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Original Research Paper

On Pointfree Countably Disconnectivity

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Abstract. We introduce pointfree counterparts of countably disconnectivity, specifically countably basically disconnected frames (briefly, c-basically disconnected frames) and countably F-frames (briefly, F_c-frames), along with their frame-theoretic characterizations. We present characterizations of zero-dimensional extremally disconnected frames, zero-dimensional c-basically disconnected frames, and zero-dimensional F_c-frames L using the ring-theoretic properties of the ring $\mathcal{R}_cL$ of continuous real-valued functions with countable images on L. We show that a zero-dimensional frame L is extremally disconnected if and only if any annihilator ideal of $\mathcal{R}_cL$ is generated by an idempotent, c-basically disconnected if and only if the annihilator of each element of $\mathcal{R}_cL$ is generated by an idempotent, and an F_c-frame if and only if the annihilator of each element of $\mathcal{R}_cL$ is pure.

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1 Introduction

A Tychonoff (completely regular and Hausdorff) topological space X is *basically disconnected* if each cozero-set has an open closure, *extremally disconnected* if each open set has an open closure, and an *F-space* if each cozero-set is C^*-embedded (see [18] for more details). These concepts have been expanded to pointfree topology such that a Tychonoff topological space X has a specific disconnectivity property if and only if its frame of open subsets, $\mathfrak{O}(X)$, has the corresponding characteristic.

The pointfree counterparts of the three types of disconnectivity are given in [7]. Let $\mathrm{Coz}L$ denote the *cozero part* of a frame L, where $a \in \mathrm{Coz}L$ if and only if there is a sequence (a_n) in L such that $a_n \prec\!\!\prec a$ for all n and $a = \bigvee a_n$. The *pseudocomplement* of an element $x \in L$ is denoted by x^*. Now, recall from [7] that a frame L is called

- *basically disconnected* if, for any $a \in \mathrm{Coz}L$, the map $x \mapsto x \vee a^*$ is a frame isomorphism from $\downarrow a^{**}$ to $\uparrow a^*$,
- *extremally disconnected* if, for any $a \in L$, the map $x \mapsto x \vee a^*$ is a frame isomorphism from $\downarrow a^{**}$ to $\uparrow a^*$, and
- an *F-frame* if, for each $a \in L$, the open quotient map $L \to \downarrow a$ is a C^*-quotient map.

The concepts of countably disconnectivity (briefly, c-disconnectivity) are introduced in [6, 22]. Let $C(X)$ be the ring of all real valued continuous functions on a Tychonoff topological space X (refer to [17] for more details). We denote by $C_c(X)$ the subring of $C(X)$ consisting of those functions whit countable image (see [16] for more details). For each $f \in C(X)$, the *zero-set* of f, denoted by $Z(f)$, is the set of zeros of f, and $Coz(f) = X \setminus Z(f)$ is the *cozero-set* of f. Now, recall from [6] that a space X is *countably F-space* (briefly, F_c-space) if $\mathbf{O}_c^p = \{f \in C_c(X) \mid p \in \mathrm{int}_{\beta_0 X}\mathrm{cl}_{\beta_0 X} Z(f)\}$ is a prime ideal in $C_c(X)$ for each $p \in \beta_0 X$, the Banaschewski compactification of X. Next, recall from [22] that a space X is *countably basically disconnected* (briefly, c-basically disconnected) if $\mathrm{int}_X Z(f)$ is closed for each $f \in C_c(X)$.

Several frame-theoretic characterizations of basically disconnected frames and F-frames are given in [7]. Recall from [7] that a frame L is

1. *basically disconnected* if $a^* \vee a^{**} = \top$ for each $a \in \mathrm{Coz}L$, and
2. an F-frame if whenever $a \wedge b = \bot$ in $\mathrm{Coz}L$, then there are $c, d \in \mathrm{Coz}L$ such that $a \wedge c = b \wedge d = \bot$, and $c \vee d = \top$.

For a frame L, let $\mathrm{Coz}_c L$ denote the sublattice of L consisting of all countable joins of complemented elements. Building on of the two types of c-disconnectivity and the two frame-theoretic characterizations discussed, we introduce two pointfree c-disconnectivity concepts in this article. A frame L is called

1. *c-basically disconnected* if $a^* \vee a^{**} = \top$ for each $a \in \mathrm{Coz}_c L$, and
2. an F_c-frame if whenever $a \wedge b = \bot$ in $\mathrm{Coz}_c L$, then there are $c, d \in \mathrm{Coz}_c L$ such that $a \wedge c = b \wedge d = \bot$, and $c \vee d = \top$.

Let $\mathcal{R}L$ be the ring of real-valued continuous functions on a completely regular frame L (see [7, 8]). For any $\varphi \in \mathcal{R}L$, let $\mathrm{coz}(\varphi) = \varphi((-,0) \vee (0,-))$, where

$$(0,-) = \bigvee\{(0,q)) \mid 0 < q \in \mathbb{Q}\}, \ \text{and}\ (-,0) = \bigvee\{(p,0)) \mid 0 > p \in \mathbb{Q}\},$$

and let $R_\varphi := \{r \in \mathbb{R} \mid \mathrm{coz}(\varphi - \mathbf{r}) = \varphi(-,r) \vee \varphi(r,-) \neq \top\}$ as defined [14], where

$$(-,r) := \bigvee_{\substack{q \in \mathbb{Q} \\ q < r}} (-,q) \qquad \text{and} \qquad (r,-) := \bigvee_{\substack{p \in \mathbb{Q} \\ r < p}} (p.-).$$

Then R_φ is the extension to arbitrary $\mathcal{R}L$ of the familiar correspondence between functions on spaces and their images (see [14]). An element φ in the ring $\mathcal{R}L$ has a *countable pointfree image* if R_φ is countable. We denote the subring consisting of those functions with countable pointfree images by $\mathcal{R}_c L$. This ring, viewed as the pointfree topology counterpart of $C_c(X)$, is introduced and analyzed in [25], where the authors prove that $\mathcal{R}_c(\mathfrak{O}X) \cong C_c(X)$ for any space X (see [12, 13, 21] for more details).

The articles [10] and [11] characterize basically disconnected frames, extremally disconnected frames, and F-frames L using the ring-theoretic properties of the ring $\mathcal{R}L$. In this article, we extend the characterizations from [11] to include extremally disconnected frames, c-basically disconnected frames, and F_c-frames L, focusing on the ring-theoretic properties of the ring $\mathcal{R}_cL$.

Here is an overview of the article. In the preliminaries, we outline several properties of frames that we will use. In Section 2, we use two lemmas from [3] and [11] to give ring-theoretic characterizations of zero-dimensional extremally disconnected frames, see Theorem 3.3. Specifically, a zero-dimensional frame L is extremally disconnected if and only if $\mathcal{R}_cL$ is a Baer ring, meaning that every annihilator ideal is generated by an idempotent. Theorem 3.4 demonstrate that the Banaschewski compactification of a frame is always extremally disconnected.

Section 3 discusses c-basically disconnected frames, with characterizations provided in Proposition 4.1. By referencing a Lemma from [2], we present a new lemma regarding the annihilator of a subset in $\mathcal{R}L$. These two lemmas allow us to provide ring-theoretic characterizations of zero-dimensional c-basically disconnected frames in Theorem 4.5. In particular, a zero-dimensional frame L is c-basically disconnected if and only if $\mathcal{R}_cL$ is a PP-ring, which means that the annihilator of each element is generated by an idempotent.

Section 4 focuses on F_c-frames and their related results. We first characterize F_c-space using cozero sets (see Theorem 5.2). This leads to the result that a zero-dimensional space X is a F_c-space if and only if $\mathfrak{O}(X)$ is a F_c-frame. In Proposition 5.6, we identify pure ideals in $\mathcal{R}_cL$, which enable us to characterize F_c-frames as those L for which $\mathcal{R}_cL$ is a PF-ring (Theorem 5.7), meaning that the annihilator of each element in $\mathcal{R}_cL$ is a pure ideal. Theorem 5.10 characterizes a zero-dimensional L as a F_c-frame when certain ideals of $\mathcal{R}_cL$ are prime, extending the spatial result that X is a F_c-space if and only if the ideal $\mathbf{O}_c^p$ is prime for all $p \in \beta_0 X$. This theorem allows us to identify F_c-frames as those L where every maximal ideal of $\mathcal{R}_cL$ contains a unique minimal prime ideal or where the localization $\mathcal{R}_cL_P$ is domain for ever prime ideal P of $\mathcal{R}_cL$ (see Corollaries 5.11 and 5.13). Proposition 4.1 confirms that c-basically disconnected frames are indeed F_c-frames. We will now investigate the situation in which the converse is also valid. By defining a c-cozero-complementation frame, analogous to the concept of a c-cozero-complementation space, Theorem 5.15 characterizes c-basically disconnected frames as those L that are both F_c-frames and c-cozero-complementation frames. Next, we consider the space of minimal prime ideals of $\mathcal{R}_cL$, denoted by $\mathrm{Min}(\mathcal{R}_cL)$. Proposition 5.16 shows that a zero-dimensional frame L is c-cozero-complementation if and only if the space $\mathrm{Min}(\mathcal{R}_cL)$ is compact. Thus, combining these results, we conclude that a zero-dimensional frame L is c-basically disconnected if and only if it is an F_c-frame and $\mathrm{Min}(\mathcal{R}_cL)$ is compact (Corollary 5.17).

2 Preliminaries

2.1 Frame.

A frame is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee_{s \in S} a \wedge s.$$

holds for all $a \in L$ and $S \subseteq L$. We write $\perp$ and $\top$ for the bottom and the top element of L, respectively. The frame of open subsets of a topological space X is denoted by $\mathfrak{O}(X)$. The *pseudocomplement* of an element $a \in L$ is the element $a^* = \bigvee\{x \in L \mid a \wedge x = \perp\}$. When $a \vee a^* = 1$, then a is called a *complemented* element in L. We denote the set of all complemented elements of L by $B(L) = \{a \in L \mid a \vee a^* = \top\}$. A frame L is called *zero-dimensional* if $a = \bigvee\{x \in B(L) \mid x \leq a\}$ for any $a \in L$.

