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Exploring the Presence of Positive Solutions in a Fractional Equation Featuring Dual Integral and Derivative Boundary Conditions under the p-Laplacian Operator

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Abstract. In this article, we investigate the existence of a positive solution for a fractional differential equation with dual derivative and integral boundary conditions and the uniqueness of this solution. In this regard, we use fixed point theorems in collective operators of Banach spaces. Using the results of the main theorem, in addition to finding a positive solution, we can use an iterative scheme to approximate this differential equation and show this issue with an example.

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1 Introduction

Today, the use of fractional differential equations ($\mathbb{FDE}$s) in sciences such as engineering, chemistry, mechanics, biology, economics, etc. is not hidden from anyone's eyes [3, 5, 7, 8, 10–14, 16, 19–24, 26, 27, 29–31]. Recently, investigating the existence of positive solutions of fractional differential equations with multiple conditions has become very important and has attracted significant comments [3, 7, 12, 21, 27, 30]. Also, investigating the uniqueness of these solutions for nonlinear $\mathbb{FDE}$s has been significant for authors and researchers and has been studied [5, 13, 24, 26, 29].

Zhao *et al.* were able to check the existence of positive solutions for $\mathbb{FDE}$s with integral boundary condition for continuous operators by using the Guo–Krasnosel'skii's fixed point theorem. They stated that this class of equations with these conditions can satisfy sub-linear or super-linear conditions [31]. But they could not express repeated plans to approximate the existence of a suitable solution. After that, Sun *et al.*, in [22], built a completely continuous operator and using the iteration method, they investigated the following $\mathbb{FDE}$ with an integral boundary condition

$$\begin{cases} \mathcal{D}_{0^+}^{\varsigma} \mathrm{w}(\varrho) + \hbar(\varrho) + V(\varrho, \mathrm{w}(\varrho)) = 0, & \varrho \in \mathsf{Y} := (0,1), \\ \mathrm{w}(0) = \mathrm{w}'(0) = 0, \quad \mathrm{w}(1) = \int_{\overline{\mathsf{Y}}:=[0,1]} \mathrm{z}(\upsilon)\,\mathrm{d}\upsilon, \end{cases}$$

in which $\mathcal{D}_{0^+}^{\varsigma}$ is the standard Riemann–Liouville fractional derivative of order $2 < \varsigma \leq 3$. The researchers presented the existence of a positive solution to the problem and were able to express a sequence by iteration with an approximation for an initial value. But the uniqueness of the answer is still not provided [22, 31]. Roomi *et al.* obtained some existence results of solutions for the following fractional differential inclusions involving Caputo type fractional derivative, with boundary conditions,

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}^{\varsigma_1} \mathrm{w}(\varrho) \in V\big(\varrho, \mathrm{w}(\varrho), {}^{\mathrm{C}}\mathcal{D}^{\varsigma_2} \mathrm{w}(\varrho)\big), & \varsigma_i, \varrho \in \mathsf{Y}, \\ \mathrm{w}(1) = \mathrm{w}'(1) = \int_0^{\kappa} \mathrm{z}(\upsilon)\,\mathrm{d}\upsilon, \quad \mathrm{w}(0) = 0, & \kappa \in \mathsf{Y}, \end{cases}$$

based on the fixed point theorems, where $V : [0,1] \times \mathbb{R}^2 \to 2^{\mathbb{R}}$ denotes a compact valued multifunction [17]. Afshari *et al.* examined the existence of solutions of the initial boundary value problem by utilizing the new generalized contraction of the form

$$\begin{cases} {}^{\mathrm{ABC}}_{\ \ a}\mathcal{D}^{\varsigma_1}\mathrm{w}(\varrho) = V_1\big(\varrho, \mathrm{w}(\varrho)\big) + \int_0^{\kappa} V_2\big(\varrho, \upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon, = 0, \\ \mathrm{w}(a) = \mathrm{w}_a, \quad V_1\big(\varrho, \mathrm{w}(\varrho)\big)\Big|_a = 0, \end{cases}$$

where $\varsigma \in \mathsf{Y} \cup \{1\}$, $V_1 \in C\big([a,b], \times\mathbb{R}\big)$ and $V_2 \in C\big([a,b], \times\mathbb{R}^2\big)$ [1, 2, 4, 6, 15]. Further, they investigated the existence of solutions for a class of χ-Caputo $\mathbb{FDE}$s and an inclusion problem of the form,

$$\begin{cases} {}^{\mathrm{C}}\mathcal{D}_0^{\varsigma,\chi}\mathrm{w}(\varrho) = V_1\big(\varrho, \mathrm{w}(\varrho)\big), \\ {}^{\mathrm{C}}\mathcal{D}_0^{\varsigma,\chi}\mathrm{w}(\varrho) \in V_2\big(\varrho, \mathrm{w}(\varrho)\big), \end{cases}$$

for $\varrho \in [a,b]$, $1 < \varsigma < 2$, equipped with nonlocal χ-integral boundary conditions

$$\mathrm{w}(a) = 0, \qquad \mathrm{w}(b) = \sum_{i=1}^{n} \beta_i \,{}^{\mathrm{RL}}\mathcal{D}_0^{\varsigma_i,\chi}\mathrm{w}(\sigma_i),$$

for $\varsigma_i > 0$, $\beta_i \in \mathbb{R}$, $a < \sigma_i < b$, $V_1 : [a,b] \times \mathbb{R} \to \mathbb{R}$, $V_2 : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued compact map [18].

Now, motivated the idea from [22], in this article, we consider the following system of $\mathbb{FDE}$ with dual derivative and integral boundary conditions with p-Laplacian operator,

$$\begin{cases} \Phi_p\left(\mathcal{D}_{0^+}^{\varsigma_1}\mathrm{w}_1(\varrho)\right) + V_1\big(\varrho, \mathrm{w}_1(\varrho)\big) = 0, \\ \Phi_p\left(\mathcal{D}_{0^+}^{\varsigma_2}\mathrm{w}_2(\varrho)\right) + V_2\big(\varrho, \mathrm{w}_2(\varrho)\big) = 0, \\ \quad\vdots \\ \Phi_p\left(\mathcal{D}_{0^+}^{\varsigma_k}\mathrm{w}_\mathrm{k}(\varrho)\right) + V_k\big(\varrho, \mathrm{w}_\mathrm{k}(\varrho)\big) = 0, \end{cases} \tag{1}$$

