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Original Research Paper

Neutral Integro-Differential Equations with Nonlocal Conditions via Densifiability Techniques

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Abstract. The existence of solutions to neutral integro-differential equations in a Banach space is investigated in this article using a novel fixed-point theorem based on the degree of nondensifiability (DND). Aside from that, an example is provided to support our main findings. This research paper improves and expands on previous findings in the area.

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1 Introduction

In this paper, the core of our findings relies heavily on the excellent results presented by García [31, 30] concerning the investigation of mild solution existence for neutral integro-differential equations within Banach spaces structured as follows:

$$\begin{cases} \frac{d}{d\varsigma}\left[\xi(\varsigma)+\int_0^\varsigma N(\varsigma,\kappa)\xi(\kappa)d\kappa\right] = \aleph(\varsigma)\left[\xi(\varsigma)+\int_0^\varsigma \beta(\varsigma,\kappa)\xi(\kappa)d\kappa\right] \\ \qquad\qquad\qquad\qquad +\Phi(\varsigma,\xi(\vartheta(\varsigma))), \quad \text{for } \varsigma \geq 0, \\ \xi(0)=\xi_0 \in \Xi, \end{cases} \tag{1}$$

where $\xi(\cdot)$ is the state variable taking values in a Banach space $(\Xi, \|\cdot\|_\Xi)$, and $\Phi : \Theta \times \Xi \to \Xi$, $(\Theta = [0, \varkappa])$ is a continuous function. The operators $\aleph(\varsigma) : \mathfrak{G}(\aleph(\varsigma)) \subset \Xi \to \Xi$ and $\beta(\varsigma, \kappa)$ are closed linear operators on Ξ, with dense domain $\mathfrak{G}(\aleph(\varsigma))$, which is independent of ς, and $\mathfrak{G}(\aleph(\kappa)) \subset \mathfrak{G}(\beta(\varsigma, \kappa))$. The operator $N(\varsigma)$ is the neutral term in a family of bounded linear operators on Ξ. The function $\vartheta(\cdot) : [0, \varkappa] \to [0, \varkappa]$ is continuous and satisfies $0 \leq \vartheta(\varsigma) \leq \varsigma$.

Next, we investigate the existence of mild solutions for neutral integro-differential equations with a nonlocal initial condition having the form:

$$\begin{cases} \frac{d}{d\varsigma}\left[\xi(\varsigma)+\int_0^\varsigma N(\varsigma,\kappa)\xi(\kappa)d\kappa\right] = \aleph(\varsigma)\left[\xi(\varsigma)+\int_0^\varsigma \beta(\varsigma,\kappa)\xi(\kappa)d\kappa\right] \\ \qquad\qquad\qquad\qquad +\Phi(\varsigma,\xi(\vartheta(\varsigma))), \quad \text{for } \varsigma \geq 0, \\ \xi(0)+g(\xi)=\xi_0 \in \Xi, \end{cases} \tag{2}$$

where $g : C(\Theta, \Xi) \to \Xi$ is continuous function and the set $C(\Theta, \Xi)$ is given later.

Integro-differential equations can describe natural phenomena across various fields, including electronics, fluid dynamics, biological models, and chemical kinetics. Classical differential equations cannot explain these phenomena, see [9, 10]. Integro-differential equations have recently gained popularity among physicists, mathematicians, and engineers. For more general results on differential equations, see [6, 4, 2, 3] and the references therein. Volterra suggests that the dynamics of elastic materials can be described by a partial integro-differential response diffusion equation, as shown below: In [43], the author proposes using a partial integro-differential response diffusion equation to describe the kinetics of certain elastic materials:

$$\frac{\partial}{\partial\varsigma}z(\theta,\varsigma)=\Delta z(\theta,\varsigma)+\int_0^\varsigma\phi(\varsigma,\kappa)\Delta z(\theta,\kappa)d\kappa+\varphi(\theta,\varsigma),\quad\text{for } (\theta,\varsigma)\in\mathbb{R}\times\mathbb{R}_+,$$

where ϕ and φ are appropriates functions.

The authors of [15, 22] used the following linear partial integro-differential equation to study the electric displacement field in Maxwell Hopkinson dielectric:

$$\left\{\ \frac{\partial^2}{\partial\varsigma^2}z(\theta,\varsigma)=\frac{1}{\eta^\gamma}\Delta z(\theta,\varsigma)+\int_0^\varsigma\frac{1}{\eta^\gamma}\psi(\varsigma-\kappa)\Delta z(\theta,\kappa)d\kappa,\quad\text{for } (\theta,\varsigma)\in\tilde{\Omega}\times[0,\varkappa),\right.$$

for $\varkappa>0$ and $\tilde{\Omega}\subset\mathbb{R}^3$, where $\eta,\gamma\in\mathbb{R}$ and ψ is a vector of scalar function.