A *frame homomorphism* (or *frame map*) is a lattice homomorphism $h : L \to M$ that preserves the bottom element, top element, binary meets, and arbitrary Joins. The homomorphism h is *dense* when $h(x) = \perp_M$ implies $x = \perp_L$. A *quotient* of a frame L is the codomain M of a frame surjection $h : L \to M$ whit h being referred to as the quotient map. An element $a \in L$ generates two new frames, the *open* quotient $\downarrow a = \{x \in L \mid x \leq a\}$ and the *closed* quotient $\uparrow a = \{x \in L \mid a \leq x\}$. They are associated with the homomorphisms

$$L \to \downarrow a, x \mapsto x \wedge a \qquad \text{and} \qquad L \to \uparrow a, x \mapsto x \vee a.$$

An element a of a frame L is said to be *rather below* an element b, denoted $a \prec b$, if there exists a separating element c such that $a \wedge c = \perp$ and $c \vee b = \top$. Furthermore, a is *completely below* b, denoted $a \prec\!\!\prec b$, if there are elements a_p indexed by the rational numbers $\mathbb{Q} \cap [0,1]$ such that $a = a_0$, $b = a_1$, and $a_p \prec a_q$ for $p < q$. A frame L is *completely regular* if $a = \bigvee\{x \in L \mid x \prec\!\!\prec a\}$ for any $a \in L$. For an element a of a frame L, $a \in B(L)$ if and only if $a \prec a$ if and only if $a \prec\!\!\prec a$. In compact completely regular frames, the rather below relation coincides with the completely below relation.

2.2 The Banaschewski compactification.

The *Banaschewski compactification* of a frame L is denoted by $\beta_0 L$. It is defined as:

$$\beta_0 L = Id(B(L)),$$

where $Id(B(L))$ denotes the frame of all ideals of $B(L)$. The frame $\beta_0 L$ is *compact* and *zero-dimensional*, and the join map $j_0 : \beta_0 L \to L$ is a *dense frame homomorphism*. Moreover, the join map $j_0 : \beta_0 L \to L$ is a compactification for a frame L if and only if L is zero-dimensional (refer to [4]). We use r_0 to denote the right adjoint of the join map $j_0 : \beta_0 L \to L$, and recall from [9] that $r_0(a) = \{x \in B(L) \mid x \leq a\} = \downarrow a \cap B(L)$ for any $a \in L$. The following is shown in [2]:

1. If $a = \bigvee\limits_{s \in S} s$ for some $S \subseteq B(L)$, then $\bigvee r_0(a) = a$ and $r_0(a^*) = \big(r_0(a)\big)^*$.
2. If $a \in \mathrm{Coz}_c L$, then $\bigvee r_0(a) = a$ and $r_0(a^*) = \big(r_0(a)\big)^*$.
3. If $J \prec\!\!\prec I$ in $\beta_0 L$, then $\bigvee J \prec\!\!\prec b$ for some $b \in I$.
4. For any zero-dimensional L, $r_0(a) \prec\!\!\prec I$ in $\beta_0 L$ iff there exists $b \in I$ such that $a \prec\!\!\prec b$ in L.
5. If $a, b \in \mathrm{Coz}_c L$, then $r_0(a \vee b) = r_0(a) \vee r_0(b)$.

2.3 The rings $\mathcal{R}L$ and $\mathcal{R}_c L$

For any frame L, *the real-valued continuous functions* on L are the homomorphisms $\mathcal{L}(\mathbb{R}) \to L$, where $\mathcal{L}(\mathbb{R})$ denotes the frame of reals. The set $\mathcal{R}L$ of all real-valued continuous functions on L is an f-ring (see [7, 8]. Let $\mathcal{R}^* L$ denote the subring of *bounded* functions, that is, the functions φ such that $\varphi(p,q) = \top$ for some $p,q \in \mathbb{Q}$. The identity element and the zero element of $\mathcal{R}L$ are denoted by $\mathbf{1}$ and $\mathbf{0}$, respectively. For every $r \in \mathbb{R}$, the constant frame map $\mathbf{r} \in \mathcal{R}L$ is defined by

$$\mathbf{r}(p,q) = \begin{cases} \top & \text{if } p < r < q \\ \bot & \text{otherwise.} \end{cases}$$

An important feature of $\mathcal{R}L$ is its *cozero map* $\operatorname{coz} : \mathcal{R}L \to L$ taking every $\varphi \in \mathcal{R}L$ to $\operatorname{coz}(\varphi) = \varphi((-,0) \vee (0,-))$. Some properties of the cozero map that we shall use are:

1. $\operatorname{coz}(\varphi) = \bot$ iff $\varphi = \mathbf{0}$.
2. $\operatorname{coz}(\varphi) = \operatorname{coz}(|\varphi|) = \operatorname{coz}(\varphi^n)$, for any $n \in \mathbb{N}$.
3. $\operatorname{coz}(\varphi\delta) = \operatorname{coz}(\varphi) \wedge \operatorname{coz}(\delta)$.
4. $\operatorname{coz}(\varphi + \delta) \leq \operatorname{coz}(\varphi) \vee \operatorname{coz}(\delta)$, with equality when $\varphi, \delta \geq \mathbf{0}$.
5. $\varphi \in \mathcal{R}L$ is invertible iff $\operatorname{coz}(\varphi) = \top$.

A *cozero element* of a frame L is an element of the form $\operatorname{coz}(\varphi)$ for some $\varphi \in \mathcal{R}L$. The set $\operatorname{Coz}L = \{\operatorname{coz}(\varphi) \mid \varphi \in \mathcal{R}L\}$ is a sub-σ-frame of L.

Let $\mathcal{R}_cL = \{\alpha \in \mathcal{R}L \mid R_\alpha \text{ is countable}\}$ and let $\mathcal{R}^*_cL = \mathcal{R}_cL \cap \mathcal{R}^*L$. In [24, 25], it is shown that $\mathcal{R}_cL$ is a f-ring and $\mathcal{R}_c(\mathfrak{O}X) \cong C_c(X)$ for any topological space X. Moreover, the authors show that for any frame L, there exists a zero-dimensional frame M such that $\mathcal{R}_cL \cong \mathcal{R}_cM$. Therefore, when analyzing the rings $\mathcal{R}_cL$, one might assume that all frames are zero-dimensional. For $S \subseteq \mathcal{R}_cL$, let $\operatorname{Coz}[S] = \{\operatorname{coz}(\alpha) \mid \alpha \in S\}$. For any frame L, [15] establishes the following:

1. For any $\alpha \in \mathcal{R}_cL$, $\operatorname{coz}(\alpha)$ is a countable join of complemented elements of L, which implies $\operatorname{Coz}_cL = \operatorname{Coz}[\mathcal{R}_cL] = \{\operatorname{coz}(\alpha) \mid \alpha \in \mathcal{R}_cL\}$.
2. The set Coz_cL is a sublattice of L that includes $\bot$ and $\top$.
3. L is a zero-dimensional frame if and only if it is generated by Coz_cL.

4. $\mathrm{Coz}_c L$ is normal, that is, for any $a, b \in \mathrm{Coz}_c L$ with $a \vee b = \top$, there exist u and v in $\mathrm{Coz}_c L$ such that $u \wedge v = \perp$ and $a \vee u = \top = b \vee v$.

Now, we recall from [2] what maximal ideals of $\mathcal{R}_c L$ look like. For every frame homomorphism $h : \beta_0 L \to M$, the ideals $\mathbf{M}_c^h$ and $\mathbf{O}_c^h$ are defined by

$$\mathbf{M}_c^h = \{\alpha \in \mathcal{R}_c L \mid h(r_0(\mathrm{coz}(\alpha))) = \perp_M\} \quad \text{and}$$
$$\mathbf{O}_c^h = \{\alpha \in \mathcal{R}_c L \mid h(r_0((\mathrm{coz}(\alpha)^*))) = \top_M\}.$$

In particular, if $I \in \beta_0 L$ and $h : \beta_0 L \to \uparrow I$ is the homomorphism defined by the correspondence $J \mapsto J \vee I$, then the ideals $\mathbf{M}_c^h$ and $\mathbf{O}_c^h$ are denoted by $\mathbf{M}_c^I$ and $\mathbf{O}_c^I$, respectively, so that

$$\mathbf{M}_c^I = \{\alpha \in \mathcal{R}_c L \mid r_0(\mathrm{coz}(\alpha)) \subseteq I\}$$

and

$$\mathbf{O}_c^I = \{\alpha \in \mathcal{R}_c L \mid I \vee r_0((\mathrm{coz}(\alpha)^*)) = \top_{\beta_0 L}\}.$$

The following is shown in [2]:

1. For any $I \in \beta_0 L$, we have
$$\begin{aligned}\mathbf{O}_c^I &= \{\alpha \in \mathcal{R}_c L \mid r_0(\mathrm{coz}(\alpha)) \prec\!\!\prec I\} \\ &= \{\alpha \in \mathcal{R}_c L \mid \mathrm{coz}(\alpha) \prec\!\!\prec \mathrm{coz}(\delta) \in I \text{ for some } \delta \in \mathcal{R}_c L\}.\end{aligned}$$

2. A subset D of $\mathcal{R}_c L$ is a maximal ideal iff there exists a unique maximal element I of $\beta_0 L$ such that $D = \mathbf{M}_c^I$.

3. If P is a prime ideal of $\mathcal{R}_c L$, then there is a unique maximal I of $\beta_0 L$ such that $\mathbf{O}_c^I \subseteq P \subseteq \mathbf{M}_c^I$.

3 Extremally Disconnected Frames

A frame L is *extremally disconnected* if for every $x \in L$, $x^* \vee x^{**} = \top$. This concept extends its point-sensitive counterpart, as a topological space X is extremally disconnected if and only if the frame $\mathfrak{O}(X)$ is extremally disconnected. We will now present characterizations of zero-dimensional extremally disconnected frames and review some relevant facts before proceeding.

Let $a \in B(L)$. For each $p, q \in \mathbb{Q}$, we define

$$e_a(p,q) = \begin{cases} \bot & \text{if } p < q \leq 0 \text{ or } 1 \leq p < q \\ a^* & \text{if } p < 0 < q \leq 1 \\ a & \text{if } 0 \leq p < 1 < q \\ \top & \text{if } p < 0 < 1 < q. \end{cases}$$

According to [7, 8.4], we have $e_a \in \mathcal{R}L$ with $\mathrm{coz}(e_a) = a$, $\mathrm{coz}(e_a - \mathbf{1}) = a^*$, and ${e_a}^2 = e_a$. Moreover, e_a also belongs to $\mathcal{R}_cL$ since $R_{e_a} \subseteq \{0,1\}$.

Let R be a ring, $x \in R$, and $S \subseteq R$. We denote the annihilator of S by $\mathrm{Ann}_R(S)$ or $\mathrm{Ann}(S)$, while the annihilator of the singleton $\{x\}$ is abbreviated as $\mathrm{Ann}_R(x)$ or $\mathrm{Ann}(x)$. A ring R is a *Baer ring* if, for any subset S of R, its annihilator $\mathrm{Ann}_R(S)$ is generated by an idempotent element. An ideal I is *essential* in another ideal J if $I \subseteq J$ and $I \cap H \neq \{0\}$ for every nonzero ideal H of R that is contained in J. An ideal I of R is *essentially closed* if it is not essential in any strictly larger ideal. A ring is a *CS-ring* if every essentially closed ideal is generated by an idempotent. Ideal generation will be denoted by $\langle . \rangle$.