for $\varrho \in \mathsf{Y}, 2 < \varsigma_i \leq 3$ under,

$$\begin{aligned} &\mathrm{w}_\mathrm{i}(0) = 0, \quad {\mathrm{w}_\mathrm{i}}'(1) = 0 \\ &\mathrm{w}_\mathrm{i}(1) = \mathcal{I}_{0^+}^{\alpha_i} Z_i\big(\upsilon, \mathrm{w}_\mathrm{i}(\upsilon)\big)\Big|_{p_i=\eta} = \int_0^{\eta} \tfrac{(\eta-\upsilon)^{\alpha_i-1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \mathrm{w}_\mathrm{i}(\upsilon)\big)\,\mathrm{d}\upsilon, \end{aligned}$$

where $\mathcal{D}_{0^+}^{\varsigma_i}$ is the standard Riemann-Liouville fractional derivative of order $2 < \varsigma_i \leq 3$ and $\alpha_i \in \mathsf{Y}$, $\eta \in \mathsf{Y}$ and $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $\Phi_q = (\Phi_p)^{-1}$, $p + q = pq$.
In this paper, we are interested in presenting several alternative solutions to the main results of these papers [22, 31]. In this regard, we use two fixed point theorems to express the existence and uniqueness of positive solutions for $\mathbb{FDE}$s (1). Also, we can construct multiple sequences to approximate the uniqueness of the solution. By comparing the main results of this paper with the results of paper [19, 22, 23, 31], we can obtain the uniqueness of positive solutions for $\mathbb{FDE}$s (1). In a particular set for each initial value we can construct an iterative scheme to approximate the unique solution. Also, in this article, we do not assume requirements such as being super-linear or sub-linear or bounded.

In Section 2, we recall some essential definition of fractional quantum calculus. Section 3 contains our main results in this work, while an example is presented to support the validity of our obtained results. stability results are extensively discussed in Section 4. An illustrative example with some needed algorithms for the problem are given in Section 5. Finally, the conclusion are presented.

2 Preliminaries

For ease of further work, we state here the definitions, main lemmas and fixed point theorems that help to prove our main result.

Suppose that $\zeta > 0$ and $\Gamma(\zeta)$ shows the gamma function. The Riemann-Liouville fractional integral and derivative of order ζ for the function $\mathrm{w}(z)$ defined in the interval $[0, \infty)$ are defined as follows

$$\mathcal{I}_{0^+}^{\zeta}\mathrm{w}(z) = \int_0^z \tfrac{(z-\upsilon)^{\zeta-1}}{\Gamma(\zeta)}\mathrm{w}(\upsilon)\,\mathrm{d}\upsilon, \qquad z > 0,$$
$$\mathcal{D}_{0^+}^{\zeta}\mathrm{w}(z) = \left(\tfrac{\mathrm{d}}{\mathrm{d}z}\right)^m \int_0^z \tfrac{(z-\upsilon)^{m-\zeta-1}}{\Gamma(m-\zeta)}\mathrm{w}(\upsilon)\,\mathrm{d}\upsilon,$$

where $m = [\zeta] + 1$ [20].

Lemma 2.1 ([20]). *Let* $\mathrm{w} \in L^1(\mathsf{Y}) \cap C(\mathsf{Y})$ *with a fractional derivative of order* $\zeta > 0$ *that belongs to* $L^1(\mathsf{Y}) \cap C(\mathsf{Y})$. *Then*

$$\mathcal{I}_{0^+}^{\zeta}\left(\mathcal{D}_{0^+}^{\zeta}\mathrm{w}(\varrho)\right) = \mathrm{w}(\varrho) + \sum_{i=0}^{m} \tilde{d}_i \varrho^i, \qquad m = [\zeta] + 1.$$

Now, we will introduce some familiar signs (refer to [9, 25, 28] for more details). Consider $(\mathcal{F}, \|\cdot\|)$ to be a real Banach space and w to be zero element of $\mathcal{F}$ space. An infinite and convex subset like $\mathcal{Q}$ of $\mathcal{F}$ is called a cone if it has the following conditions.

a) $\mathrm{w} \in \mathcal{F}$, $\lambda \geq 0 \to \lambda\mathrm{w} \in \mathcal{Q}$;

b) $\mathrm{w} \in \mathcal{Q} \to -\mathrm{w} \in \mathcal{Q} \to \mathrm{w} = \theta$, with the zero element θ of $\mathcal{F}$.

The subset W is partially ordered by cone $\mathcal{Q}$, i.e., $\mathrm{w} \leq \acute{\mathrm{w}}$ if and only if $\acute{\mathrm{w}} - \mathrm{w} \in \mathcal{Q}$. A cone like $\mathcal{Q}$ is called normal if there exists a fixed number like $\mathcal{N}$ such that for every $\mathrm{w}, \acute{\mathrm{w}} \in \mathcal{F}$, such that $\theta \leq \mathrm{w} \leq \acute{\mathrm{w}}$, we have $\|\mathrm{w}\| \leq \mathcal{N}\|\acute{\mathrm{w}}\|$. In this case, $\mathcal{N}$ is called the normal constant of the cone $\mathcal{Q}$. We consider the operator $\Omega : \mathcal{F} \to \mathcal{F}$ to be increasing (decreasing) when $\Omega(\mathrm{w}) \leq \Omega(\acute{\mathrm{w}})$ results from $\mathrm{w} \leq \acute{\mathrm{w}}$, $(\Omega(\acute{\mathrm{w}}) \leq \Omega(\mathrm{w}))$. For each $\mathrm{w}, \acute{\mathrm{w}} \in \mathcal{F}$, we define the symbol $*$ such that there exists $\lambda > 0$ and $\nu > 0$ such that $\lambda\mathrm{w} \leq \acute{\mathrm{w}} \leq \nu\mathrm{w}$. Clearly, $*$ defines an equivalence relation. Given $j > 0$, we denote by $\mathcal{Q}_j$ the set $\mathcal{Q}_j = \left\{\mathrm{w} \in \mathcal{F} : \mathrm{w} * j\right\}$. It is clear to show $\mathcal{Q}_j \subset \mathcal{Q}$.

