(x) + 0 < 1$, i.e., $x \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. It follows from (30) that $0 \in (\hbar_A, 1)_\in \cap (\eth_A, 0)_\in$. Thus $0 \in H_{(0,1)}$. Let $x \in H_{(0,1)}$ and $f_y(x)|f_y(x) \in H_{(0,1)}$. Then $\hbar_A(x) + 1 > 1$, $\eth_A(x) + 0 < 1$, $\hbar_A(f_y(x)|f_y(x)) + 1 > 1$, and $\eth_A(f_y(x)|f_y(x)) + 0 < 1$, that is, $x \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$ and $f_y(x)|f_y(x) \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. Using (31), we have $y \in (\hbar_A, 1)_\in \cap (\eth_A, 0)_\in$ which implies that $\hbar_A(y) \neq 0$ and $\eth_A(y) \neq 1$. Hence $y \in H_{(0,1)}$, and therefore $H_{(0,1)}$ is an ideal of $\mathbb{H} := (H, |)$. □

Theorem 3.20. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies:*

$$(\hbar_A, s)_q \cap (\eth_A, t)_q \neq \emptyset \;\Rightarrow\; 0 \in (\hbar_A, s)_q \cap (\eth_A, t)_q, \tag{32}$$

$$\left(\begin{array}{l} x \in (\hbar_A, s)_q \cap (\eth_A, t)_q,\ f_y(x)|f_y(x) \in (\hbar_A, s)_q \cap (\eth_A, t)_q \\ \Rightarrow y \in (\hbar_A, s)_q \cap (\eth_A, t)_q \end{array} \right) \tag{33}$$

for all $x, y \in H$, then the nonempty $(0,1)$-set is an ideal of $\mathbb{H} := (H, |)$.

Proof. Let $x \in H_{(0,1)}$. Then $\hbar_A(x) \neq 0$ and $\eth_A(x) \neq 1$, and so $\hbar_A(x) + 1 > 1$ and $\eth_A(x) + 0 < 1$, i.e., $x \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. It follows from (32) that $0 \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. Hence $\hbar_A(0)+1 > 1$ and $\eth_A(0)+0 < 1$ which shows that $0 \in H_{(0,1)}$. Let $x \in H_{(0,1)}$ and $f_y(x)|f_y(x) \in H_{(0,1)}$. Then $\hbar_A(x) + 1 > 1$, $\eth_A(x) + 0 < 1$, $\hbar_A(f_y(x)|f_y(x)) + 1 > 1$, and $\eth_A(f_y(x)|f_y(x)) + 0 < 1$, that is, $x \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$ and $f_y(x)|f_y(x) \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. It follows from (33) that $y \in (\hbar_A, 1)_q \cap (\eth_A, 0)_q$. Hence $\hbar_A(y) + 1 > 1$ and $\eth_A(y) + 0 < 1$, and so $y \in H_{(0,1)}$. Therefore $H_{(0,1)}$ is an ideal of $\mathbb{H} := (H, |)$. □

We provide conditions for the intuitionistic level set and intuitionistic q-set to be ideals.

Theorem 3.21. *If an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies:*

$$\hbar_A(x) \leq \max\{\hbar_A(0), 0.5\},\ \eth_A(x) \geq \min\{\eth_A(0), 0.5\}, \tag{34}$$

$$\left(\begin{array}{l} \max\{\hbar_A(x), 0.5\} \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \\ \min\{\eth_A(x), 0.5\} \leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\} \end{array}\right) \tag{35}$$

for all $x, y \in H$. then its nonempty intuitionistic level set $(A^*, (s,t))_\in$ is an ideal of $\mathbb{H} := (H, |)$ for all $(s,t) \in (0.5, 1] \times [0, 0.5)$.

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(A^*, (s,t))_\in \neq \emptyset$. Then there exists $\mathfrak{a} \in (A^*, (s,t))_\in = (\hbar_A, s)_\in \cap (\eth_A, t)_\in$, and so $\max\{\hbar_A(0), 0.5\} \geq \hbar_A(\mathfrak{a}) \geq s > 0.5$ and $\min\{\eth_A(0), 0.5\} \leq \eth_A(\mathfrak{a}) \leq t < 0.5$ by (34). Thus $\hbar_A(0) \geq s$ and $\eth_A(0) \leq t$, i.e., $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in = (A^*, (s,t))_\in$. Let $x, y \in H$ be such that $y \in (A^*, (s,t))_\in = (\hbar_A, s)_\in \cap (\eth_A, t)_\in$ and $f_x(y)|f_x(y) \in (A^*, (s,t))_\in = (\hbar_A, s)_\in \cap (\eth_A, t)_\in$. Using (35), we have

$$\max\{\hbar_A(x), 0.5\} \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} \geq s > 0.5$$

and $\min\{\eth_A(x), 0.5\} \leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\} \leq t < 0.5$. It follows that $\hbar_A(x) \geq s$ and $\eth_A(x) \leq t$, that is, $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in = (A^*, (s,t))_\in$. Consequently, $(A^*, (s,t))_\in$ is an ideal of $\mathbb{H} := (H, |)$. □