Lemma 3.1. *[11, Lemma 2.3] If A is reduced, then for any ideals $I \subseteq J$, I is essential in J if and only if $\mathrm{Ann}(I) = \mathrm{Ann}(J)$.*

Lemma 3.2. *[3, Lemma 3.8] Suppose $\mathrm{coz}(\lambda) \prec\!\!\prec \mathrm{coz}(\mu)$ in Coz_cL for some $\lambda, \mu \in \mathcal{R}_cL$. Then there exists $\rho \in \mathcal{R}_cL$ such that $\rho^2 + \mu^2$ is invertible and $\lambda = \lambda\mu^2(\rho^2 + \mu^2)^{-1}$.*

Theorem 3.3. *The following are equivalent for a zero-dimensional frame L.*

1. *L is extremally disconnected.*
2. *$\mathcal{R}_cL$ is a Baer ring.*
3. *Every nonzero ideal in $\mathcal{R}_cL$ is essential in an ideal generated by an idempotent.*
4. *$\mathcal{R}_cL$ is a CS-ring.*

Proof. (1) $\Rightarrow$ (2). Assume L is extremally disconnected and S is a subset of $\mathcal{R}_cL$. Now, take $a = \bigvee \mathrm{Coz}[S]$. Given that a^* is complemented by assumption, there exists an idempotent $e_{a^*} \in \mathcal{R}_cL$ such that $a^* = \mathrm{coz}(e_{a^*})$. We claim that $\mathrm{Ann}(S) = \langle e_{a^*} \rangle$. Suppose $\alpha \in S$. Then

$$\begin{aligned}\mathrm{coz}(\alpha) \leq \bigvee_{\beta \in S} \mathrm{coz}(\beta) = a &\Rightarrow \mathrm{coz}(\alpha) \wedge a^* = \perp \\ &\Rightarrow \mathrm{coz}(\alpha) \wedge \mathrm{coz}(e_{a^*}) = \perp \\ &\Rightarrow \mathrm{coz}(\alpha e_{a^*}) = \perp \\ &\Rightarrow \alpha e_{a^*} = \mathbf{0} \\ &\Rightarrow e_{a^*} \in \mathrm{Ann}(S).\end{aligned}$$

Thus, we deduce that $\langle e_{a^*} \rangle \subseteq \mathrm{Ann}(S)$. To show the reverse inclusion, let $\varphi \in \mathrm{Ann}(S)$. Then, for every $\alpha \in S$, $\varphi\alpha = \mathbf{0}$, showing $\mathrm{coz}(\varphi) \wedge \mathrm{coz}(\alpha) = \perp$, which shows that $\bigvee_{\alpha \in S} \big(\mathrm{coz}(\varphi) \wedge \mathrm{coz}(\alpha)\big) = \perp$. Next, by the frame distributive law, we can conclude that

$$\perp = \mathrm{coz}(\varphi) \wedge \bigvee_{\alpha \in S} \mathrm{coz}(\alpha) = \mathrm{coz}(\varphi) \wedge a,$$

implying that $\mathrm{coz}(\varphi) \leq a^* = \mathrm{coz}(e_{a^*})$. Since $\mathrm{coz}(e_{a^*}) \prec\!\!\prec \mathrm{coz}(e_{a^*})$ in Coz_cL, Lemma 3.2 shows that there is $\beta \in \mathcal{R}_cL$ such that $\varphi = \beta e_{a^*}$. This prove the $\mathrm{Ann}(S) \subseteq \langle e_{a^*} \rangle$. Consequently, $\mathrm{Ann}(S) = \langle e_{a^*} \rangle$, this means that $\mathcal{R}_cL$ is a Baer ring.

(2) $\Rightarrow$ (1). Suppose that $\mathcal{R}_cL$ is a Baer ring. Let $a \in L$, then, by zero-dimensionality, there is a $S \subseteq \mathcal{R}_cL$ such that $a = \bigvee_{\alpha \in S} \mathrm{coz}(\alpha)$. Since $\mathcal{R}_cL$ is Baer ring, then there is an element of idempotent $e \in \mathcal{R}_cL$ such that $\mathrm{Ann}(S) = e\mathcal{R}_cL$. We prove that $a^* = \mathrm{coz}(e)$. For any $\alpha \in S$, $\alpha e = \mathbf{0}$, and hence $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(e) = \perp$, which shows, by the frame distributive law, $\perp = \mathrm{coz}(e) \wedge \bigvee_{\alpha \in S} \mathrm{coz}(\alpha) = \mathrm{coz}(e) \wedge a$, and so $\mathrm{coz}(e) \leq a^*$. To prove the revers inequality, let $\beta \in \mathcal{R}_cL$ be such that $\mathrm{coz}(\beta) \leq a^*$. Then $\mathrm{coz}(\beta) \wedge a = \perp$, which implies $\mathrm{coz}(\beta) \wedge \mathrm{coz}(\alpha) = \perp$ for any $\alpha \in S$. So, $\beta\alpha = \mathbf{0}$ for any $\alpha \in S$, implying that $\beta \in \mathrm{Ann}(S) = e\mathcal{R}_cL$, and so β is a multiple of e. But this means that $\mathrm{coz}(\beta) \leq \mathrm{coz}(e)$. Thus, the property of zero-dimensionality results in $a^* \leq \mathrm{coz}(e)$, and so

$a^* = \mathrm{coz}(e)$. Consequently, a^* is complemented, and hence L is extremally disconnected.

(2)⇒(3). Suppose that I be a nonzero ideal in $\mathcal{R}_cL$. Then the there is an idempotent element e in $\mathcal{R}_cL$ such that

$$\mathrm{Ann}(I) = e\mathcal{R}_cL = \mathrm{Ann}((\mathbf{1}-e)\mathcal{R}_cL),$$

which shows that

$$\alpha = \alpha(\mathbf{1}-e) \in (\mathbf{1}-e)\mathcal{R}_cL \cap I,$$

for all $\alpha \in I$. Therefor, by Lemma 3.1, I is an essential ideal in $(\mathbf{1}-e)\mathcal{R}_cL$.

(3)⇒(4). Obvious.

(4)⇒(2). Let $S \subseteq \mathcal{R}_cL$. We claim that the ideal $\mathrm{Ann}(S)$ is a closed ideal in $\mathcal{R}_cL$. Suppose $\mathrm{Ann}(S)$ is essential in a larger ideal I. Then $SI \neq \{\mathbf{0}\}$ implies that $SI \cap \mathrm{Ann}(S) \neq \{\mathbf{0}\}$, but $(SI \cap \mathrm{Ann}(S))^2 = \{\mathbf{0}\}$, which is impossible, since $\mathcal{R}_cL$ is a reduced ring. This shows that $\mathrm{Ann}(S)$ is a closed ideal, so the hypothesis shows that an idempotent generates it. □

The Stone-Čech compactification of a frame L is denoted by βL. Recall that a completely regular frame L is extremally disconnected if and only if βL is extremally disconnected (see [7, 11]). We state the following theorem.

Theorem 3.4. *For any zero-dimensional frame L, $\beta_0 L$ is extremally disconnected.*

Proof. Since $B(L)$ is a Boolean algebra, for any ideal I in $B(L)$, we have $I^* =\downarrow (\vee^{B(L)} I)'$. Thus,

$$I^{**} = (\downarrow (\vee^{B(L)} I)')^* =\downarrow ((\vee^{B(L)} I)')' =\downarrow (\vee^{B(L)} I)$$

Consequently, We can express

$$I^* \vee I^{**} = (\downarrow (\vee^{B(L)} I)') \vee \downarrow (\vee^{B(L)} I) =\downarrow ((\vee^{B(L)} I)' \vee \vee^{B(L)} I) =\downarrow \top = B(L).$$

This confirms that $\beta_0 L$ is an extremally disconnected frame. □

Corollary 3.5. *If A is a Boolean algebra, then the frame of its ideals, denoted $Id(A)$, is extremally disconnected.*

Proof. Assuming $I \in Id(A)$ and applying the properties of complemented Boolean algebra, we find that:

$$I^{*} \vee I^{**} = \downarrow (\vee I)' \vee \downarrow ((\vee I)')' = \downarrow (\vee I)' \vee \downarrow (\vee I) = \downarrow (\vee I)' \vee (\vee I) = \downarrow \top = A.$$

This implies that $Id(A)$ is extremally disconnected. $\square$

4 c-Basically Disconnected Frames

Let a_1 and a_2 be elements of a frame L. We say that a_1 and a_2 are *c-completely separated* if there exist $b_1, b_2 \in \mathrm{Coz}_c L$ such that $b_1 \vee b_2 = \top$ and $a_1 \wedge b_1 = \bot = a_2 \wedge b_2$. A frame is *$c$-basically disconnected* if it has the attributes outlined in the next proposition.

Proposition 4.1. *The following are equivalent for a frame L.*

1. *For any $a \in \mathrm{Coz}_c L$, $a^* \vee a^{**} = \top$.*
2. *For any $a \in \mathrm{Coz}_c L$ and $b \in L$, $a \wedge b = \bot$ implies $a^* \vee b^* = \top$.*
3. *For any $a \in \mathrm{Coz}_c L$ and $b \in L$, $a \wedge b = \bot$ implies a and b are c-completely separated.*
4. *For any $a \in \mathrm{Coz}_c L$, the map $x \mapsto x \vee a^*$ is a frame isomorphism from $\downarrow a^{**}$ to $\uparrow a^*$.*

Proof. (1) $\Rightarrow$ (2). Let $a \in \mathrm{Coz}_c L$ and $b \in L$ with $a \wedge b = \bot$. Then $b \leq a^*$, which implies $a^{**} \leq b^*$. So, $a^* \vee b^* \geq a^* \vee a^{**}$, leading to $a^* \vee b^* = \top$ by (1).

(2) $\Rightarrow$ (3). Let $a \in \mathrm{Coz}_c L$ and $b \in L$ with $a \wedge b = \bot$. By (2), it follows that $a^* \vee b^* = \top$. Since $a \wedge a^* = \bot = b \wedge b^*$, we conclude that (2) implies (3).