The resolvent operator, which replaces the role of the C_0-semigroup in evolution equations, is critical in solving problem (1) in both weak and strict senses. Many authors have used resolvent operator theory to study semi-linear integro-differential evolution equations, including existence, regularity, stability, and control problems (references [18, 21, 25, 28, 34, 38, 44, 11, 12, 13, 14]).

Conversely, in numerous scenarios, employing a nonlocal initial condition proves to be more effective than the classical initial condition $\vartheta(0)=\vartheta_0$ in elucidating certain physical phenomena. The investigation of nonlocal Cauchy problems for evolution equations dates back to 1991 when Byszewski *et al.* delved into the subject [16], while the significance of nonlocal conditions across various domains has been extensively discussed in [16, 23] and the accompanying references. Further insights can be gleaned from [1, 7, 8]. Subsequently, many scholars have explored evolution equations featuring nonlocal conditions, yielding a plethora of intriguing findings on various aspects of nonlocal problems over the years, as documented in works such as [5, 17, 26, 35, 36, 37, 42], among others. Moreover, in recent years, there has been a surge of interest in investigating integro-differential evolution equations with nonlocal conditions, as evidenced by works such as [27, 36, 45]. In [36], the authors considered:

$$\begin{cases} \xi'(\varsigma) = \aleph\left[\xi(\varsigma) + \int_0^\varsigma F(\varsigma-\kappa)\xi(\kappa)d\kappa\right] + \Phi(\varsigma, \xi(\varsigma)), & \text{for } \varsigma \in [0, \varkappa], \\ \xi(0) + g(\varsigma_1, \ldots, \varsigma_p, \xi) = \xi_0. \end{cases}$$

The discussion on the existence and regularity of solutions for a neutral integro-differential evolution equation was tackled in [27], where the approach involved utilizing the theory of resolvent operators and analytic semigroups.

Cherruault and Guillez [19] first introduced the concept of ζ-dense curves in the 1980s. Cherruault [20] and Mora [39] were primarily responsible for its inception. Mora and Mira [40] introduced the concept of (DND), based on ζ-dense curves. García [29, 31] demonstrated a new fixed-point result using the DND. See [24], for more results.

We note that our work is considered as the natural continuation of the results presented in [46]. While the authors of [46] used the theory of fractional power, ζ-norm and Schauder's fixed point theorem to prove their results, we apply a new theorem based on the (DND) which is more generalized.

This paper is organized as follows. In Section 2, some necessary concepts and important definitions and lemmas are given. In Section 3, we show the existence of mild solutions for neutral integro-differential equations with local and nonlocal initial conditions for the problems (1) and (2). An example is also given in Section 4 to illustrate the theory of the abstract main result.

2 Preliminaries

Let Ξ be a real Banach space with the norm $\|\cdot\|_\Xi$ and M_Ξ is the class of non-empty and bounded subsets of Ξ. Let $\mathfrak{Y}(\Xi)$ be the space of all bounded linear operators from Ξ into Ξ, with the norm

$$\|\mathcal{N}\|_{\mathfrak{Y}(\Xi)} = \sup_{\xi\in\Xi} \|\mathcal{N}(\xi)\|_{\Xi}.$$

We denote by $(L^1(\Theta,\Xi), \|\cdot\|_{L^1})$ is the Bochner integrable mappings ξ from $\Theta := [0,\varkappa]$ into Ξ, with the norm

$$\|\xi\|_{L^1} = \int_0^{\varkappa} \|\xi(\varsigma)\|_{\Xi} d\varsigma.$$

We denote by $(L^\infty(\Xi), \|\cdot\|_{L^\infty})$ the Banach space of measurable function $\xi : \Theta \to \Xi$ which are essentially bounded with

$$\|\xi\|_{L^\infty} = \inf\{\gamma > 0 : \|\xi(\varsigma)\|_{\Xi} \leq \gamma, \quad a.e \quad \varsigma \in \Theta\}.$$