Let δ be a real number such that $0 < \delta < 1$. An operator $\Omega : \mathcal{Q} \to \mathcal{Q}$ said to be $\delta -$ concave if it satisfies $\Omega(\lambda\mathrm{w}) > \lambda^{\delta}\Omega(\mathrm{w})$ for each $\mathrm{w} \in \mathcal{Q}$ and $0 < \lambda < 1$. An operator $\Omega : \mathcal{F} \to \mathcal{F}$ is said to be homogeneous if it satisfies $\Omega(\lambda\mathrm{w}) = \lambda\Omega(\mathrm{w})$ for all $\lambda > 0$ and $\mathrm{w} \in \mathcal{F}$. An operator $\Omega : \mathcal{Q} \to \mathcal{Q}$ is said to be sub-homogeneous if it satisfies $\Omega(\lambda\mathrm{w}) > \lambda\Omega(\mathrm{w})$ for all $\lambda > 0$, $\mathrm{w} \in \mathcal{Q}$. In [25, 28], the authors have considered the following addition operator equation,

$$\Omega_1(\mathrm{w}) + \Omega_2(\mathrm{w}) = \mathrm{w}, \tag{2}$$

where $\Omega_i : \mathcal{Q} \to \mathcal{Q}$ are monotone operators.

Theorem 2.2 ([28]). *Consider* $\mathcal{Q} \subset \mathcal{F}$ *be a normal cone in a real Banach space* $\mathcal{F}$, $\Omega_1 : \mathcal{Q} \to \mathcal{Q}$ *be an increasing* $\delta -$ concave *operator, and* $\Omega_2 : \mathcal{Q} \to \mathcal{Q}$ *be an increasing sub-homogeneous operator. Let*

1) *there exists* $j > \delta$ *such that* $\Omega_1 j \in \mathcal{Q}_j$ *and* $\Omega_2 j \in \mathcal{Q}_j$;
2) *there exists* $\theta_0 > 0$ *such that* $\Omega_1(\mathrm{w}) \geq \theta_0 \Omega_2(\mathrm{w})$ *for all* $\mathrm{w} \in \mathcal{Q}$.

Then, the operator Eq. (2) *has a unique solution* $\mathrm{w}^* \in \mathcal{Q}_j$. *Also, constructing successively the sequence*

$$\mathrm{w}_m = \Omega_1(\mathrm{w}_{m-1}) + \Omega_2(\mathrm{w}_{m-1}), \qquad m \in \mathbb{N},$$

for any initial value $\mathrm{w}_0 \in \mathcal{Q}_j$, *we have* $\mathrm{w}_m \to \mathrm{w}^*$ *as* $m \to \infty$.

Theorem 2.3 ([25]). *Consider* $\mathcal{Q}$ *be a normal cone in a real Banach space* $\mathcal{F}$, $\Omega_1 : \mathcal{Q} \to \mathcal{Q}$ *be an increasing operator, and* $\Omega_2 : \mathcal{Q} \to \mathcal{Q}$ *be a decreasing operator. Let*

1) *for any* $\mathrm{w} \in \mathcal{Q}$ *and* $p \in \mathsf{Y}$, *there exist* $\varphi_i(p) \in (p, 1)$, $i = 1, 2$, *such that*
$$\Omega_1(p\mathrm{w}) \geq \varphi_1(p)\Omega_1(\mathrm{w}), \qquad \Omega_2(p\mathrm{w}) \leq \tfrac{\Omega_2(\mathrm{w})}{\varphi_2(p)}; \tag{3}$$
2) *there exists* $j_0 \in \mathcal{Q}_j$ *such that* $\Omega_1 j_0 + \Omega_2 j_0 \in \mathcal{Q}_j$.

Then, the operator Eq. (2) *has a unique solution* $\mathrm{w}^* \in \mathcal{Q}_j$. *Moreover, for any initial values* $\mathrm{w}_0, \acute{\mathrm{w}}_0 \in \mathcal{Q}_j$, *constructing successively the sequences*

$$\begin{aligned} \mathrm{w}_m &= \Omega_1(\mathrm{w}_{m-1}) + \Omega_2(\mathrm{w}_{m-1}), \\ \acute{\mathrm{w}}_m &= \Omega_1(\acute{\mathrm{w}}_{m-1}) + \Omega_2(\mathrm{w}_{m-1}), \end{aligned}$$

$m \in \mathbb{N}$, *we have* $\mathrm{w}_m \to \mathrm{w}^*, \acute{\mathrm{w}}_m \to \acute{\mathrm{w}}^*$ *as* $m \to \infty$.

Remark 2.4. When Ω_2 is a null operator, Theorems 2.2 and 2.3 also hold.

3 Existence Results

In this part, by applying Theorems 2.2 and 2.3, we will examine the solution for $\mathbb{FDE}$ (1) and prove the uniqueness of this positive solution.

Let $\mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F} \subset C(\overline{\mathsf{Y}}) \times C(\overline{\mathsf{Y}}) \times \cdots \times C(\overline{\mathsf{Y}}), \mathsf{Y}_0 := [0, \infty)$. We denote the

$$\mathcal{F} = \Big\{ \mathrm{w} \in C(\mathsf{Y}) \,:\, \mathrm{w}(\varrho) > 0, \varrho \in \overline{\mathsf{Y}},\, i = 1, 2, \ldots, k \Big\},$$

with the norm

$$\|\mathrm{w_i}(\varrho)\| = \max\left\{\max_{\varrho\in\mathsf{Y}} |\mathrm{w_i}(\varrho)| \,:\, i = 1, 2, \ldots, k\right\}.$$

For $(\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_k) \in \mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F}$ and so

$$\|(\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_k)\| = \max\left\{\|\mathrm{w}_1\|, \|\mathrm{w}_2\|, \ldots, \|\mathrm{w}_k\|\right\}.$$

It is clear that $\big(\mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F}, \|(\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w}_k)\|\big)$ is a Banach space. Define set

$$\mathcal{Q} = \left\{\mathrm{w} \in \mathcal{F} \,:\, \mathrm{w}(\varrho) \geq 0,\, i = 1, 2, \ldots, k,\, \varrho \in \overline{\mathsf{Y}}\right\}.$$

and $\mathcal{Z} = \mathcal{Q} \times \mathcal{Q} \times \cdots \times \mathcal{Q}$, then $\mathcal{Z}$ is a normal cone is endowed with an order relation:

$$\big(\mathrm{w}_1, \mathrm{w}_2, \ldots, \mathrm{w_k}\big) \leq \big(\acute{\mathrm{w}}_1, \acute{\mathrm{w}}_2, \ldots, \acute{\mathrm{w}}_\mathrm{k}\big),$$

with

$$\acute{\mathrm{w}}_\mathrm{i} \in \mathcal{F} \Leftrightarrow \mathrm{w}_1(\varrho) \leq \acute{\mathrm{w}}_1(\varrho), \mathrm{w}_2(\varrho) \leq \acute{\mathrm{w}}_2(\varrho), \ldots, \mathrm{w_k}(\varrho) \leq \acute{\mathrm{w}}_\mathrm{k}(\varrho),$$

for $0 \leq \varrho \leq 1$. We state the next key lemma.