(3) $\Rightarrow$ (4). We first demonstrate that the map $x \mapsto x \vee a^*$ is a frame homomorphism. This map is homomorphism if it sends the top element a^{**} of $\downarrow a^{**}$ to the top element $\top$ of $\uparrow a^*$. Therefore, it suffices to show that $a^* \vee a^{**} = \top$ for all $a \in \mathrm{Coz}_c L$. For $a \in \mathrm{Coz}_c L$, since $a \wedge a^* = \bot$, by (3), there exist $c, d \in L$ such that $c \vee d = \top$ and $a \wedge c = \bot = a^* \wedge d$. This implies $c \leq a^*$ and $d \leq a^{**}$, thus $a^* \vee a^{**} \geq c \vee d = \top$. We now prove

that the map $x \mapsto x \vee a^*$ is bijective. To establish injectivity, assume $x, y \in \downarrow a^{**}$ such that $x \vee a^* = y \vee a^*$. Then we have

$$\begin{aligned} x = x \vee \bot = x \vee (a^* \wedge a^{**}) &= (x \vee a^*) \wedge (x \vee a^{**}) \\ &= (y \vee a^*) \wedge a^{**} \\ &= (y \wedge a^{**}) \vee (a^* \wedge a^{**}) \\ &= y \vee \bot = y. \end{aligned}$$

Next, we demonstrate that the map is surjective. For any $y \in \uparrow a^*$, let $x = y \wedge a^{**}$. Then $x \in \downarrow a^{**}$ and

$$x \vee a^* = (y \wedge a^{**}) \vee a^* = (y \vee a^*) \wedge (a^* \vee a^{**}) = y \wedge \top = y.$$

(4) $\Rightarrow$ (1). For any $a \in \mathrm{Coz}_c L$, statement (4) asserts that the map $x \mapsto x \vee a^*$ is a frame homomorphism, which means it maps the top element a^{**} of $\downarrow a^{**}$ to the top element $\top$ of $\uparrow a^*$. Thus, $a^* \vee a^{**} = \top$ for all $a \in \mathrm{Coz}_c L$. $\square$

For a topological space X, we have $U^* = (\mathrm{cl}_X U)^c$ and $\mathrm{int}_X Z(f) = (Coz(f))^*$, where $U \in \mathfrak{O}X$ and $f \in C(X)$. Hence, the first statement of the previous proposition indicates that a topological space X is a c-basically disconnected if and only if the frame $\mathfrak{O}(X)$ is a c-basically disconnected.

We will demonstrate that zero-dimensional c-basically disconnected frames can be characterized by the ring-theoretic properties of the rings $\mathcal{R}_c L$. Before we continue, we present the following lemmas.

Lemma 4.2. *[2, Lemma 3.3] If $\alpha, \beta \in \mathcal{R}_c L$ and $(coz(\alpha))^* \leq (coz(\beta))^*$, then* $\mathrm{Ann}(\alpha) \subseteq \mathrm{Ann}(\beta)$*. The converse is true if L is zero-dimensional.*

Corollary 4.3. *[2] For a zero-dimensional frame L, an element $\varphi \in \mathcal{R}_c L$ is a zero-divisor if and only if $coz(\varphi)$ is not dense.*

Proof. The cozero element $\mathrm{coz}(\varphi)$ is dense if and only if $(\mathrm{coz}(\varphi))^* = \bot$, which is equivalent to $(\mathrm{coz}(\varphi))^* \leq \bot = (\mathrm{coz}(\mathbf{1}))^*$. This happens if and only if $\mathrm{Ann}(\varphi) \subseteq \mathrm{Ann}(\mathbf{1}) = \{\mathbf{0}\}$, which implies that $\mathrm{Ann}(\varphi) = \{\mathbf{0}\}$, and thus φ is not a zero-divisor. $\square$

Lemma 4.4. *Let $S \subseteq \mathcal{R}_c L$ and assume $\bigvee Coz[S] = coz(\delta)$ for some $\delta \in \mathcal{R}_c L$. Then, we have* $\mathrm{Ann}(S) = \mathrm{Ann}(\delta)$*.*

Proof. For any $\alpha \in S$, we have $\operatorname{coz}(\alpha) \leq \operatorname{coz}(\delta)$, leading to $(\operatorname{coz}(\delta))^* \leq (\operatorname{coz}(\alpha))^*$. Thus, Lemma 4.2 implies that $\operatorname{Ann}(\delta) \subseteq \operatorname{Ann}(\alpha)$ for each $\alpha \in S$. Consequently, $\operatorname{Ann}(\delta) \subseteq \bigcap_{\alpha\in S} \operatorname{Ann}(\alpha) = \operatorname{Ann}(S)$. To prove the reverse inclusion, let $\beta \in \operatorname{Ann}(S)$. Then, for each $\alpha \in S$, we have $\operatorname{coz}(\beta) \wedge \operatorname{coz}(\alpha) = \perp$ which implies

$$\operatorname{coz}(\beta) \wedge \operatorname{coz}(\delta) = \bigvee \{\operatorname{coz}(\beta) \wedge \operatorname{coz}(\alpha) | \alpha \in S\} = \perp.$$

Hence, $\operatorname{coz}(\beta\delta) = \perp$, which indicates $\beta\delta = \mathbf{0}$, showing that $\beta \in \operatorname{Ann}(\delta)$. Therefore, we conclude that $\operatorname{Ann}(S) \subseteq \operatorname{Ann}(\delta)$. $\square$

A proper ideal H of a ring R is called a *d-ideal* if $P_x \subseteq H$ for every $x \in H$, where P_x denote the intersection of all minimal prime ideals of R containing x. Recall from [2] that an ideal D of $\mathcal{R}_cL$ is called a *d_c-ideal* if, for every $\alpha \in \mathcal{R}_cL$ and $\beta \in D$, the condition $(\operatorname{coz}(\alpha))^* = (\operatorname{coz}(\beta))^*$ implies that $\alpha \in D$. It is demonstrated in [2] that, for any zero-dimensional frame L, an ideal of $\mathcal{R}_cL$ is a d_c-ideal if and only if it is a d-ideal.

Theorem 4.5. *The following are equivalent for a zero-dimensional frame L.*

(1) L is c-basically disconnected.

(2) Every nonzero countably generated ideal of $\mathcal{R}_cL$ is essential in an ideal generated by an idempotent.

(3) If $S \subseteq \mathcal{R}_cL$ is countable, then an idempotent generates $\operatorname{Ann}(S)$.

(4) $\mathcal{R}_cL$ is a PP-ring.

(5) Every d_c-ideal of $\mathcal{R}_cL$ is generated by idempotents.

(6) For every quotient $h : \beta_0 L \to M$, the ideal $\mathbf{O}^h_c$ is generated by idempotents.

Proof. (1) $\Rightarrow$ (2). Let D be an ideal of $\mathcal{R}_cL$ generated by elements $\{\alpha_n\}_{n\in\mathbb{N}} \subseteq \mathcal{R}_cL$. Define $a = \bigvee \operatorname{coz}(\alpha_n)$, which belongs to $\operatorname{Coz}_c L$ and satisfies $a = \bigvee \operatorname{coz}[D]$. Assume $a = \operatorname{coz}(\alpha)$ for some $\alpha \in \mathcal{R}_cL$. By Lemma 4.4, we have $\operatorname{Ann}(D) = \operatorname{Ann}(\alpha)$. Since L is a c-basically disconnected, this implies $a^{**} \in B(L)$ and thus $a^{**} = \operatorname{coz}(e_{a^{**}})$. Consequently, we find that $(\mathrm{coz}(\alpha))^* = a^* = a^{***} = (\mathrm{coz}(e_{a^{**}}))^*$. By applying Lemma 4.2, we get $\mathrm{Ann}(D) = \mathrm{Ann}(\alpha) = \mathrm{Ann}(e_{a^{**}})$, hence $D \subseteq \mathrm{Ann}(\mathbf{1} - e_{a^{**}}) = \langle e_{a^{**}} \rangle$. Therefore, by Lemma 3.1, the ideal D is essential in the principal ideal $\langle e_{a^{**}} \rangle$, where $e_{a^{**}}$ is an idempotent element of $\mathcal{R}_c L$.

(2) $\Rightarrow$ (3). Let $D = \langle S \rangle$. The current hypothesis shows that D is essential in $\langle e \rangle$ for some an idempotent $e \in \mathcal{R}_c L$. Thus, we have $\mathrm{Ann}(S) = \mathrm{Ann}(D) = \mathrm{Ann}(e) = \langle \mathbf{1} - e \rangle$, establishing that (2) implies (3).

(3) $\Rightarrow$ (4) is trivial.

(4) $\Rightarrow$ (5). Let D be a d_c-ideal of $\mathcal{R}_c L$. The current hypothesis states that for each $\alpha \in D$, there exists an idempotent $e_\alpha \in \mathcal{R}_c L$ such that $\mathrm{Ann}(\alpha) = \langle e_\alpha \rangle = \mathrm{Ann}(\mathbf{1} - e_\alpha)$. By Lemma 4.2, we have $(\mathrm{coz}(\alpha))^* = (\mathrm{coz}(\mathbf{1} - e_\alpha))^*$, which shows that $\mathbf{1} - e_\alpha \in D$ since D is a d_c-ideal. Also $\alpha e_\alpha = \mathbf{0}$ implies that $\alpha = \alpha(\mathbf{1} - e_\alpha)$. Thus, the set $\{\mathbf{1} - \mathbf{e}_\alpha \mid \alpha \in \mathrm{Q}\ \}$ of idempotent elements generates the ideal D.

(5) $\Rightarrow$ (6). It is evident since $\mathbf{O}_c^h$ is a d_c-ideal according to [2, Example 1].