By $C(\Theta,\Xi)$ we denote the Banach space of all continuous functions from Θ into Ξ with

$$\|\xi\|_\infty = \sup_{\varsigma\in\Theta} \|\xi(\varsigma)\|_{\Xi}.$$

We present the basic theory of resolvent operators for the following neutral integro-differential equation associated with problem (1):

$$\begin{cases} \frac{d}{d\varsigma}\left[\xi(\varsigma) + \int_0^\varsigma N(\varsigma,\kappa)\xi(\kappa)d\kappa\right] = \aleph(\varsigma)\left[\xi(\varsigma) + \int_0^\varsigma \beta(\varsigma,\kappa)\xi(\kappa)d\kappa\right], \quad \text{for } \varsigma \geq 0, \\ \xi(0) = \xi_0 \in \Xi. \end{cases} \tag{3}$$

The discussion regarding the existence and characteristics of a resolvent operator has been elaborated upon in [46]. Consider:

$(A1)$ $\aleph(\varsigma)$ generates a uniformly continuous semigroup of evolution operators in Ξ.

$(A2)$ Assume that X is the Banach space formed from $\mathfrak{G}(\aleph(\varsigma))$ with the graph norm. $\aleph(\varsigma)$ and $\beta(\varsigma,\kappa)$ are closed operators. It follows that $\aleph(\varsigma)$ and $\beta(\varsigma,\kappa)$ are in the set of bounded operators from X to Ξ, $\beta(X,\Xi)$ for $0 \leq \varsigma \leq \varkappa$ and $0 \leq \kappa \leq \varsigma \leq \varkappa$, respectively. Furthermore, $\aleph(\varsigma)$ and $\beta(\varsigma,\kappa)$ are continuous on $0 \leq \varsigma \leq \varkappa$ and $0 \leq \kappa \leq \varsigma \leq \varkappa$, respectively, into $\beta(X,\Xi)$.

Definition 2.1. ([46]) A two-parameters family of bounded linear operators $R(\varsigma,\kappa) \in \mathfrak{Y}(\Xi)$ for $0 \leq \kappa \leq \varsigma \leq \varkappa$, is called a resolvent operator for problem (3) if it verifies the following conditions:

(1) For each $\xi \in \Xi$, $\varsigma \to R(\varsigma,\kappa)\xi$ is strongly continuous in ς and κ, $R(\kappa,\kappa) = I$, $0 \leq \kappa \leq \varkappa$ (the identity map of Ξ) and $\|R(\varsigma,\kappa)\|_{\mathfrak{Y}(\Xi)} \leq Me^{\eta(\varsigma-\kappa)}$ for some constants $M > 0$ and $\eta \in \mathbb{R}$.

(2) $R(\varsigma,\kappa)X \subset X$, $R(\varsigma,\kappa)$ is strongly continuous in ς and κ on X.

(3) For each $\xi \in \mathfrak{G}(\aleph(\varsigma))$, $R(\varsigma,\kappa)\xi$ is strongly continuously differentiable in ς and κ and

$$\frac{d}{d\varsigma}\left[R(\varsigma,\kappa)\xi + \int_0^\varsigma N(\varsigma,\kappa)R(\varsigma,\kappa)\xi d\kappa\right] = \aleph(\varsigma)\left[R(\varsigma,\kappa)\xi + \int_0^\varsigma \beta(\varsigma,\kappa)R(\varsigma,\kappa)\xi d\kappa\right],$$

$$\frac{d}{d\varsigma}\left[R(\varsigma,\kappa)\xi + \int_0^\varsigma R(\varsigma,\kappa)N(\kappa)\xi d\kappa\right] = R(\varsigma,\kappa)\aleph(\varsigma)\xi + \int_0^\varsigma R(\varsigma,\kappa)\aleph(\varsigma)\beta(\kappa)\xi d\kappa,$$

with $\frac{d}{d\varsigma}R(\varsigma,\kappa)\xi$ is strongly continuous on $0 \leq \kappa \leq \varsigma \leq \varkappa$. Here, $R(\varsigma,\kappa)$ can be extracted from the evolution operator of the generator $\aleph(\varsigma)$.

The next theorem presents a satisfactory answer to the problem of the existence of resolvent operator to (3).

Theorem 2.2. *([46]) Assume that* $(A1) - (A2)$ *hold, then there exists a unique resolvent operator for the Cauchy problem (3).*

Definition 2.3. ([39, 41]) Let $\zeta \geq 0$ and $\mathfrak{W} \in M_\Xi$. A continuous mapping $\Im : \mho := [0,1] \to \Xi$ is called $\zeta-$dense curve in $\mathfrak{W}$ if:

- $\Im(\mho) \subset \mathfrak{W}$.
- For any $\xi_1 \in \mathfrak{W}$, there is $\xi_2 \in \Im(\mho)$ such that $\|\xi_1 - \xi_2\|_\Xi \leq \zeta$.