Lemma 3.1. *Consider* $\mathrm{w} : \overline{\mathsf{Y}} \to \mathsf{Y}_0$ *with* $\mathrm{w} \in L^1(\overline{\mathsf{Y}})$*. The following problem with the boundary condition of the first and second order derivatives and the integral with p-Laplacian operator*

$$\begin{cases} \Phi_p\left(\mathcal{D}^{\varsigma}_{0^+}\mathrm{w}(\varrho)\right) + V(\varrho, \mathrm{w}(\varrho)) = 0, & \varrho \in \overline{\mathsf{Y}}, \\ \mathrm{w}(0) = 0, \quad \mathrm{w}'(1) = 0, \\ \mathrm{w}(1) = \mathcal{I}^{\alpha}_{0^+} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\Big|_{p=\eta} = \displaystyle\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon, \end{cases}$$

has the solution

$$\begin{aligned} \mathrm{w}(\varrho) = &\int_{\overline{\mathsf{Y}}} K(\varrho, \upsilon) h_\mathrm{w}(\upsilon)\,\mathrm{d}\upsilon \\ &+ \varrho^{\varsigma-1}\left(2 + (\varsigma - 1)\varrho^{-1}\right) \int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon. \end{aligned}$$

where $2 < \varsigma \leq 3$, $0 < \alpha < 1$ *and for* $\varrho \in \overline{\mathsf{Y}}, \upsilon \in \mathsf{Y}$,

$$K(\varrho, \upsilon) = \frac{1}{\Gamma(\varsigma)} \begin{cases} \varrho^{\varsigma-1}\Big[\Big((\varsigma-2) - \varrho^{-1}(\varsigma-1)\Big)(1-\upsilon)^{\varsigma-1} \\ \quad +(\varsigma-1)(\varrho^{-1}-1)(1-\upsilon)^{\varsigma-2}\Big] + (\varrho-\upsilon)^{\varsigma-1}, & \upsilon < \varrho, \\ \varrho^{\varsigma-1}\Big[\Big((\varsigma-2) - \varrho^{-1}(\varsigma-1)\Big)(1-\upsilon)^{\varsigma-1} \\ \quad +(\varsigma-1)(\varrho^{-1}-1)(1-\upsilon)^{\varsigma-2}\Big], & \varrho < \upsilon. \end{cases} \tag{4}$$

and so $h_{\mathrm{w}}(\varrho) = \Phi_q\left(V\big(\varrho, \mathrm{w}(\varrho)\big)\right)$,

Proof. At first, we have (1) according to

$$\Phi_p\big(\mathcal{D}_{0+}^{\varsigma}\mathrm{w}(\varrho)\big) = -V\big(\varrho, \mathrm{w}(\varrho)\big) \Rightarrow \mathcal{D}_{0+}^{\varsigma}\mathrm{w}(\varrho) = \Phi_q\big(V(\varrho, \mathrm{w}(\varrho))\big) = h_{\mathrm{w}}(\varrho),$$

we have

$$\mathcal{I}^{\varsigma}\left(\mathcal{D}_{0+}^{\varsigma}\mathrm{w}(\varrho)\right) = \mathcal{I}^{\varsigma}h_{\mathrm{w}}(\varrho).$$

Therefore, according to Lemma (1), we have

$$\mathrm{w}(\varrho) = \mathcal{I}^{\varsigma}h_{\mathrm{w}}(\varrho) + \tilde{d}_1\varrho^{\varsigma-1} + \tilde{d}_2\varrho^{\varsigma-2} + \tilde{d}_3\varrho^{\varsigma-3}.$$

According to the first condition of problem $\mathrm{w}(0) = 0$, we have that $\tilde{d}_3 = 0$. In the following, we have a problem according to the second condition

$$\begin{aligned} \mathrm{w}(1) &= \int_{\overline{\mathsf{Y}}} \tfrac{(1-\upsilon)^{\varsigma-1}}{\Gamma(\varsigma)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon + \tilde{d}_1 + \tilde{d}_2 \\ &= \int_0^{\eta} \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon. \end{aligned}$$

Hence,

$$\tilde{d}_1 + \tilde{d}_2 = \int_0^{\eta} \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon - \int_{\overline{\mathsf{Y}}} \tfrac{(1-\upsilon)^{\varsigma-1}}{\Gamma(\varsigma)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon.$$

According to the third condition, we have the problem that

$$\tilde{d}_1(\varsigma-1) + \tilde{d}_2(\varsigma-2) = -\int_{\overline{\mathsf{Y}}} \tfrac{(1-\upsilon)^{\varsigma-2}}{\Gamma(\varsigma-1)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon.$$

Therefore, the above relations imply that

$$\begin{aligned}\tilde{d}_1 = 2\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon \\ + (\varsigma-2)\int_{\overline{\Upsilon}} \tfrac{(1-\upsilon)^{\varsigma-1}}{\Gamma(\varsigma)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon - \int_{\overline{\Upsilon}} \tfrac{(1-\upsilon)^{\varsigma-2}}{\Gamma(\varsigma-1)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon,\end{aligned}$$

and

$$\begin{aligned}\tilde{d}_2 = \int_{\overline{\Upsilon}} \tfrac{(1-\upsilon)^{\varsigma-2}}{\Gamma(\varsigma-1)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon + (\varsigma-1)\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon \\ - (\varsigma-1)\int_{\overline{\Upsilon}} \tfrac{(1-\upsilon)^{\varsigma-1}}{\Gamma(\varsigma)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon.\end{aligned}$$