(6) $\Rightarrow$ (1). Let $a \in \mathrm{Coz}_c L$ with $a = \mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}_c L$. We aim to show that $a^* \vee a^{**} = \top$. Consider the open quotient

$$h(I \to I \cap r_0(a^*)) : \beta_0 L \to \downarrow r_0(a^*).$$

The hypothesis implies that the ideal

$$\mathbf{O}_c^h = \{\varphi \in \mathcal{R}_c L \mid h(r_0(\mathrm{coz}(\varphi)^*) = \top_{\downarrow r_0(a^*)}\}$$

is generated by idempotent elements in $\mathcal{R}_c L$. Since

$$h(r_0(a^*)) = r_0(a^*) \cap r_0(a^*) = r_0(a^*) = \top_{\downarrow r_0(a^*)},$$

we conclude that $\alpha \in \mathbf{O}_c^h$. Thus, there exist idempotents $e_1, \cdots, e_m$ in $\mathbf{O}_c^h$ and elements $\alpha_1, \cdots, \alpha_m$ in $\mathcal{R}_c L$ such that $\alpha = \alpha_1 e_1 + \cdots + e_m \alpha_m$. This implies

$$a = \mathrm{coz}(\alpha) \leq \bigvee_{i=1}^{m} \mathrm{coz}(\alpha_i e_i) \leq \bigvee_{i=1}^{m} \mathrm{coz}(e_i) = \mathrm{coz}(e_1 + \cdots + e_n) = \mathrm{coz}(e),$$

where $e = e_1 + \cdots + e_n$. Thus, we have $(\mathrm{coz}(e))^* \leq a^*$. Given that $e_1, \cdots, e_n \in \mathbf{O}_c^h$, we find $e = e_1 + \cdots + e_n \in \mathbf{O}_c^h$ as well, leading to

$h(r_0((\mathrm{coz}(e))^*) = \top_{\downarrow r_0(a^*)}$, which means

$$r_0((\mathrm{coz}(e))^*) \cap r_0(a^*) = r_0(a^*).$$

This shows $r_0(a^*) \leq r_0((\mathrm{coz}(e))^*)$, thus

$$a^* = \bigvee r_0(a^*) \leq \bigvee r_0((\mathrm{coz}(e))^*) = (\mathrm{coz}(e))^*,$$

confirming $a^* = (\mathrm{coz}(e))^*$. Since $\mathrm{coz}(e)$ is a finite join of complemented elements, we have $\mathrm{coz}(e) \in B(L)$, meaning that $\mathrm{coz}(e) = (\mathrm{coz}(e))^{**}$ and $(\mathrm{coz}(e)) \vee (\mathrm{coz}(e))^* = \top$. Therefore,

$$a^* \vee a^{**} = (\mathrm{coz}(e))^* \vee (\mathrm{coz}(e))^{**} = (\mathrm{coz}(e))^* \vee \mathrm{coz}(e) = \top.$$

□

Proposition 15 in [23] states every element of a commutative ring R can be expressed as a product of a non-zero-divisor element and an idempotent if and only if R is a PP-ring. Consequently, based on the equivalence of (1) and (4) in the previous characterizations, the following corollary is immediate.

Corollary 4.6. *A zero-dimensional frame L is c-basically disconnected if and only if every element of $\mathcal{R}_cL$ is a product of an idempotent and a non-zero-divisor.*

5 F_c-Frames

We recall from [6] that a space X is an F_c-space if, for all $p \in \beta_0 X$, $\mathbf{O}_c^P$ is a prime ideal in $C_c(X)$. The same reference provides the following characterizations of these spaces. We remind the reader that the subring of $C(X)$ consisting of functions with finite images is denoted by $C^F(X)$.

Theorem 5.1. *[6] For every zero-dimensional space X, the following statements are equivalent.*

(1) X is an F_c-space.

(2) Every localization $C_c(X)_P$ of $C_c(X)$ at a prime ideal P is a domain.

(3) *For all $f \in C_c(X)$, the ideal $\langle f, |f|\rangle$ is principal.*

(4) *For all $f \in C_c(X)$, $\mathrm{Coz}(f) = X \setminus Z(f)$ is C^F-embedded in X, that is, every function in $C^F(\mathrm{Coz}(f))$ can be extended to a function in $C^F(X)$.*

We now present the following characterizations of F_c-spaces. Recall that an ideal I of a commutative ring is pure if for every $x \in I$ there exists $y \in I$ such that $x = xy$. A ring R classified as a PF-ring if $\mathrm{Ann}(x)$ is pure for all $x \in R$.

Theorem 5.2. *For any zero-dimensional space X, the following statements are equivalent.*

(1) *$C_c(X)$ is a PF-ring.*

(2) *X is an F_c-space.*

(3) *For every $f, g \in C_c(X)$ with $\mathrm{Coz}(f) \cap \mathrm{Coz}(g) = \emptyset$, there exist $\overline{f}, \overline{g} \in C_c(X)$ such that $\mathrm{Coz}(\overline{f}) \cup \mathrm{Coz}(\overline{g}) = X$, and*

$$\mathrm{Coz}(\overline{f}) \cap \mathrm{Coz}(f) = \emptyset = \mathrm{Coz}(\overline{g}) \cap \mathrm{Coz}(g).$$

Proof. (1) $\Rightarrow$ (2). Using Theorem 5.1, we aim to demonstrate that, for any $f \in C_c(X)$, the ideal $\langle f, |f|\rangle$ is principal. For $f \in C_c(X)$, we claim that $\langle f, |f|\rangle = \langle |f|\rangle$. Since $(f - |f|)(f + |f|) = 0$, it follows that $(f - |f|) \in \mathrm{Ann}(f + |f|)$. Therefore, by (1), there exists $g \in \mathrm{Ann}(f + |f|)$ such that

$$f - |f| = g(f - |f|) = gf - g|f|.$$

Since $g \in \mathrm{Ann}(f + |f|)$, we have $g(f + |f|) = 0$, which shows that $gf + g|f| = 0$, leading to $gf = -g|f|$. Consequently, we can write

$$f - |f| = -g|f| - g|f| \Rightarrow f = |f| - 2g|f| = (1 - 2g)|f|.$$

This implies $\langle f, |f|\rangle = \langle |f|\rangle$.

(2) $\Rightarrow$ (3). Let $f, g \in C_c(X)$ with $\mathrm{Coz}(f) \cap \mathrm{Coz}(g) = \emptyset$. Define the function $h : \mathrm{Coz}(f) \cup \mathrm{Coz}(g) \to X$ as follows:

$$h(x) = \begin{cases} 1 & x \in \mathrm{Coz}(f) \\ 0 & x \in \mathrm{Coz}(g). \end{cases}$$

Since $\mathrm{Coz}(f) \cup \mathrm{Coz}(g) = \mathrm{Coz}(f^2+g^2)$ and $h \in C^F(\mathrm{Coz}(f^2+g^2))$, by the current assumption, theorem 5.1 guarantees the existence of $\overline{h} \in C^F(X) \subseteq C_c(X)$ such that $\overline{h}|_{\mathrm{Coz}(f)\cup\mathrm{Coz}(g)} = h$. Define $\overline{g} = \overline{h}$ and $\overline{f} = \overline{h} - 1$. Thus, we have $\mathrm{Coz}(\overline{f}) \cup \mathrm{Coz}(\overline{g}) = X$ and

$$\mathrm{Coz}(\overline{f}) \cap \mathrm{Coz}(f) = \emptyset = \mathrm{Coz}(\overline{g}) \cap \mathrm{Coz}(g).$$

(3) $\Rightarrow$ (1). We need to show that $\mathrm{Ann}(f)$ is pure for any $f \in C_c(X)$. Let $f \in C_c(X)$ and $g \in \mathrm{Ann}(f)$. Since $gf = 0$, it follows that $\mathrm{Coz}(f) \cap \mathrm{Coz}(g) = \emptyset$. Thus, by the current assumption, there exist $\overline{f}, \overline{g} \in C_c(X)$ such that $\mathrm{Coz}(\overline{f}) \cup \mathrm{Coz}(\overline{g}) = X$ and

$$\mathrm{Coz}(\overline{f}) \cap \mathrm{Coz}(f) = \emptyset = \mathrm{Coz}(\overline{g}) \cap \mathrm{Coz}(g).$$

Without loss of generality, we may also assume that $\overline{f}, \overline{g}$ are nonnegative because $\mathrm{Coz}(\overline{f}) = \mathrm{Coz}(|\overline{f}|)$ and $\mathrm{Coz}(\overline{g}) = \mathrm{Coz}(|\overline{g}|)$. Thus,

$$X = \mathrm{coz}(\overline{f}) \cup \mathrm{coz}(\overline{g}) = \mathrm{coz}(\overline{f} + \overline{g}),$$

which shows that $\overline{f} + \overline{g}$ invertible in $C_c(X)$. Define $h = (\overline{f} + \overline{g})^{-1}\overline{f}$, so $h \in C_c(X)$. We claim that $h \in \mathrm{Ann}(f)$ and $g = hg$. Since

$$\emptyset = \mathrm{coz}(\overline{f}) \cap \mathrm{coz}(f) = \mathrm{coz}(\overline{f}f),$$

we have $\overline{f}f = 0$, implying $h \in \mathrm{Ann}(f)$. Similarly, since

$$\emptyset = \mathrm{coz}(\overline{g}) \cap \mathrm{coz}(g) = \mathrm{coz}(\overline{g}g),$$

we have $\overline{g}g = 0$, leading to $(\overline{f} + \overline{g})g = \overline{f}g + \overline{g}g = \overline{f}g$. Thus, we conclude that $g = (\overline{f} + \overline{g})^{-1}\overline{f}g = hg$, proving our claims. Consequently, $\mathrm{Ann}(f)$ is a pure ideal. $\square$

Based on the above characterization of F_c-spaces and the frame-theoretic description of F-frames discussed in the introduction, we present the following key definition for this article.

Definition 5.3. A frame L is called an F_c-frame if whenever $a \wedge b = \bot$ in $\mathrm{Coz}_c L$, then there are $c, d \in \mathrm{Coz}_c L$ such that $a \wedge c = b \wedge d = \bot$ and $c \vee d = \top$.

By combining this definition with the earlier theorem, we conclude that a zero-dimensional space X is a F_c-space if and only if $\mathfrak{O}(X)$ is a F_c-frame. Furthermore, according to the third statement of Proposition 4.1, a c-basically disconnected frame is an F_c-frame.

In this section, we aim to study and characterize F_c-frames. Similarly to the characterization of F-frames (see [11]), we will first present several algebraic and frame-theoretic characterizations of F_c-frames. Our initial characterization necessitates identifying the pure ideals of $\mathcal{R}_cL$. The following lemmas will help identify pure ideals of $\mathcal{R}_cL$.

Lemma 5.4. *Let D be an ideal of $\mathcal{R}_cL$ and $\alpha \in \mathcal{R}_cL$. If*

$$r_0(coz(\alpha)) \prec\!\!\prec \bigvee\{r_0(coz(\beta)) \mid \beta \in D\},$$

then $\alpha \in D$.