If for $\zeta > 0$, there is an $\zeta-$dense curve in $\mathfrak{W}$, then $\mathfrak{W}$ is called densifiable.

Definition 2.4. ([40, 32]) Let $\zeta > 0$, and denote by $\Gamma_{\zeta,\mathfrak{W}}$ the class of all $\zeta-$dense curves in $\mathfrak{W} \in M_\Xi$. The DND is a mapping $\wp : M_\Xi \to \mathbb{R}_+$ given by:

$$\wp(\mathfrak{W}) = \inf\{\zeta \geq 0 : \Gamma_{\zeta,\mathfrak{W}} \neq \varnothing\},$$

for each $\mathfrak{W} \in M_\Xi$.

Lemma 2.5 ([33, 32])**.** *Let $\mathfrak{W}_1, \mathfrak{W}_2 \in M_\Xi$. Then, $\wp$ verifies:*

(a) $\wp(\mathfrak{W}_1) = 0 \iff \mathfrak{W}_1$ *is a precompact set, for each nonempty, bounded and arc-connected subset* $\mathfrak{W}_1$ *of* Ξ.

(b) $\wp(\bar{\mathfrak{W}}_1) = \wp(\mathfrak{W}_1)$, *where* $\bar{\mathfrak{W}}_1$ *denotes the closure of* $\mathfrak{W}_1$.

(c) $\wp(\lambda\mathfrak{W}_1) = |\lambda|\wp(\mathfrak{W}_1)$, *for* $\lambda \in \mathbb{R}$.

(d) $\wp(\vartheta + \mathfrak{W}_1) = \wp(\mathfrak{W}_1)$, *for all* $\vartheta \in \Xi$.

(e) $\wp(Conv\mathfrak{W}_1) \leq \wp(\mathfrak{W}_1)$ *and* $\wp(Conv\mathfrak{W}_1 \cup \mathfrak{W}_2) \leq \max\{\wp(Conv\mathfrak{W}_1), \wp(Conv\mathfrak{W}_2)\}$, *where* $\wp(Conv\mathfrak{W}_1)$ *represent the convex hull of* $\mathfrak{W}_1$.

(f) $\wp(\mathfrak{W}_1 + \mathfrak{W}_2) \leq \wp(\mathfrak{W}_1) + \wp(\mathfrak{W}_2)$.

Now, we consider:

$$\mathcal{A} = \left\{ \begin{array}{c} \varpi : \mathbb{R}_+ \to \mathbb{R}_+ : \varpi \text{ is monotone increasing} \\ \text{and } \lim\limits_{n\to\infty} \varpi^n = 0 \text{ for any } \varsigma \in \mathbb{R}_+ \end{array} \right\},$$

where $n \in \mathbb{N}$ and $\varpi^n(\varsigma)$ denotes the $n-$th composition of ϖ with itself.

Theorem 2.6. *[31] Let Q be a nonempty, bounded, closed and convex subset of a Banach space Ξ, and let $\varkappa : Q \to Q$ be a continuous operator. Suppose that $\exists \varpi \in \mathcal{A}$ where:*

$$\wp(\varkappa(\mathfrak{W})) \leq \varpi(\wp(\mathfrak{W}))$$

for any non-empty subset $\mathfrak{W}$ of Q. Then, $\varkappa$ has at least one fixed point in Q.

Lemma 2.7. *([31]) Let $\mathfrak{W} \subset C(\Theta, \Xi)$ be non-empty and bounded. Then:*

$$\sup_{\varsigma\in\Theta} \wp(\mathfrak{W}(\varsigma)) \leq \wp(\mathfrak{W}).$$

3 Existence of Mild Solutions for Neutral Integro-Differential Equations

Definition 3.1. A continuous function $\xi(\cdot) \in C(\Theta, \Xi)$ is a mild solution of (1), if ξ verifies

$$\xi(\varsigma) = R(\varsigma, 0)\xi_0 + \int_0^{\varsigma} R(\varsigma, \kappa)\Phi(\kappa, \xi(\vartheta(\kappa)))d\kappa, \quad \text{for each } \varsigma \in \Theta.$$