Theorem 3.22. *If $A^* := (\hbar_A, \eth_A)$ is an intuitionistic fuzzy ideal of $\mathbb{H} := (H, |)$, then its nonempty intuitionistic q-set $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$.*

Proof. Let $x \in (A^*, (s,t))_q$. Then $x \in (\hbar_A, s)_q \cap (\eth_A, t)_q$, which imply from (20) that $\hbar_A(0) \geq \hbar_A(x) > 1 - s$ and $\eth_A(0) \leq \eth_A(x) < 1 - t$. Hence $0 \in (\hbar_A, s)_q \cap (\eth_A, t)_q = (A^*, (s,t))_q$ Let $y \in (A^*, (s,t))_q = (\hbar_A, s)_q \cap (\eth_A, t)_q$ and $f_x(y)|f_x(y) \in (A^*, (s,t))_q = (\hbar_A, s)_q \cap (\eth_A, t)_q$. Then $\hbar_A(y) + s > 1$, $\hbar_A(f_x(y)|f_x(y)) + s > 1$, $\eth_A(y) + t < 1$, and $\eth_A(f_x(y)|f_x(y)) + t < 1$. It follows from (21) that

$$\hbar_A(x) \geq \min\{\hbar_A(y), \hbar_A(f_x(y)|f_x(y))\} > 1 - s$$

and $\eth_A(x) \leq \max\{\eth_A(y), \eth_A(f_x(y)|f_x(y))\} < 1 - t$. Hence $x \in (\hbar_A, s)_q \cap (\eth_A, t)_q = (A^*, (s,t))_q$, and therefore $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$. □

Proposition 3.23. *Given an intuitionistic fuzzy set* $A^* := (\hbar_A, \eth_A)$ *in* H*, if its intuitionistic q-set* $(A^*, (s,t))_q$ *is an ideal of* $\mathbb{H} := (H, |)$ *for all* $(s,t) \in (0, 0.5] \times [0.5, 1)$*, then* $0 \in (A^*, (s,t))_\in$ *and*

$$\langle y_{(s,t)} \rangle \, q \, A^*, \langle (f_x(y)|f_x(y))_{(s,t)} \rangle \, q \, A^* \;\Rightarrow\; x \in (A^*, (s,t))_\in \tag{36}$$

for all $x, y \in H$ *and* $(s,t) \in (0, 0.5] \times [0.5, 1)$.

Proof. Assume that $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$ for all $(s,t) \in (0, 0.5] \times [0.5, 1)$. Since $0 \in (A^*, (s,t))_q = (\hbar_A, s)_q \cap (\eth_A, t)_q$, we have $\hbar_A(0) > 1 - s \geq s$ and $\eth_A(0) < 1 - t \leq t$, and so $0 \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in = (A^*, (s,t))_\in$. Let $x, y \in H$ and $(s,t) \in (0, 0.5] \times [0.5, 1)$ be such that $\langle y_{(s,t)} \rangle \, q \, A^*$ and $\langle (f_x(y)|f_x(y))_{(s,t)} \rangle \, q \, A^*$. Then $y \in (A^*, (s,t))_q$ and $f_x(y)|f_x(y) \in (A^*, (s,t))_q$, and thus $x \in (A^*, (s,t))_q = (\hbar_A, s)_q \cap (\eth_A, t)_q$. It follows that $\hbar_A(x) > 1 - s \geq s$ and $\eth_A(x) < 1 - t \leq t$. Hence $x \in (\hbar_A, s)_\in \cap (\eth_A, t)_\in = (A^*, (s,t))_\in$. $\square$

Proposition 3.24. *Given an intuitionistic fuzzy set* $A^* := (\hbar_A, \eth_A)$ *in* H*, if its intuitionistic q-set* $(A^*, (s,t))_q$ *is an ideal of* $\mathbb{H} := (H, |)$ *for all* $(s,t) \in (0.5, 1] \times [0, 0.5)$*, then*

$$\langle y_{(s,t)} \rangle \in A^*, \langle (f_x(y)|f_x(y))_{(s,t)} \rangle \in A^* \;\Rightarrow\; x \in (A^*, (s,t))_q \tag{37}$$

for all $x, y \in H$ *and* $(s,t) \in (0.5, 1] \times [0, 0.5)$.