Proof. If $r_0(\mathrm{coz}(\alpha)) \prec\!\!\prec \bigvee\{r_0(\mathrm{coz}(\beta)) \mid \alpha \in D\}$, then

$$\big(r_0(\mathrm{coz}(\alpha))\big)^* \vee \bigvee\{r_0(\mathrm{coz}(\beta)) \mid \alpha \in D\} = \top.$$

Consequently, Since $\beta_0 L$ is compact, there exist finitely many elements $\beta_1, \cdots, \beta_n \in D$ such that

$$\big(r_0(\mathrm{coz}(\alpha))\big)^* \vee r_0(\mathrm{coz}(\beta_1)) \vee ... \vee r_0(\mathrm{coz}(\beta_n)) = \top.$$

This lead to

$$r_0(\mathrm{coz}(\alpha)) \prec\!\!\prec r_0(\mathrm{coz}(\beta_1^2)) \vee \cdots \vee r_0(\mathrm{coz}(\beta_n^2)) \leq r_0(\mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2)).$$

Therefore, there exists $b \in r_0(\mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2))$ such that

$$\mathrm{coz}(\alpha) \prec\!\!\prec b \leq \mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2),$$

implying $\mathrm{coz}(\alpha) \prec\!\!\prec \mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2)$ in $\mathrm{Coz}_c L$. In consequence, by Lemma 3.2, α is a multiple of the element $\beta_1^2 + \cdots + \beta_n^2$ in D, which means $\beta \in D$. $\square$

Recall that if $c, d \in \mathrm{Coz}_c L$ and c is rather below d in $\mathrm{Coz}_c L$ if and only if $c \prec\!\!\prec d$ in $\mathrm{Coz}_c L$. So, by [1, Lemma 3.7], the following lemma is immediate.

Lemma 5.5. *An ideal D of $\mathcal{R}_cL$ (or $\mathcal{R}_c^*L$) is pure if and only if for every $\mu \in D$, there exists $\lambda \in D$ such that $coz(\mu) \prec\!\!\prec coz(\lambda)$ in Coz_cL.*

Recall from [2, Theorem 4.2] that for any $I \in \beta_0L$, $\alpha \in \mathbf{O}_c^I$ if and only if there exist $\beta \in \mathcal{R}_cL$ and an idempotent $e \in \mathbf{O}_c^I$ such that $\alpha = e\beta$.

Proposition 5.6. *Pure ideals of $\mathcal{R}_cL$ are exactly the ideals $\mathbf{O}_c^I$ for $I \in \beta_0L$.*

Proof. For any $I \in \beta_0L$, $\mathbf{O}_c^I$ is a pure ideal since, if $\alpha \in \mathbf{O}_c^I$, there exists $\beta \in \mathcal{R}_cL$ and an idempotent $e \in \mathbf{O}_c^I$ such that $\alpha = e\beta$. This implies that $\operatorname{coz}(\alpha) \prec\!\!\prec \operatorname{coz}(e)$ in Coz_cL, thus the result follows from the previous lemma. To proof of the conversely, let D be a pure ideal of $\mathcal{R}_cL$. We claim that $D = \mathbf{O}_c^I$, where $I = \bigvee\{r_0(\operatorname{coz}(\beta)) \mid \beta \in D\}$. Assume $\delta \in D$. By Lemma 5.5, there exists $\lambda \in D$ such that $\operatorname{coz}(\delta) \prec\!\!\prec \operatorname{coz}(\lambda)$ in Coz_cL. Thus, there are $a, b \in \operatorname{Coz}_cL$ with $a \wedge \operatorname{coz}(\delta) = \perp$ and $b \vee \operatorname{coz}(\lambda) = \top$, leading to $r_0(a) \wedge r_0(\operatorname{coz}(\delta)) = \perp_{\beta_0L}$ and $r_0(b) \vee r_0(\operatorname{coz}(\lambda)) = \top_{\beta_0L}$. This shows that $r_0(\operatorname{coz}(\delta)) \prec\!\!\prec r_0(\operatorname{coz}(\lambda)) \leq I$, thus $r_0(\operatorname{coz}(\delta)) \prec\!\!\prec I$. Therefore, $\delta \in \mathbf{O}_c^I$, implying $D \subseteq \mathbf{O}_c^I$. To establish the reverse inclusion, let $\alpha \in \mathbf{O}_c^I$. Then

$$r_0(\operatorname{coz}(\alpha)) \prec\!\!\prec \bigvee_{\beta \in D} r_0(\operatorname{coz}(\beta)),$$

and by Lemma 5.4, we have $\alpha \in D$. This proves the reverse inclusion, completing the proof. $\square$

Theorem 5.7. *The following statements are equivalent for a zero-dimensional frame L.*

(1) L is an F_c-frame.

(2) For each $\varphi \in \mathcal{R}_cL$, $\operatorname{Ann}(\varphi) = \mathbf{O}_c^{r_0((coz(\varphi))^*)}$.

(3) $\mathcal{R}_cL$ is a PF-ring.

(4) For every $c \in Coz_c(L)$, $\mathbf{M}_c^{r_0(c^*)} = \mathbf{O}_c^{r_0(c^*)}$.

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in \mathbf{O}_c^{r_0((\operatorname{coz}(\varphi))^*)}$. Then there exists $\beta \in \mathcal{R}_cL$ such that $\operatorname{coz}(\alpha) \prec\!\!\prec \operatorname{coz}(\beta) \in r_0((\operatorname{coz}(\varphi))^*)$, which implies

$\mathrm{coz}(\alpha) \lll (\mathrm{coz}(\varphi))^*$, implying $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\varphi) = \bot$, which leads to $\alpha\varphi = \mathbf{0}$. Thus, $\alpha \in \mathrm{Ann}(\varphi)$, confirming that $\mathbf{O}_c^{r_0((\mathrm{coz}(\varphi))^*)} \subseteq \mathrm{Ann}(\varphi)$. Now, let $\alpha \in \mathrm{Ann}(\varphi)$; then $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\varphi) = \bot$. Since L is an F_c-frame, there exist $a, b \in \mathrm{Coz}_c L$ such that $a \vee b = \top$ and

$$a \wedge \mathrm{coz}(\alpha) = \bot = b \wedge \mathrm{coz}(\varphi).$$

Because r_0 preserves meets and finite joints in $\mathrm{Coz}_c L$, we have $r_0(a) \vee r_0(b) = \top_{\beta_0 L}$ and

$$r_0(a) \wedge r_0(\mathrm{coz}(\alpha)) = \bot_{\beta_0 L} = r_0(b) \wedge r_0(\mathrm{coz}(\varphi)).$$

It follows that $r_0(b) \leq (r_0(\mathrm{coz}(\varphi))^*$, leading to

$$r_0(a) \vee (r_0(\mathrm{coz}(\varphi))^* \geq r_0(a) \vee r_0(b) = \top_{\beta_0 L}.$$

Thus, $r_0(a)$ separate $r_0(\mathrm{coz}(\alpha))$ and $(r_0(\mathrm{coz}(\varphi))^*$, meaning that $r_0(\mathrm{coz}(\alpha)) \prec (r_0(\mathrm{coz}(\varphi))^*$. Since r_0 preserves pseudocomplements and $\prec = \lll$ in $\beta_0 L$, we conclude that $r_0(\mathrm{coz}(\alpha)) \lll (r_0(\mathrm{coz}(\varphi)^*)$. Therefore, $\alpha \in \mathbf{O}_c^{r_0(\mathrm{coz}(\varphi))^*}$, which completes the reverse inclusion proof.

(2) $\Rightarrow$ (3). This follows immediately from Proposition 5.6.

(3) $\Rightarrow$ (4). Let $c \in \mathrm{Coz}_c(L)$ such that $c = \mathrm{coz}(\alpha)$ for some $\alpha \in \mathcal{R}_c L$. We aim to show that $\mathbf{M}_c^{r_0(c^*)} \subseteq \mathbf{O}_c^{r_0(c^*)}$, given that $\mathbf{O}_c^{r_0(c^*)} \subseteq \mathbf{M}_c^{r_0(c^*)}$. Let $\varphi \in \mathbf{M}_c^{r_0(c^*)}$. Then $r_0(\mathrm{coz}(\varphi)) \leq r_0(c^*)$, which implies

$$\mathrm{coz}(\varphi) = \bigvee r_0(\mathrm{coz}(\varphi)) \leq \bigvee r_0(c^*) = c^*.$$

Thus, we have

$$\bot = \mathrm{coz}(\varphi) \wedge c = \mathrm{coz}(\varphi) \wedge \mathrm{coz}(\alpha),$$

leading to $\alpha\varphi = \mathbf{0}$. This means $\varphi \in \mathrm{Ann}(\alpha)$, and by current hypothesis, there exists $\delta \in \mathcal{R}_c L$ such that $\alpha\delta = \mathbf{0}$ and $\varphi = \varphi\delta$. Consequently,

$$\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\delta) = \bot = \mathrm{coz}(\varphi) \wedge \mathrm{coz}(\mathbf{1} - \delta).$$

Since $\mathrm{coz}(\delta) \vee \mathrm{coz}(\mathbf{1} - \delta) = \top$, by normality of $\mathrm{Coz}_c L$, we can find $u, v \in \mathrm{Coz}_c L$ such that $u \wedge v = \bot$ and

$$u \vee \mathrm{coz}(\delta) = \top = v \vee \mathrm{coz}(\mathbf{1} - \delta).$$

Similar to the discussion in (1) $\Rightarrow$ (2), we conclude that

$$r_0(\mathrm{coz}(\varphi)) \prec\!\!\prec r_0((\mathrm{coz}(\alpha))^*) = r_0(c^*),$$

which implies $\varphi \in \mathbf{O}_c^{r_0(c^*)}$. Therefore, we have $\mathbf{M}_c^{r_0(c^*)} \subseteq \mathbf{O}_c^{r_0(c^*)}$.

(4) $\Rightarrow$ (1). Assume that $a \wedge b = \perp$ for some $a, b \in \mathrm{Coz}_c L$. Let $\alpha \in \mathcal{R}_c L$ such that $\mathrm{coz}(\alpha) = a$. Since $a \leq b^*$, we have $r_0(\mathrm{coz}(\alpha)) \leq r_0(b^*)$, leading to $\alpha \in \mathbf{M}_c^{r_0(b^*)}$. By hypothesis, $\alpha \in \mathbf{O}_c^{r_0(b^*)}$, implying $a \prec\!\!\prec u \leq b^*$ for some $u \in r_0(b^*) \subseteq B(L)$. So, we have $u \vee u^* = \top$, $a \wedge u^* \leq u \wedge u^* = \perp$, and $b \wedge u \leq b \wedge b^* = \perp$. This demonstrates that L is an F_c-frame, completing the proof. $\square$

We aim to characterize F_c-frames via $\mathbf{O}_c^I$-ideals by showing that $\mathcal{R}_c L$ is a PF-ring if and only if $\mathcal{R}_c^* L$ is also a PF-ring.