These imply that

$$\begin{aligned}\mathrm{w}(\varrho) = \int_0^\varrho \tfrac{(\varrho-\upsilon)^{\varsigma-1}}{\Gamma(\varsigma)} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ + \tfrac{\varrho^{\varsigma-1}}{\Gamma(\varsigma)}\Big[-(\varsigma-1)\varrho^{-1}\int_{\overline{\Upsilon}} (1-\upsilon)^{\varsigma-1} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ + (\varsigma-2)\int_{\overline{\Upsilon}} (1-\upsilon)^{\varsigma-1} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ - (\varsigma-1)\int_{\overline{\Upsilon}} (1-\upsilon)^{\varsigma-2} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ + (\varsigma-1)\varrho^{-1}\int_{\overline{\Upsilon}} (1-\upsilon)^{\varsigma-2} h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ + \tfrac{\varrho^{\varsigma-1}}{\Gamma(\alpha)}\Big[2\int_0^\eta (\eta-\upsilon)^{\alpha-1} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon \\ + (\varsigma-1)\varrho^{-1}\int_0^\eta (\eta-\upsilon)^{\alpha-1} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon\Big],\end{aligned}$$

so

$$\begin{aligned}\mathrm{w}(\varrho) = \int_{\overline{\Upsilon}} K(\varrho, q) h_{\mathrm{w}}(\upsilon)\,\mathrm{d}\upsilon \\ + \varrho^{\varsigma-1}\left(2 + (\varsigma-1)\varrho^{-1}\right)\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha-1}}{\Gamma(\alpha)} Z\big(\upsilon, \mathrm{w}(\upsilon)\big)\,\mathrm{d}\upsilon.\end{aligned}$$

This complete the proof. □

It is clear that the function $K(\varrho, \upsilon)$ is defined by (4) and $h_{\mathrm{w}}(\varrho) = \Phi_q\big(V(\varrho, \mathrm{w}(\varrho))\big)$ has the following properties [27],

$$\frac{\varrho^{\varsigma-1}\big(\varrho^{-1}(\varsigma-1)-(\varsigma-2)\big)(1-\upsilon)^{\varsigma-1}}{\varsigma^5\Gamma(\varsigma)} \le K(\varrho, \upsilon) \le \frac{\varrho^{\varsigma-1}\big((1-\upsilon)^{\varsigma-1}+(1-\upsilon)^{\varsigma-2}\big)}{\Gamma(\varsigma)}. \tag{5}$$

This relationship has also been verified in MATLAB. Now, for the continuous functions $V_i(\varrho, \mathrm{w})$, we have

$$(\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_k) \in C(\overline{\mathsf{Y}}) \times C(\overline{\mathsf{Y}}) \times \cdots \times C(\overline{\mathsf{Y}}),$$

is a solution of system of $\mathbb{FDE}$ (1) if and only if $(\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_k)$ is a solution of the integral equations,

$$\begin{cases} \mathrm{w}_1(\varrho) = \displaystyle\int_{\overline{\mathsf{Y}}} K_1(\varrho, \upsilon) h_{\mathrm{w}_1}(\upsilon)\,\mathrm{d}\upsilon \\ \quad +\varrho^{\varsigma_1-1}\left(2 + (\varsigma_1 - 1)\varrho^{-1}\right) \\ \quad \times \displaystyle\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha_1-1}}{\Gamma(\alpha_1)} Z_1\big(\upsilon, \mathrm{w}_1(\upsilon)\big)\,\mathrm{d}\upsilon, \\ \mathrm{w}_2(\varrho) = \displaystyle\int_{\overline{\mathsf{Y}}} K_2(\varrho, \upsilon) h_{\mathrm{w}_2}(\upsilon)\,\mathrm{d}\upsilon \\ \quad +\varrho^{\varsigma_2-1}\left(2 + (\varsigma_2 - 1)\varrho^{-1}\right) \\ \quad \times \displaystyle\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha_2-1}}{\Gamma(\alpha_2)} Z_2\big(\upsilon, \mathrm{w}_2(\upsilon)\big)\,\mathrm{d}\upsilon, \\ \quad \vdots \\ \mathrm{w}_{n-1}(\varrho) = \displaystyle\int_{\overline{\mathsf{Y}}} K_{n-1}(\varrho, \upsilon) h_{\mathrm{w}_{n-1}}(\upsilon)\,\mathrm{d}\upsilon \\ \quad +\varrho^{\varsigma_{n-1}-1}\left(2 + (\varsigma_{n-1} - 1)\varrho^{-1}\right) \\ \quad \times \displaystyle\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha_{n-1}-1}}{\Gamma(\alpha_{n-1})} Z_{n-1}\big(\upsilon, \mathrm{w}_{n-1}(\upsilon)\big)\,\mathrm{d}\upsilon, \\ \mathrm{w}_n(\varrho) = \displaystyle\int_{\overline{\mathsf{Y}}} K_n(\varrho, \upsilon) h_{\mathrm{w}_n}(\upsilon)\,\mathrm{d}\upsilon \\ \quad +\varrho^{\varsigma_n}\left(2 + (\varsigma_n - 1)\varrho^{-1}\right) \\ \quad \times \displaystyle\int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha_n-1}}{\Gamma(\alpha_n)} Z_n\big(\upsilon, \mathrm{w}_n(\upsilon)\big)\,\mathrm{d}\upsilon, \end{cases} \tag{6}$$

where $K_i(\varrho,\upsilon)$ is given for $\upsilon \in \mathsf{Y}, \varrho \in \overline{\mathsf{Y}}$ by,

$$K_i(\varrho,\upsilon) = \frac{1}{\Gamma(\varsigma_i)} \times \begin{cases} \varrho^{\varsigma_i-1}\Big[\Big((\varsigma_i-2)-\varrho^{-1}(\varsigma_i-1)\Big)(1-\upsilon)^{\varsigma_i-1} \\ \quad +(\varsigma_i-1)(\varrho^{-1}-1)(1-\upsilon)^{\varsigma_i-2}\Big]+(\varrho-\upsilon)^{\varsigma_i-1}, & \upsilon<\varrho, \\ \varrho^{\varsigma_i-1}\Big[\Big((\varsigma_i-2)-\varrho^{-1}(\varsigma_i-1)\Big)(1-\upsilon)^{\varsigma_i-1} \\ \quad +(\varsigma_i-1)(\varrho^{-1}-1)(1-\upsilon)^{\varsigma_i-2}\Big], & \varrho<\upsilon, \end{cases} \tag{7}$$

with $\mathrm{i}(\varrho) = \varrho^{\varsigma_i-1}$ and so $h_{\mathrm{w}_i}(\varrho) = \Phi_q\big(V(\varrho,\mathrm{w}_i(\upsilon))\big)$. We define the operator Ω_i, $i=1,2$ for $\varrho\in\overline{\mathsf{Y}}$, as

$$\begin{cases} \Omega_{1i}(\mathrm{w_i})(\varrho) = \displaystyle\int_{\overline{\mathsf{Y}}} K_i(\varrho,\upsilon)h_{\mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon, \qquad i=1,2,\dots,n \\ \Omega_{2i}(\mathrm{w_i})(\varrho) = \varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right) \\ \quad\times\displaystyle\int_0^\eta \frac{(\eta-\upsilon)^{\alpha_i-1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon,\mathrm{w_i}(\upsilon)\big)\,\mathrm{d}\upsilon. \end{cases} \tag{8}$$

Now, $\mathrm{w_i}$ answer to integral equation (6) if and only if $\mathrm{w_i} = \Omega_{1i}(\mathrm{w_i}) + \Omega_{2i}(\mathrm{w_i})$.