Now, we assume the following hypotheses:

$(H1)$ The function $\Phi : \Theta \times \Xi \to \Xi$ satisfies the Carathéodory conditions, and there exist $p_f \in L^1(\Theta, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing continuous function such that

$$\|\Phi(\varsigma, \xi))\|_{\Xi} \leq p_f(\varsigma)\psi(\|\xi\|_{\Xi}), \quad \text{for } \xi \in \Xi, \text{ and for a.e. } \ \varsigma \in \Theta.$$

$(H2)$ The resolvent operator is uniformly continuous and there exist $\mathfrak{Z} \geq 1$ such that

$$\|R(\varsigma, \kappa)\|_{\mathfrak{Y}(\Xi)} \leq \mathfrak{Z}, \text{ for every } 0 \leq \kappa \leq \varsigma \leq \varkappa.$$

$(H3)$ There exist $K \in L^{\infty}(\Theta, \mathbb{R}_+)$ and $h \in \mathcal{A}$ such that for any non-empty, bounded and convex subset $\mathfrak{W} \subset \Xi$,

$$\wp(\Phi(\varsigma, \mathfrak{W})) \leq K(\varsigma)h(\wp(\mathfrak{W})), \text{ for a.e } \varsigma \in \Theta.$$

$(H4)$ There exist $r > 0$ such that

$$r \geq \mathfrak{Z}\Big[r + \psi(r)\|p_f\|_{L^1}\Big].$$

Theorem 3.2. *Assume that the conditions* $(H1) - (H4)$ *are satisfied, and that*

$$\varkappa\mathfrak{Z}\|K\|_{L^{\infty}} \leq 1,$$

then, the system (1) has at least one solution defined on Θ*.*

Proof. Firstly, transform the problem (1) into a fixed point problem and define the operator

$$\varkappa\xi(\varsigma) = R(\varsigma,0)\xi_0 + \int_0^{\varsigma} R(\varsigma,\kappa)\Phi(\kappa,\xi(\vartheta(\kappa)))d\kappa, \quad \text{for each } \varsigma \in \Theta.$$

We consider the set

$$Q = \left\{ \xi \in C(\Theta,\Xi) : \|\xi\|_{\infty} \leq r \right\}.$$

We note that Q is bounded, closed and convex subset.

Step 1 : We prove that $\varkappa Q \subset Q$.

Indeed for any $\xi \in Q$ and under $(H_1) - (H_4)$ we obtain

$$\begin{aligned}
\|\varkappa\xi(\varsigma)\|_{\Xi} &= \|R(\varsigma,0)\xi_0 + \int_0^{\varsigma} R(\varsigma,\kappa)\Phi(\kappa,\xi(\vartheta(\kappa)))d\kappa\|_{\Xi} \\
&\leq \|R(\varsigma,0)\|_{\mathfrak{Y}(\Xi)}\|\xi_0\|_{\Xi} + \int_0^{\varsigma} \|R(\varsigma,\kappa)\|_{\mathfrak{Y}(\Xi)}\|\Phi(\kappa,\xi(\vartheta(\kappa)))\|_{\Xi}d\kappa \\
&\leq \mathfrak{Z}\|\xi_0\|_{\Xi} + \mathfrak{Z}\int_0^{\varsigma} p_f(\kappa)\psi(\|\xi(\vartheta(\varsigma))\|_{\Xi})d\kappa \\
&\leq \mathfrak{Z}r + \mathfrak{Z}\psi(r)\|p_f\|_{L^1} \\
&\leq r.
\end{aligned}$$

Thus $\varkappa(Q) \subset Q$.
By (H_1) and the Lebesgue dominated convergence theorem, we can deduce that $\varkappa$ is continuous on Q.

Step 2 : We prove that $\varkappa$ is contractive.
Let $\mathfrak{H}$ be any non-empty and convex subset of Q, and for each $\varsigma \in \Theta$, let $\zeta_{\varsigma} = \wp(\mathfrak{H}(\varsigma))$. By (H_3), there are $K \in L^{\infty}(\Theta,\mathbb{R}_+)$ and $h \in \mathcal{A}$ where for a.e $\varsigma \in \Theta$,

$$\wp(\Phi(\varsigma,\mathfrak{H}(\varsigma))) \leq K(\varsigma)h(\wp(\zeta_{\varsigma})).$$