Proposition 5.8. *For any frame L, $\mathcal{R}_c^* L$ is a PF-ring if and only if $\mathcal{R}_c L$ is a PF-ring.*

Proof. ($\Rightarrow$). Assume $\alpha \in \mathcal{R}_c L$ and $\beta \in \mathrm{Ann}_{\mathcal{R}_c L}(\alpha)$. Then $\alpha\beta = \mathbf{0}$, leading to $\frac{\beta^2}{\mathbf{1}+\beta^2}\frac{\alpha^2}{\mathbf{1}+\alpha^2} = \mathbf{0}$. Since both $\frac{\beta^2}{\mathbf{1}+\beta^2}$ and $\frac{\alpha^2}{\mathbf{1}+\alpha^2}$ are in $\mathcal{R}_c^* L$, it follows that $\frac{\beta^2}{\mathbf{1}+\beta^2} \in \mathrm{Ann}_{\mathcal{R}_c^* L}(\frac{\alpha^2}{\mathbf{1}+\alpha^2})$. By the hypothesis, $\mathrm{Ann}_{\mathcal{R}_c^* L}(\frac{\alpha^2}{\mathbf{1}+\alpha^2})$ is pure, so by Lemma 5.5, there exists $\delta \in \mathrm{Ann}_{\mathcal{R}_c^* L}\left(\frac{\alpha^2}{\mathbf{1}+\alpha^2}\right)$ such that $\mathrm{coz}\left(\frac{\beta^2}{\mathbf{1}+\beta^2}\right) \prec\!\!\prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c L$. Since

$$\mathrm{coz}\left(\frac{\beta^2}{\mathbf{1}+\beta^2}\right) = \mathrm{coz}(\beta^2) \wedge \mathrm{coz}((\mathbf{1}+\beta^2)^{-1}) = \mathrm{coz}(\beta) \wedge \top = \mathrm{coz}(\beta),$$

we conclude that $\mathrm{coz}(\beta) \preceq \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c L$. Additionally, since $\delta \in \mathrm{Ann}_{\mathcal{R}_c^* L}\left(\frac{\alpha^2}{\mathbf{1}+\alpha^2}\right)$, it follows that $\delta\alpha^2 = \mathbf{0}$, which implies $(\delta\alpha)^2 = \mathbf{0}$, leading to $\delta\alpha = \mathbf{0}$ because $\mathcal{R}_c L$ is reduced. Therefore, $\mathrm{Ann}_{\mathcal{R}_c L}(\alpha)$ is pure by Lemma 5.5.

($\Leftarrow$). Let $\alpha \in \mathcal{R}_c^* L$ and $\beta \in \mathrm{Ann}_{\mathcal{R}_c^* L}(\alpha)$. Then, $\beta \in \mathrm{Ann}_{\mathcal{R}_c L}(\alpha)$. Since $\mathrm{Ann}_{\mathcal{R}_c L}(\alpha)$ is pure by the hypothesis, by Lemma 5.5, there exists $\delta \in \mathrm{Ann}_{\mathcal{R}_c L}(\alpha)$ such that $\mathrm{coz}(\beta) \prec\!\!\prec \mathrm{coz}(\delta)$ in $\mathrm{Coz}_c L$. Thus, $\frac{\delta^2}{\mathbf{1}+\delta^2}$ belongs to $\mathrm{Ann}_{\mathcal{R}_c^* L}(\alpha)$ and satisfies $\mathrm{coz}(\beta) \prec\!\!\prec \mathrm{coz}\left(\frac{\delta^2}{\mathbf{1}+\delta^2}\right)$ in $\mathrm{Coz}_c L$. Consequently, $\mathrm{Ann}_{\mathcal{R}_c^* L}(\alpha)$ is pure by Lemma 5.5. $\square$

Before starting the next characterization, we review some facts from [2, 6, 20, 25] and present a definition.

1. For any maximal $I \in \beta_0 L$ and $\varphi \in \mathcal{R}_c L$, $\varphi \in \mathbf{O}_c^I$ if and only if $\alpha\varphi = \mathbf{0}$ for some $\alpha \notin \mathbf{M}_c^I$ (see [2]).
2. Every compact regular frame is spatial (see [20], III Proposition III 1.10).
3. For any frame L, we have $\mathcal{R}_c^* L \cong \mathcal{R}_c(\beta L)$ (see [25, Proposition 3.2]). Consequently, by Proposition 7.1 in the same reference, it follows that $\mathcal{R}_c^* L \cong \mathcal{R}_c M$ for some compact zero-dimensional frame M.
4. It is straightforward to demonstrate that $r_0 : L \to \beta_0 L$ is an isomorphism when L is a compact frame (see Lemma 4.1 of [2]).
5. If X is a compact zero-dimensional space, then the ideals $\mathbf{O}_c^q$, for $q \in \beta_0 X$, are exactly(up to isomorphism) the ideals $\mathbf{O}_{cp} = \{f \in C_c(X) \mid p \in \mathrm{int}Z(f)\}$ (see [6]).
6. For any open sets U and V in a space X, $U \prec V$ in $\mathfrak{O}(X)$ if and only if $\mathrm{cl}_X U \subseteq V$ (see [20]).

Definition 5.9. An ideal I of $B(L)$ is called c-coz-prime if for any cozero elements a and b in $\mathrm{Coz}_c L$ such that $a \wedge b \leq c \in I$, it holds that $a \leq u$ or $b \leq u$ for some $u \in I$.

Theorem 5.10. *The following statements are equivalent for a frame L.*

1. *L is an F_c-frame.*
2. *All maximal elements of $\beta_0 L$ are c-coz-prime.*
3. *For each maximal $I \in \beta_0 L$, the ideal $\mathbf{O}_c^I$ is prime.*

Proof. (1) $\Rightarrow$ (2). Let I be a maximal element of $\beta_0 L$ and let $a, b \in \mathrm{Coz}_c L$ such that $a \wedge b \leq c \in I$. Assume $a = \mathrm{coz}(\alpha)$ and $b = \mathrm{coz}(\beta)$ for some $\alpha, \beta \in \mathcal{R}_c L$. Then, since

$$\mathrm{coz}(\alpha\beta) = \mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) \leq c \in I,$$

it follows that $\alpha\beta \in \mathbf{O}_c^I$. We claim that $\alpha \in \mathbf{O}_c^I$ or $\beta \in \mathbf{O}_c^I$, which implies that $a = \mathrm{coz}(\alpha) \prec\!\!\prec u \in I$ or $b = \mathrm{coz}(\beta) \prec\!\!\prec v \in I$. To prove this claim, let $\beta \notin \mathbf{O}_c^I$. Since $\alpha\beta \in \mathbf{O}_c^I$, there exists $\delta \notin \mathbf{M}_c^I$ such that $\alpha\beta\delta = \mathbf{0}$. This lead to $\operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta\delta) = \bot$. According to the current hypothesis, there exist $\varphi, \gamma \in \mathcal{R}_c L$ such that $\operatorname{coz}(\varphi) \vee \operatorname{coz}(\gamma) = \top$ and

$$\operatorname{coz}(\alpha) \wedge \operatorname{coz}(\varphi) = \bot = \operatorname{coz}(\gamma) \wedge \operatorname{coz}(\beta\delta),$$

that is, $\alpha\varphi = \gamma\beta\delta = \mathbf{0}$. Furthermore, since $\beta \notin \mathbf{O}_c^I$ and $\delta \notin \mathbf{M}_c^I$, it follows that $\gamma \in \mathbf{M}_c^I$. Consequently, $\varphi \notin \mathbf{M}_c^I$ because

$$\top = \operatorname{coz}(\varphi^2 + \gamma^2) = \operatorname{coz}(\varphi) \vee \operatorname{coz}(\gamma).$$

Thus, we have $\alpha\varphi = \mathbf{0}$ with $\varphi \notin \mathbf{M}_c^I$, leading to the conclusion that $\alpha \in \mathbf{O}_c^I$, thereby proving the claim.

(2) $\Rightarrow$ (3). Let I be a maximal element of $\beta_0 L$, and let $\alpha, \beta \in \mathcal{R}_c L$ such that $\alpha\beta \in \mathbf{O}_c^I$. Then, we have $\operatorname{coz}(\alpha) \wedge \operatorname{coz}(\beta) = \operatorname{coz}(\alpha\beta) \leq c$ for some $c \in I$. By the hypothesis, there exists $u \in I$ such that $\operatorname{coz}(\alpha) \leq u$ or $\operatorname{coz}(\beta) \leq u$. Thus, it follows that $\alpha \in \mathbf{O}_c^I$ or $\beta \in \mathbf{O}_c^I$. Consequently, $\mathbf{O}_c^I$ is a prime ideal.

(3) $\Rightarrow$ (1). According to the previous proposition and the third part of the above discussion, we can assume L is compact zero-dimensional, making it spatial. Thus, let $L \cong \mathfrak{O}(X)$ for some compact zero-dimensional space X. Define the ring isomorphism $\Psi : C(X) \to \mathcal{R}(\mathfrak{O}(X))$ by mapping f to $\tilde{f}$, where $\tilde{f} \in \mathcal{R}(\mathfrak{O}(X))$ is given by

$$\tilde{f}(p,q) = f^{-1}\big(\{x \in \mathbb{R} \mid p < x < q\}\big).$$

Since X is compact, the map $r_0 : \mathfrak{O}(X) \to \beta_0(\mathfrak{O}(X))$ is an isomorphism, and the ideals $\mathbf{O}_c^q$, for $q \in \beta_0 X$, correspond exactly to the ideals $\mathbf{O}_{cp}$ for $p \in X$. Let $p \in X$. The set $X \setminus \{p\}$ is a maximal element of $\mathfrak{O}(X)$, making $r_0(X \setminus \{p\})$ is a maximal element of $\beta_0(\mathfrak{O}(X))$. Now, we have

$$\begin{aligned}
f \in \mathbf{O}_{cp} &\Leftrightarrow p \in \operatorname{int} Z(f) \\
&\Leftrightarrow p \in X \setminus \operatorname{cl}\big(f^{-1}\big((-,0) \vee (0,-)\big)\big) \\
&\Leftrightarrow \operatorname{cl}\big(f^{-1}\big((-,0) \vee (0,-)\big)\big) \subseteq X \setminus \{p\}) \\
&\Leftrightarrow f^{-1}\big((-,0) \vee (0,-)\big) \prec X \setminus \{p\} \\
&\Leftrightarrow \operatorname{coz}(\tilde{f}) \prec X \setminus \{p\} \\
&\Leftrightarrow r_0(\operatorname{coz}(\tilde{f})) \prec r_0(X \setminus \{p\}), \ \text{ since isomorphisms preserve } \prec \\
&\Leftrightarrow r_0(\operatorname{coz}(\tilde{f})) \prec\!\!\prec r_0(X \setminus \{p\}), \ \text{since } \prec = \prec\!\!\prec \ \text{ in } \beta_0 L \\
&\Leftrightarrow \tilde{f} \in \mathbf{O}_c^{r_0(X \setminus \{p\})}.
\end{aligned}$$

This implies that $\Psi(\mathbf{O}_{cp}) = \mathbf{O}_c^{r_0(X\setminus\{p\})}$. Since Ψ is a ring isomorphism, the current hypothesis shows that each ideal $\mathbf{O}_{cp}$ of $C_c(X)$ is prime. Consequently, by Theorem 4.1, X is an F_c-space, and so $L = \mathfrak{O}(X)$ is an F_c-frame. $\square$

Before presenting the following corollary, we recall from Lemma 5.2 in [2] if P is a prime ideal of $\mathcal{R}_cL$, then there exists a unique maximal element I of $\beta_0 L$ such that $\mathbf{O}_c^I \subseteq P \subseteq \mathbf{M}_c^I$.