Theorem 3.2. *Suppose the following conditions are met*

- K_1) V_i, $Z_i : \overline{\mathsf{Y}}\times\mathsf{Y}_0\to\mathsf{Y}_0$ *are continuous and increasing with respect to the second argument,* $Z_i(\varrho,0)\neq 0$*;*
- K_2) $Z_i(\varrho,\lambda\mathrm{w}) \geq \lambda Z_i(\varrho,\mathrm{w})$ *for* $\lambda\in\mathsf{Y}, \varrho\in\overline{\mathsf{Y}}$, $\mathrm{w}\in\mathsf{Y}_0$, *and there exists a constant* $\delta\in\mathsf{Y}$ *such that* $V_i(\varrho,\lambda\mathrm{w}) \geq \lambda^\delta V_i(\varrho,\mathrm{w})$*;*
- K_3) *there exists a constant* $\delta_0>0$ *such that* $V_i(\varrho,\mathrm{w_i}) \geq \delta_0 Z_i(\varrho,\mathrm{w})$, $\varrho\in\overline{\mathsf{Y}}$, $\mathrm{w}\geq 0$.

Then, system of $\mathbb{FDE}$ (1) *has a unique positive solution*

$$\big(\mathrm{w_1}^*,\mathrm{w_2}^*,\dots,\mathrm{w_n}^*\big)\in\mathcal{Q}_j\times\mathcal{Q}_j\times\cdots\times\mathcal{Q}_j,$$

where $\mathrm{i}(\varrho)=\varrho^{\varsigma_i-1}, \varrho\in\overline{\mathsf{Y}}$. *And, for any initial value*

$$\big(\mathrm{w}_{01},\mathrm{w}_{02},\dots,\mathrm{w}_{0n}\big)\in\mathcal{Q}_j\times\mathcal{Q}_j\times\cdots\times\mathcal{Q}_j,$$

constructing successively the sequence

$$\begin{aligned}\mathrm{w}_{i(m+1)}(\varrho) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{\mathrm{w}_{im}}(\varrho)\,\mathrm{d}\upsilon \\ &+ \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \\ &\times \int_0^{\eta} \tfrac{(\eta - q)^{\alpha_i - 1}}{\Gamma(\alpha_i)} Z_i(\upsilon, \mathrm{w}_{im}(\upsilon))\,\mathrm{d}\upsilon, \qquad m \in \{0\} \cup \mathbb{N}.\end{aligned}$$

Then, we have $\mathrm{w}_{im}(\varrho) \to \mathrm{w_i}^*(\varrho)$ *as* $m \to \infty$.

Proof. According to Lemma 3.1, we know that system of $\mathbb{FDE}$ (1) has an integral solution expressed by

$$\begin{aligned}\mathrm{w_i}(\varrho) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{\mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon \\ &+ \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \int_0^{\eta} \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \mathrm{w_i}(\upsilon)\big)\,\mathrm{d}\upsilon,\end{aligned}$$

where $K_i(\varrho, \upsilon)$ is given by (4) . Define two operators $\Omega_{1i} : \mathcal{Q} \to \mathcal{Q}$ and $\Omega_{2i} : \mathcal{Q} \to \mathcal{Q}$ by

$$\begin{aligned}\Omega_{1i}(\varrho) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{\mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon, \\ \Omega_{2i}(\varrho) &= \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \int_0^{\eta} \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \mathrm{w_i}(\upsilon)\big)\,\mathrm{d}\upsilon.\end{aligned}$$

It can be seen that w is the solution of system of $\mathbb{FDE}$ (1) if and only if $\mathrm{w_i} = \Omega_{1i}(\mathrm{w_i}) + \Omega_{2i}(\mathrm{w_i})$. From (K_1), we know that $\Omega_{1i} : \mathcal{Q} \to \mathcal{Q}$ and $\Omega_{2i} : \mathcal{Q} \to \mathcal{Q}$. Next, we have to check that Ω_{1i} and Ω_{2i} satisfy all the assumptions of Theorem 2.2. First, we prove that Ω_{1i}, Ω_{2i} are two increasing operators. For $\varpi_1, \varpi_2 \in \mathcal{Q}$ with $\varpi_1 \geq \varpi_2$, we have $\varpi_1(\varrho) \geq \varpi_2(\varrho)$, $\varrho \in \overline{\mathsf{Y}}$, and, by (K_1),

$$\begin{aligned}\Omega_{1i}(\varpi_1(\varrho)) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{i\varpi_1}(\upsilon)\,\mathrm{d}\upsilon \\ &\geq \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{i\varpi_2}(\upsilon)\,\mathrm{d}\upsilon = \Omega_{1i}\big(\varpi_2(\varrho)\big).\end{aligned}$$

That is, $\Omega_{1i}\big(\varpi_1(\varrho)\big) \geq \Omega_{1i}(\varpi_2(\varrho))$. By the same way, $\Omega_{2i}\big(\varpi_1(\varrho)\big) \geq \Omega_{2i}\big(\varpi_2(\varrho)\big)$. In the second part, we prove that Ω_{1i} is a $\delta -$ concave operator and Ω_{2i} is a sub-homogeneous operator. For any $0 < \lambda < 1$ and $\mathrm{w_i} \in \mathcal{Q}$, from ($\mathrm{K}_2$), we obtain

$$\begin{aligned}\Omega_{1i}(\lambda \mathrm{w_i})(\varrho) &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon) h_{i\lambda \mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon \\ &\geq \lambda^{\delta} \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon) h_{i\mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon = \lambda^{\delta}\Omega_{1i}\mathrm{w_i}(\varrho).\end{aligned}$$