Therefor, given any $\gamma \leq 0$, there is a continuous mapping $\Im_{\varsigma} : \mho \to \Xi$, with $\Im_{\varsigma}(\mho) \subset \Phi(\varsigma,\mathfrak{H}(\varsigma))$, such that for all $\xi \in \mathfrak{H}$, there is $\eta \in \mho$ with

$$\|\Phi(\varsigma,\xi(\vartheta(\varsigma))) - \Im_{\varsigma}(\eta)\|_{\Xi} \leq K(\varsigma)h(\zeta_{\varsigma}) + \gamma, \quad \text{for a.e } \varsigma \in \Theta. \tag{4}$$

Let $\tilde{\Im} : \mho \to ((C(\Theta, \Xi)), \|\cdot\|_\infty)$ defined as:

$$\eta \in \mho \to \tilde{\Im}(\eta, \varsigma) = R(\varsigma, 0)\xi_0 + \int_0^\varsigma R(\varsigma, \kappa)\Im_\kappa(\eta)d\kappa, \quad \text{for a.e } \varsigma \in \Theta.$$

Clearly, $\tilde{\Im}$ is continuous and $\tilde{\Im}(\mho) \subset \varkappa(\mathfrak{H})$. By (4), given $\xi \in \mathfrak{H}$ we can find $\eta \in \mho$ where

$$\begin{aligned}\|\varkappa\xi(\varsigma) - \tilde{\Im}_\varsigma(\eta)\|_\Xi &\leq \int_0^\varsigma \|R(\varsigma, \kappa)\|_{\mathfrak{Y}(\Xi)}\|\Phi(\kappa, \xi(\vartheta(\kappa))) - \Im_\kappa(\eta)\|_\Xi d\kappa \\ &\leq \mathfrak{Z}\int_0^\varsigma K(\kappa)h(\zeta_\kappa) + \gamma d\kappa.\end{aligned}$$

Setting $\zeta := \wp(\mathfrak{H})$, we can deduce that $h(\zeta_\varsigma) \leq h(\zeta)$ for a.e $\varsigma \in \Theta$, we obtain

$$\begin{aligned}\|\varkappa\xi(\varsigma) - \tilde{\Im}_\varsigma(\eta)\|_\Xi &\leq \varkappa\mathfrak{Z}\|K\|_{L^\infty}h(\zeta) \\ &\leq h(\zeta),\end{aligned}$$

which means, from the arbitrariness of $\varsigma \in \Theta$, that $\wp(\varkappa\mathfrak{H}) \leq h(\zeta)$. □

4 Neutral Integro-Differential Equations with Nonlocal Condition

Definition 4.1. We say that a continuous function $\xi(\cdot) \in C(\Theta, \Xi)$ is a mild solution of problem (2), if ξ satisfies the following integral equation

$$\xi(\varsigma) = R(\varsigma, 0)[\xi_0 - g(\xi)] + \int_0^\varsigma R(\varsigma, \kappa)\Phi(\kappa, \xi(\vartheta(\kappa)))d\kappa, \quad \text{for each } \varsigma \in \Theta.$$

Let us recall the following assumptions:

($C1$) The function $g : C(\Theta, \Xi) \to \Xi$ is continuous, and there exists a constant $L > 0$ such that

$$\|g(\xi)\|_\Xi \leq L\|\xi\|_\infty, \quad \text{for } \xi \in C(\Theta, \Xi).$$

($C2$) There exists $r > 0$ such that

$$r \geq \mathfrak{Z}\Big[r + Lr + \psi(r)\|p_f\|_{L^1}\Big].$$

Theorem 4.2. *Assume that the conditions* $(H1)-(H3)$ *and* $(C1)-(C2)$ *are satisfied, and that*

$$\varkappa\mathfrak{Z}\|K\|_{L^\infty} \leq 1,$$

then, the system (2) has at least one solution defined on Θ.

Proof. We define the operator

$$\mathcal{M}\xi(\varsigma) \quad = \quad R(\varsigma,0)[\xi_0 - g(\xi)] + \int_0^\varsigma R(\varsigma,\kappa)\Phi(\kappa,\xi(\vartheta(\kappa)))d\kappa, \qquad \text{for each } \varsigma \in \Theta.$$