Corollary 5.11. *The following statements are equivalent for a zero-dimensional frame L.*

1. *L is an F_c-frame.*
2. *The minimal prime ideals of $\mathcal{R}_cL$ are precisely the ideals $\mathbf{O}_c^I$ for I maximal in $\beta_0 L$.*
3. *Every maximal ideal of $\mathcal{R}_cL$ contains a unique minimal prime ideal.*

Proof. (1) $\Rightarrow$ (2). Let L be an F_c-frame. By Theorem 5.10, for each maximal element I in $\beta_0 L$, the ideal $\mathbf{O}_c^I$ is prime. Since every element of $\mathbf{O}_c^I$ is annihilated by an element outside $\mathbf{O}_c^I$, it follows that $\mathbf{O}_c^I$ is a minimal prime ideal. Now, consider a minimal prime ideal P of $\mathcal{R}_cL$. Then there exists a maximal element $I \in \beta_0 L$ such that $\mathbf{O}_c^I \subseteq P \subseteq \mathbf{M}_c^I$. Therefore, the minimality of P implies $P = \mathbf{O}_c^I$. This proves (2).

(2) $\Rightarrow$ (3). Given the current hypothesis that $\mathbf{O}_c^I$ is the unique minimal prime ideal contained in $\mathbf{M}_c^I$ for each maximal I in $\beta_0 L$, this implication is immediate.

(3) $\Rightarrow$ (1). By Theorem 5.10, We need to show that, for any maximal element I in $\beta_0 L$, the ideal $\mathbf{O}_c^I$ is prime. Consider a maximal element I in $\beta_0 L$. To show that $\mathbf{O}_c^I$ is prime, it suffices to prove that its radical $\sqrt{\mathbf{O}_c^I} = \{\alpha \in \mathcal{R}_cL \mid \alpha^n \in \mathbf{O}_c^I \text{ for some } n \in \mathbb{N}\}$ is prime since it is clear that $\mathbf{O}_c^I = \sqrt{\mathbf{O}_c^I}$. Since $\sqrt{\mathbf{O}_c^I}$ is the intersection of all prime ideals containing $\mathbf{O}_c^I$ and every such prime ideal contains a minimal prime ideal that also contains $\mathbf{O}_c^I$, we conclude that $\sqrt{\mathbf{O}_c^I}$ is the intersection of all minimal prime ideals containing $\mathbf{O}_c^I$. Furthermore, any minimal prime ideals containing $\mathbf{O}_c^I$ is contained in the maximal ideal $\mathbf{M}_c^I$. Given our hypothesis that there is only one minimal prime ideal contained in $\mathbf{M}_c^I$,

it follows that $\sqrt{\mathbf{O}_c^I}$ is a minimal prime ideal, thus confirming that $\sqrt{\mathbf{O}_c^I}$ is prime. $\square$

For each prime ideal P in $\mathcal{R}_cL$, define

$$O_P = \{\alpha \in \mathcal{R}_cL \mid \alpha\beta = \mathbf{0} \text{ for some } \beta \notin P\}.$$

Thus, by Part (1) preceding Definition 5.9, we have $\mathbf{O}_c^I = O_{\mathbf{M}_c^I}$ for any maximal $I \in \beta_0 L$. Thus, we directly obtain the following result from Lemma 2.1 in [5].

Lemma 5.12. *Let L be a zero-dimensional. Then, for any maximal element I of $\beta_0 L$,*

$$\frac{\mathcal{R}_cL}{\mathbf{O}_c^I} \cong \mathcal{R}_cL_{\mathbf{M}_c^I}.$$

The following result is immediate because of [5, Theorem 2.2, Corollary 2.4], Theorem 5.10, and the preceding lemma.

Corollary 5.13. *The following statements are equivalent for a zero-dimensional frame L.*

1. *L is an F_c-frame.*
2. *$\mathcal{R}_cL$ is locally a domain, that is, the localization $\mathcal{R}_cL_P$ is domain for ever prime ideal P of $\mathcal{R}_cL$.*
3. *The localization $\mathcal{R}_cL_M$ is domain for ever maximal ideal M of $\mathcal{R}_cL$.*
4. *Every prime ideal of $\mathcal{R}_cL$ contains a unique minimal prime ideal.*
5. *$\mathcal{R}_cL$ is locally uniform, that is, for any prime ideal P of $\mathcal{R}_cL$, any two nonzero ideal in $\mathcal{R}_cL_P$ intersect non-trivially.*

The paragraph following definition 5.3 confirms that c-basically disconnected frames are F_c-frames. We will now explore the situation in which the converse is also valid. We begin with the following definition.

Definition 5.14. A frame L is c-cozero-complemented if for each $a \in \mathrm{Coz}_cL$, there exists $b \in \mathrm{Coz}_cL$ such that $a \wedge b = \bot$ and $(a \vee b)^* = \bot$.

Theorem 5.15. *A frame L is c-basically disconnected if and only if it is a c-cozero complemented F_c-frame.*

Proof. Let L be a c-basically disconnected frame, which implies that $a^* \vee a^{**} = \top$ for each $a \in \mathrm{Coz}_c L$. Consequently, the following holds:

$$(a \vee a^*)^* = a^* \wedge a^{**} = a^{***} \wedge a^{**} = (a^{**} \vee a^*)^* = \top^* = \bot.$$

This shows that L is c-cozero complemented since $a \wedge a^* = \bot$. Thus, the proof of the "if" part is complete.

Conversely, let L be a c-cozero complemented F_c-frame. To show that L is a c-basically disconnected frame, consider $a \in \mathrm{Coz}_c L$. Since L is c-cozero complemented, there exists $b \in \mathrm{Coz}_c L$ such that $a \wedge b = \bot$ and $(a \vee b)^* = \bot$. The latter implies that $a^* \wedge b^* = \bot$, so $a^{**} \geq b^*$. Since L is an F_c-frame and $a \wedge b = \bot$, we can find $c, d \in \mathrm{Coz}_c L$ such that $c \vee d = \top$ and $c \wedge a = \bot = b \wedge d$. Consequently, we have $a^* \geq c$ and $b^* \geq d$. Therefore,

$$a^{**} \vee a^* \geq b^* \vee a^* \geq c \vee d = \top,$$

establishing that L is a c-basically disconnected frame. □

As usual, let $\mathrm{Min}(\mathcal{R}_c L)$ denote the space of minimal prime ideals of $\mathcal{R}_c L$ with the hull-kernel topology. In the proof of the next proposition, we use the fact that for a reduce ring R, the space of minimal prime ideals $\mathrm{Min}(R)$ is compact if and only if for any $x \in R$, there exists a finitely generated ideal $J \subseteq \mathrm{Ann}(x)$ such that $\mathrm{Ann}(J, x) = 0$ (see Theorem 4.3 in [19]).

Proposition 5.16. *A zero-dimensional frame L is c-cozero-complemented if and only if the space $\mathrm{Min}(\mathcal{R}_c L)$ is compact.*

Proof. Let L is c-cozero-complemented and $\alpha \in \mathcal{R}_c L$. We need to show the existence of a finitely generated ideal $J \subseteq \mathrm{Ann}(\alpha)$ such that $\mathrm{Ann}(J, \alpha) = \{\mathbf{0}\}$. Given that L is c-cozero-complemented, there exists $\beta \in \mathcal{R}_c L$ such that $\mathrm{coz}(\alpha) \wedge \mathrm{coz}(\beta) = \bot$ and $\mathrm{coz}(\alpha) \vee \mathrm{coz}(\beta)$ is dense. This implies that $\mathrm{coz}(\alpha^2 + \beta^2)$ is dense, showing that $\alpha^2 + \beta^2$ is not a zero-divisor by Corollary 4.3. Let $J = \langle \beta \rangle$. Since $\mathrm{coz}(\alpha\beta) = \bot$, we have $J \subseteq \mathrm{Ann}(\alpha)$. Furthermore, $\mathrm{Ann}(J, \alpha) = \{\mathbf{0}\}$ because $\langle J, \alpha \rangle$ includes $\alpha^2 + \beta^2$, which is not a zero-divisor.

Conversely, suppose that $\mathrm{Min}(\mathcal{R}_cL)$ is compact, and let $a \in \mathrm{Coz}_c L$. Choose $\alpha \in \mathcal{R}_cL$ such that $a = \mathrm{coz}(\alpha)$, leading to $a = \mathrm{coz}(\alpha^2)$. Thus, there exists a finitely generated ideal $J = \langle \beta_1, \beta_2, \ldots, \beta_n \rangle$ with $J \subseteq \mathrm{Ann}(\alpha)$ and $\mathrm{Ann}\langle J, \alpha \rangle = \{\mathbf{0}\}$. Define $b = \mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2)$. Given that $\alpha\beta_i = \mathbf{0}$ for each $i = 1, \ldots, n$, we have

$$a \wedge b = \mathrm{coz}(\alpha^2) \wedge \mathrm{coz}(\beta_1^2 + \cdots + \beta_n^2) = \mathrm{coz}(\alpha^2(\beta_1^2 + \cdots + \beta_n^2)) = \mathrm{coz}(\mathbf{0}) = \bot.$$

Furthermore, $a \vee b = \mathrm{coz}(\alpha^2 + \beta_1^2 + \cdots + \beta_n^2)$. To conclude, we need to prove that $\alpha^2 + \beta_1^2 + \cdots + \beta_n^2$ is not a zero-divisor. Assume there exists $\delta \in \mathcal{R}_cL$ such that $\delta(\alpha^2 + \beta_1^2 + \cdots + \beta_n^2) = \mathbf{0}$. This implies that $\delta \in \mathrm{Ann}\langle J, \alpha \rangle$, leading to $\delta = \mathbf{0}$. Therefore, $\alpha^2 + \beta_1^2 + \cdots + \beta_n^2$ is not a zero-divisor, completing the proof. $\square$

Theorem 5.15 and Proposition 5.16 together lead to the following corollary.

Corollary 5.17. *A zero-dimensional frame L is c-basically disconnected if and only if it is an F_c-frame and $Min(\mathcal{R}_cL)$ is compact.*

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