Therefore, it shows that for $0 < \lambda < 1$ and $\mathrm{w_i} \in \mathcal{Q}$, the relation $\Omega_{1i}(\lambda\mathrm{w_i}) \geq \lambda^{\delta}\Omega_{1i}(\mathrm{w}_i)$ is established. So the operator Ω_{1i} is a $\delta-$concave operator. Also, for any $0 < \lambda < 1$ and $\mathrm{w_i} \in \mathcal{Q}$, from ($\mathrm{K}_2$) we obtain,

$$\begin{aligned}\Omega_{2i}(\lambda \mathrm{w_i})(\varrho) &= \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \\ &\quad \times \int_0^{\eta} \tfrac{(\eta-\upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \lambda \mathrm{w_i}(\upsilon)\big)\,\mathrm{d}\upsilon \\ &\geq \lambda \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \\ &\quad \times \int_0^{\eta} \tfrac{(\eta-\upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \mathrm{w_i}(\upsilon)\big)\,\mathrm{d}\upsilon \\ &= \lambda\Omega_{2i}(\mathrm{w_i}(\varrho)).\end{aligned}$$

Therefore, it shows that for $0 < \lambda < 1$ and $\mathrm{w_i} \in \mathcal{Q}$, the relation $\Omega_{2i}(\lambda\mathrm{w_i})(\varrho) \geq \lambda\Omega_{2i}(\mathrm{w}_i)$ is established. So the operator Ω_{2i} is sub-homogeneous. In the following, we will show $\Omega_{1i}\mathrm{i_i}$, $\Omega_{2i}\mathrm{i_i} \in \mathcal{Q}_j$. From ($\mathrm{K}_1$)

$$\begin{aligned}\Omega_{1i}\mathrm{i_i}(\varrho) &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon) h_{i\mathrm{i}_i}(\upsilon)\,\mathrm{d}\upsilon \\ &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon)\Phi_q\left(V_i(\upsilon,\upsilon^{\varsigma_i-1})\right)\,\mathrm{d}\upsilon \\ &\leq \tfrac{\varrho^{\varsigma_i-1}}{\Gamma(\varsigma_i)}\int_{\overline{\Upsilon}}((1-\upsilon_i)^{\varsigma_i-1} + (1-\upsilon_i)^{\varsigma_i-2})\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon, \\ \Omega_{1i}\mathrm{i_i}(\varrho) &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon)\Phi_q\left(V_i(\upsilon,\upsilon^{\varsigma_i-1})\right)\,\mathrm{d}\upsilon \\ &\geq \tfrac{\varrho^{\varsigma-1}\left(\varrho^{-1}(\varsigma-1)-(\varsigma-2)\right)}{\varsigma^5\Gamma(\varsigma)}\int_{\overline{\Upsilon}}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon.\end{aligned}$$

From (K_1) and (K_3) we have $V_i(\upsilon,1) \geq V_i(\upsilon,0) \geq \delta_0\, Z_i(\upsilon,0) \geq 0$. So

$$\Phi_q\left(V_i(\upsilon,1)\right) \geq \Phi_q\left(V_i(\upsilon,0)\right) \geq \delta_0\, Z_i(\upsilon,0) \geq 0.$$

Note that $\varsigma_i - 1 > 0$ and $Z_i(\varrho, 0) \neq 0$, we can get

$$\begin{aligned}\int_{\overline{\Upsilon}} \big((1-\upsilon_i)^{\varsigma_i-1} + (1-\upsilon_i)^{\varsigma_i-2}\big)\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon \\ \geq \int_{\overline{\Upsilon}}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon \\ \geq \delta_0 \int_{\overline{\Upsilon}}(1-\upsilon)^{\varsigma_i-1}Z_i(\upsilon,0)\,\mathrm{d}\upsilon > 0.\end{aligned}$$

Let

$$\begin{aligned} r_{1i} &= \tfrac{\varrho^{\varsigma-1}\left(\varrho^{-1}(\varsigma-1)-(\varsigma-2)\right)}{\varsigma^5\Gamma(\varsigma)} \int_{\overline{\Upsilon}}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon > 0, \\ r_{2i} &= \tfrac{\varrho^{\varsigma_i-1}}{\Gamma(\varsigma_i)} \int_{\overline{\Upsilon}} \left((1-\upsilon_i)^{\varsigma_i-1} + (1-\upsilon_i)^{\varsigma_i-2}\right)\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon > 0. \end{aligned} \tag{9}$$

Then $r_{2i} \geq r_{1i} > 0$ and thus $r_{1i}\mathrm{i}_\mathrm{i}(\varrho) \leq \Omega_{1i}\mathrm{i}_\mathrm{i}(\varrho) \leq r_{2i}\mathrm{i}_\mathrm{i}(\varrho)$, $\varrho \in \overline{\Upsilon}$. So we have $\Omega_{1i}\mathrm{i}_\mathrm{i} \in \mathcal{Q}_j$. In the same way,

$$\begin{aligned} \Omega_{2i}\mathrm{i}_\mathrm{i}(\varrho) &= \tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}\,Z_i\left(\upsilon,\upsilon^{\varsigma_i-1}\right)\,\mathrm{d}\upsilon \\ &\leq \tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon, \\ \Omega_{2i}\mathrm{i}_\mathrm{i}(\varrho) &= \tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}Z_i\left(\upsilon,\upsilon^{\varsigma_i-1}\right)\,\mathrm{d}\upsilon \\ &\geq \tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}\,Z_i(\upsilon,0)\,\mathrm{d}\upsilon. \end{aligned}$$

and

$$\begin{aligned} r_{3i} &= \tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon, \\ r_{4i} &= \tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)} \int_0^\eta (\eta-\upsilon)^{\alpha_i-1}\,Z_i(\upsilon,0)\,\mathrm{d}\upsilon. \end{aligned} \tag{10}$$

Also from $Z_i(\varrho, 0) \neq 0$ we can easily prove $\Omega_{2i}\mathrm{i}_\mathrm{i} \in \mathcal{Q}_j$. That is, condition (1) of Theorem 2.2 holds. In the following, we will prove that condition (2) of Theorem 2.2 holds. For $\mathrm{w_i} \in \mathcal{Q}$, by ($\mathrm{K}_3$),

$$\begin{aligned}\Omega_{1i}(\mathrm{w_i}(\varrho)) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho,\upsilon) h_{i\mathrm{w}_i}(\upsilon)\,\mathrm{d}\upsilon \\ &\geq \delta_0 \tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)} \\ &\quad \times \int_0^\eta (\eta-\upsilon)^{\alpha_i-1} Z_i(\upsilon,0)\,\mathrm{d}\upsilon \\ &= \delta_0 \Omega_{2i}\big(\mathrm{w_i}(\varrho)\big).\end{aligned}$$