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$. Let $x, y \in H$ be such that $\langle y_{(s,t)} \rangle \in A^*$ and $\langle (f_x(y)|f_x(y))_{(s,t)} \rangle \in A^*$. Then $\hbar_A(y) \geq s > 1 - s$, $\eth_A(y) \leq t < 1 - t$, $\hbar_A(f_x(y)|f_x(y)) \geq s > 1 - s$, and $\eth_A(f_x(y)|f_x(y)) \leq t < 1 - t$. It follows that $y \in (\hbar_A, s)_q \cap (\eth_A, t)_q = (A^*, (s,t))_q$ and $f_x(y)|f_x(y) \in (\hbar_A, s)_q \cap (\eth_A, t)_q = (A^*, (s,t))_q$. Thus $x \in (A^*, (s,t))_q$. $\square$

Theorem 3.25. *If an intuitionistic fuzzy set* $A^* := (\hbar_A, \eth_A)$ *in* H *satisfies:*

$$\langle x_{(s,t)} \rangle \, q \, A^* \;\Rightarrow \langle 0_{(s,t)} \rangle \in\!\vee q A^*, \tag{38}$$

$$\langle y_{(s,t)} \rangle \, q \, A^*, \langle (f_x(y)|f_x(y))_{(s,t)} \rangle \, q \, A^* \;\Rightarrow\; \langle x_{(s,t)} \rangle \in\!\vee q A^* \tag{39}$$

for all $x, y \in H$ *and* $(s,t) \in (0.5, 1] \times [0, 0.5)$*, then its nonempty intuitionistic q-set* $(A^*, (s,t))_q$ *is an ideal of* $\mathbb{H} := (H, |)$ *for all* $(s,t) \in (0.5, 1] \times [0, 0.5)$.

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(A^*, (s,t))_q \neq \emptyset$. Then $\mathfrak{a} \in (A^*, (s,t))_q$ for some $\mathfrak{a} \in H$, and so $\langle \mathfrak{a}_{(s,t)} \rangle \, q \, A^*$. Hence $\langle 0_{(s,t)} \rangle \in \vee q A^*$ by (38), that is, $\langle 0_{(s,t)} \rangle \in A^*$ or $\langle 0_{(s,t)} \rangle \, q \, A^*$. If $\langle 0_{(s,t)} \rangle \, q \, A^*$, then $0 \in (A^*, (s,t))_q$. If $\langle 0_{(s,t)} \rangle \in A^*$, then $\hbar_A(0) \geq s > 1 - s$ and $\eth_A(0) \leq t < 1 - t$. Thus $0 \in (A^*, (s,t))_q$. Let $x, y \in H$ be such that $y \in (A^*, (s,t))_q$ and $f_x(y)|f_x(y) \in (A^*, (s,t))_q$. Then $\langle y_{(s,t)} \rangle \, q \, A^*$ and $\langle (f_x(y)|f_x(y))_{(s,t)} \rangle \, q \, A^*$. It follows from (39) that $\langle x_{(s,t)} \rangle \in \vee q A^*$, that is, $\langle x_{(s,t)} \rangle \in A^*$ or $\langle x_{(s,t)} \rangle \, q \, A^*$. If $\langle x_{(s,t)} \rangle \, q \, A^*$, then $x \in (A^*, (s,t))_q$. If $\langle x_{(s,t)} \rangle \in A^*$, then $\hbar_A(x) \geq s > 1 - s$ and $\eth_A(x) \leq t < 1 - t$. Thus $x \in (A^*, (s,t))_q$. Therefore $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$. $\square$

Theorem 3.26. *Given an ideal D of $\mathbb{H} := (H, |)$, if an intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$ in H satisfies $A^*(x) = (0, 1)$ for $x \in H \setminus D$ and $x \in (A^*, (0.5, 0.5))_\in$ for $x \in D$, then its nonempty intuitionistic q-set $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$ for all $(s,t) \in (0.5, 1] \times [0, 0.5)$.*