Step 1 : We prove $\mathcal{M}Q \subset Q$.
This step is similar to (Step 1) in the proof of Theorem 3.2. Indeed for any $\xi \in Q$ we obtain

$$\begin{aligned}
\|\mathcal{M}\xi(\varsigma)\|_\Xi &= \|R(\varsigma,0)[\xi_0 - g(\xi)] + \int_0^\varsigma R(\varsigma,\kappa)\Phi(\kappa,\xi(\vartheta(\kappa)))d\kappa\|_\Xi \\
&\leq \|R(\varsigma,0)\|_{\mathfrak{Y}(\Xi)}\|\xi_0 - g(\xi)\|_\Xi + \int_0^\varsigma \|R(\varsigma,\kappa)\|_{\mathfrak{Y}(\Xi)}\|\Phi(\kappa,\xi(\vartheta(\kappa)))\|_\Xi d\kappa \\
&\leq \mathfrak{Z}[\|\xi_0\|_\Xi + L\|\xi\|_\infty] + \mathfrak{Z}\int_0^\varsigma p_f(\kappa)\psi(\|\xi(\vartheta(\kappa))\|_\Xi)d\kappa \\
&\leq \mathfrak{Z}r + \mathfrak{Z}Lr + \mathfrak{Z}\psi(r)\|p_f\|_{L^1} \\
&\leq r.
\end{aligned}$$

Thus $\mathcal{M}(Q) \subset Q$. Furthermore, combining assumption (H_1) and the Lebesgue dominated convergence theorem, we show that $\mathcal{M}$ is continuous on Q.

Step 2 : We prove that $\mathcal{M}$ is contractive.
Let $\mathfrak{H}$ be any non-empty and convex subset of Q, and for each $\varsigma \in \Theta$, let $\zeta_\varsigma = \wp(\mathfrak{H}(\varsigma))$. By (H_3), there are $K \in L^\infty(\Theta,\mathbb{R}_+)$ and $h \in \mathcal{A}$ where for a.e $\varsigma \in \Theta$

$$\wp(\Phi(\varsigma,\mathfrak{H}(\varsigma))) \leq K(\varsigma)h(\wp(\zeta_\varsigma)).$$

By the same technique of the (step 2) in the Theorem 3.2, we get:
$\tilde{\Im}$ is continuous and $\tilde{\Im}(\delta) \subset \mathcal{M}(\mathfrak{H})$. By (4), given $\xi \in \mathfrak{H}$ we can find $\eta \in \delta$ where

$$\|\mathcal{M}\xi(\varsigma) - \tilde{\Im}_\varsigma(\eta)\|_\Xi \leq \int_0^\varsigma \|R(\varsigma,\kappa)\|_{\mathfrak{Y}(\Xi)}\|\Phi(\kappa,\xi(\vartheta(\kappa))) - \Im_\kappa(\eta)\|_\Xi d\kappa$$

$$\leq \mathfrak{Z}\int_0^{\varsigma} K(\kappa)h(\zeta_\kappa) + \gamma d\kappa.$$

Setting $\zeta := \wp(\mathfrak{H})$, we can deduce that $h(\zeta_\varsigma) \leq h(\zeta)$ for a.e $\varsigma \in \Theta$, we obtain

$$\begin{aligned}\|\mathcal{M}\xi(\varsigma) - \tilde{\Im}_\varsigma(\eta)\|_\Xi &\leq \varkappa\mathfrak{Z}\|K\|_{L^\infty} h(\zeta)\\ &\leq h(\zeta).\end{aligned}$$

Which means, from the arbitrariness of $\varsigma \in \Theta$, that $\wp(\mathcal{M}\mathfrak{H}) \leq h(\zeta)$. Then ξ is a fixed point of the operator $\mathcal{M}$, which is a mild solution of the problem (2). □

5 An Example

Consider the problem:

$$\begin{cases} \frac{\partial}{\partial\varsigma}\left[z(\varsigma,u) + \int_0^1 a(\varsigma,\kappa)z(\kappa,u)d\kappa\right] = \Gamma(\varsigma)\frac{\partial^2}{\partial u^2}z(\varsigma,u) - \int_0^\varsigma \Gamma(\varsigma-\kappa)\frac{\partial^2}{\partial u^2}z(\kappa,u)d\kappa \\ \qquad\qquad +g(\varsigma, z(\varsigma,u)) \ \text{ if } \varsigma \in \Theta = [0,1] \ \text{ and } \ u \in (0,1), \\ z(\varsigma,0) = z(\varsigma,1) = 0, \quad \text{for } \varsigma \in \Theta, \\ z(0,u) = e^u, \quad \text{for } u \in (0,1), \end{cases} \tag{5}$$

where $a : [0,1] \times [0,1] \to \mathbb{R}$ is a continuous function, and

$$g(\varsigma, z(\varsigma,u)) = \frac{1}{e^{2t}}\left(\frac{2}{(\varsigma+1)^2+1} + ln(1 + |z(\varsigma,u)|)\right).$$