Proof. Let $(s,t) \in (0.5, 1] \times [0, 0.5)$ be such that $(A^*, (s,t))_q \neq \emptyset$. Then there exists $x \in (A^*, (s,t))_q$, and so $\langle x_{(s,t)} \rangle \, q \, A^*$. Hence $\hbar_A(x) + s > 1$ and $\eth_A(x) + t < 1$. If $x \in H \setminus D$, then $s = 0 + s = \hbar_A(x) + s > 1$ and $1 + t = \eth_A(x) + t < 1$ which is a contradiction. Thus $x \in D$, and so $x \in (A^*, (0.5, 0.5))_\in = (\hbar_A, 0.5)_\in \cap (\eth_A, 0.5)_\in$, that is, $\hbar_A(x) \geq 0.5$ and $\eth_A(x) \leq 0.5$. If $\langle 0_{(s,t)} \rangle \,\overline{\in}\, A^*$, then $\hbar_A(0) < s$ or $\eth_A(0) > t$. If $\hbar_A(0) < s$, then $\hbar_A(0) + s > 2\hbar_A(0) \geq 1$. If $\eth_A(0) > t$, then $\eth_A(0) + t < 2\eth_A(0) \leq 1$. Hence $\langle 0_{(s,t)} \rangle \in \vee q A^*$. If $\langle y_{(s,t)} \rangle \, q \, A^*$ and $\langle (f_x(y)|f_x(y))_{(s,t)} \rangle \, q \, A^*$, then $y \in D$ and $f_x(y)|f_x(y) \in D$. Hence $y \in (A^*, (0.5, 0.5))_\in = (\hbar_A, 0.5)_\in \cap (\eth_A, 0.5)_\in$ and $f_x(y)|f_x(y) \in (A^*, (0.5, 0.5))_\in = (\hbar_A, 0.5)_\in \cap (\eth_A, 0.5)_\in$, that is, $\hbar_A(y) \geq 0.5$, $\eth_A(y) \leq 0.5$, $\hbar_A(f_x(y)|f_x(y)) \geq 0.5$, and $\eth_A(f_x(y)|f_x(y)) \leq 0.5$. If $\langle x_{(s,t)} \rangle \,\overline{\in}\, A^*$, then $\hbar_A(x) < s$ or $\eth_A(x) > t$. If $\hbar_A(x) < s$, then $\hbar_A(x) + s > 2\hbar_A(x) \geq 1$. If $\eth_A(x) > t$, then $\eth_A(x) + t < 2\eth_A(x) \leq 1$. Thus $\langle x_{(s,t)} \rangle \in \vee q A^*$. It follows from Theorem 3.25 that $(A^*, (s,t))_q$ is an ideal of $\mathbb{H} := (H, |)$. □

4 Conclusions

The Sheffer stroke is a powerful logical connective in propositional calculus and Boolean functions, functioning as the negation of conjunction—commonly described as "not both"—and has widespread applications in digital electronics and logic circuit design. Inspired by its algebraic versatility, Oner and colleagues introduced the concept of Sheffer stroke Hilbert algebras, providing fertile ground for further algebraic investigation.

In this paper, we extended the classical notion of ideals in Sheffer stroke Hilbert algebras to their intuitionistic fuzzy versions by employing intuitionistic fuzzy points. The notion of intuitionistic fuzzy ideals was formally defined, and their structural properties were investigated through various equivalent characterizations. Moreover, we explored intuitionistic fuzzy set constructs, including the $(0,1)$-set $H_{(0,1)}$ defined with respect to the intuitionistic fuzzy set $A^* := (\hbar_A, \eth_A)$, and established the conditions under which this set, as well as the intuitionistic level set and q-set, form ideals in the Sheffer stroke Hilbert algebra.

The main contributions of our study can be summarized as follows:

- We introduced a new framework for intuitionistic fuzzy ideals in Sheffer stroke Hilbert algebras based on intuitionistic fuzzy points.
- We provided several equivalent characterizations for these ideals, connecting them with various intuitionistic fuzzy set structures such as level sets and q-sets.
- We examined the ideal-theoretic nature of the $(0,1)$-set associated with an intuitionistic fuzzy set and identified precise conditions for when it forms a classical ideal.

These results not only broaden the theoretical landscape of Sheffer stroke Hilbert algebras under intuitionistic fuzzy logic but also offer a foundation for future studies in this direction. As a natural continuation, we intend to investigate the intuitionistic fuzzy versions of ideals in other types of Sheffer stroke-based logical algebras, including Sheffer stroke BCK-algebra, Sheffer stroke BL-algebra, Sheffer stroke BE-algebra, Sheffer stroke BCH-algebra, Sheffer stroke MTL-algebra, Sheffer stroke MV-algebra,. Additionally, future research may consider other fuzzy constructs—such as intuitionistic fuzzy filters or congruences—or explore categorical approaches and applications in logic, decision-making systems, and approximate reasoning frameworks.

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