Let $\mathcal{A}$ be defined by

$$(\mathcal{A}z)(u) = \frac{\partial^2}{\partial u^2}z(\varsigma,u).$$

And

$$\mathfrak{G}(\mathcal{A}) = \{z \in L^2(0,1) \ / \ z, \frac{\partial^2}{\partial u^2}z \in L^2(0,1) \ \ ; \ \ z(0) = z(1) = 0\}.$$

The operator $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on $L^2(0,1)$ with domain $\mathfrak{G}(\mathcal{A})$, and with more appropriate conditions on operator $\aleph(\cdot) = \Gamma(\cdot)\mathcal{A}$, the problem (5) has a resolvent operator $R(\varsigma, \kappa)$ on $L^2(0,1)$ which is norm continuous.

Now, define

$$\xi(\varsigma)(u) = z(\varsigma, u),$$
$$\Phi(\varsigma, \xi)(u) = g(\varsigma, z(\varsigma, u))$$

and $\Phi : \Theta \times L^2(0,1) \longrightarrow L^2(0,1)$ given by

$$\Phi(\varsigma, \xi)(u) = \frac{1}{e^{2t}}\left(\frac{2}{(\varsigma+1)^2+1} + ln(1+|z(\varsigma,u)|)\right), \quad \text{for } \varsigma \in \Theta,$$

Moreover, for each $\varsigma \in \Theta$, we obtain

$$\begin{aligned}\|\Phi(\varsigma,\xi)\|_{L^2} &= \left\| \frac{1}{e^{2t}}\left(\frac{2}{(\varsigma+1)^2+1} + ln(1+|z(\varsigma,u)|)\right)\right\|_{L^2} \\ &\leq \frac{1}{e^{2t}}\left(1 + \|z(\varsigma,u)\|_{L^2}\right) \\ &\leq p_f(\varsigma)\psi(\|z(\varsigma)\|_{L^2}).\end{aligned}$$

Therefore, assumption $(H1)$ is satisfied with

$$p_f(\varsigma) = \frac{1}{e^{2t}},\ \varsigma \in \Theta \text{ and } \psi(u) = 1+u,\ u \in (0,1).$$

Now we shall check that condition of $(H4)$ is satisfied.
Indeed, we have

$$r \geq \mathfrak{Z}r + \mathfrak{Z}(1+r).$$

Thus

$$r \geq \frac{\mathfrak{Z}}{1-2\mathfrak{Z}}.$$

For any non-empty, bounded and convex subset $\mathfrak{H}$ of $C(\Theta, L^2(0,1))$ and $\varsigma \in \Theta$ fixed, let $\Im$ be an ζ_ς-dense curve in $\mathfrak{H}(\varsigma)$ for some $\zeta_\varsigma \geq 0$. Then, for $z \in \mathfrak{H}$, there is $\eta \in \mho$ verifying:

$$\|z(\varsigma) - \Im(\eta,\varsigma)\|_{L^2} \leq \zeta_\varsigma.$$

Therefore, we have:

$$\begin{aligned}\|\Phi(\varsigma, z(\varsigma)) - \Phi(\varsigma, \Im(\eta,\varsigma))\|_{L^2} &\leq \frac{1}{e^{2t}}\|ln(1+|z(\varsigma,u)|) - ln(1+|\Im(\eta,\varsigma)|)\|_{L^2}\\ &\leq \frac{1}{e^{2t}}\left\|ln\left(1+\frac{|z(\varsigma,u)-\Im(\eta,\varsigma)|}{1+|\Im(\eta,\varsigma)|}\right)\right\|_{L^2}\\ &\leq \frac{1}{e^{2t}}ln(1+\|z(\varsigma,u)-\Im(\eta,\varsigma)\|_{L^2})\\ &\leq \frac{1}{e^{2t}}ln(1+\zeta_\varsigma),\end{aligned}$$

and $h(\varsigma) = \ln(1+\varsigma)$. Thus, $h \in \mathcal{A}$, so condition (H_3) is verified by $K(\varsigma) = \frac{1}{e^{2t}}$. Consequently, all the hypotheses of Theorem 3.2 are verified and thus (5) has at least one solution $\xi \in C(\Theta, L^2(0,1))$.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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