So we obtain $\Omega_{1i}(\mathrm{w_i}) \geq \delta_0\Omega_{2i}(\mathrm{w_i})$, $\mathrm{w_i} \in \mathcal{Q}$. In the last part according to Theorem 2.2, we know that the operator $\Omega_{1i}(\mathrm{w_i}) + \Omega_{2i}(\mathrm{w_i}) = \mathrm{w_i}$ has a unique positive solution $\left(\mathrm{w_1}^*, \mathrm{w_2}^*, \ldots, \mathrm{w_n}^*\right) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j$, where $\mathrm{i}(\varrho) = \varrho^{\varsigma_i-1}$, $\varrho \in \overline{\mathsf{Y}}$. And, for any initial value

$$\left(\mathrm{w}_{01}, \mathrm{w}_{02}, \ldots, \mathrm{w}_{0n}\right) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j,$$

constructing successively the sequence

$$\begin{aligned}\mathrm{w}_{i(m+1)}(\varrho) = &\int_{\overline{\mathsf{Y}}} K_i(\varrho,\upsilon) h_{i\mathrm{w}_{im}}(\upsilon)\,\mathrm{d}\upsilon \\ &+ \varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right) \\ &\times \int_0^\eta \tfrac{(\eta-\upsilon)^{\alpha_i-1}}{\Gamma(\alpha_i)} Z_i\big(\upsilon, \mathrm{w}_{im}(\upsilon)\big)\,\mathrm{d}\upsilon, \qquad m \in \{0\}\cup\mathbb{N},\end{aligned}$$

for $i \in \mathbb{N}$, we have $\mathrm{w}_{im}(\varrho) \to \mathrm{w_i}^*(\varrho)$ as $m \to \infty$. □

Remark 3.3. Note that, when $Z_i(\varrho, 0) = 0$ in condition (K_1) then, $\Omega_{2i}J_i \in Q_j$ can not hold.

Theorem 3.4. *Assume*

- K_4) $V_i : \overline{\mathsf{Y}} \times \mathsf{Y}_0 \to \mathsf{Y}_0$ *is continuous and increasing with respect to the second argument,* $V_i(\varrho, 0) \neq 0$*;*
- K_5) $Z_i : \overline{\mathsf{Y}} \times \mathsf{Y}_0 \to \mathsf{Y}_0$ *is continuous and decreasing with respect to the second argument,* $Z_i(\varrho, 1) \neq 0$*;*
- K_6) *there exists* $\varphi_i(\lambda) \in (\lambda, 1)$, $(i = 1, 2)$, *for* $\lambda \in \mathsf{Y}$, *such that*

$$V_i(\varrho, \lambda\mathrm{w_i}) \geq \varphi_{1i}(\lambda) V_i(\varrho, \mathrm{w_i}), \qquad Z_i(\varrho, \lambda\mathrm{w_i}) \leq \tfrac{Z_i(\varrho,\mathrm{w_i})}{\varphi_{2i}(\lambda)},$$

for $\varrho \in \overline{\mathsf{Y}}$, $\mathrm{w_i} \in \mathsf{Y}_0$.

Then problem (1) has a unique positive solution

$$\left(\mathrm{w_1}^*, \mathrm{w_2}^*, \dots, \mathrm{w_n}^*\right) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j,$$

where $\mathrm{i_i}(\varrho) = \varrho^{\varsigma_i - 1}, \varrho \in \overline{\mathsf{Y}}$. *And, for any initial value*

$$\left((\mathrm{w}_{01}, \mathrm{w}_{02}, \dots, \mathrm{w}_{0n}), (\acute{\mathrm{w}}_{01}, \acute{\mathrm{w}}_{02}, \dots, \acute{\mathrm{w}}_{0n})\right) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j,$$

constructing successively the sequences

$$\begin{aligned}
\mathrm{w}_{i(m+1)}(\varrho) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{i\mathrm{w}_{im}}(\upsilon)\, \mathrm{d}\upsilon \\
&\quad + \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \int_0^{\eta} \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, Z_i\big(\upsilon, \mathrm{w_{i}}_m(\upsilon)\big)\, \mathrm{d}\upsilon, \\
\acute{\mathrm{w}}_{i(m+1)}(\varrho) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) \acute{h}_{i\mathrm{w}_{im}}(\upsilon)\, \mathrm{d}\upsilon \\
&\quad + \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \int_0^{\eta} \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, Z_i\big(\upsilon, \acute{\mathrm{w}}_{\mathrm{i}m}(\upsilon)\big)\, \mathrm{d}\upsilon,
\end{aligned}$$

for $m \in \mathbb{N}$, *we have* $\acute{\mathrm{w}}_{im}(\varrho) \to \mathrm{w_i}^*(\varrho), \mathrm{w}_{\mathrm{i}m}(\varrho) \to \mathrm{w_i}^*(\varrho)$ *as* $m \to \infty$.

Proof. As in Theorem 3.2, we define operators $\Omega_{1i} : \mathcal{Q} \to \mathcal{Q}$ and $\Omega_{2i} : \mathcal{Q} \to \mathcal{Q}$ as follows

$$\begin{aligned}
\Omega_{1i}(\mathrm{w_i}(\varrho)) &= \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{i\mathrm{w}_i}(\upsilon)\, \mathrm{d}\upsilon, \\
\Omega_{2i}(\mathrm{w_i}(\varrho)) &= \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \int_0^{\eta} \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, Z_i\big(\upsilon, \acute{\mathrm{w}}_{\mathrm{i}m}(\upsilon)\big)\, \mathrm{d}\upsilon.
\end{aligned}$$

According to (K_4) and (K_5) we know that $\Omega_{1i} : \mathcal{Q} \to \mathcal{Q}$ is increasing and $\Omega_{2i} : \mathcal{Q} \to \mathcal{Q}$ is decreasing. Also, it can be proved from (K_5) that a and b are satisfied in condition (1) of Theorem 2.3. Therefore, we have to prove it $\Omega_{1i}\mathrm{i}_\mathrm{i}(\varrho)+\Omega_{2i}\mathrm{i}_\mathrm{i}(\varrho)\in\mathcal{Q}_j$. From ($K_4$) and ($K_5$),

$$\begin{aligned}
\Omega_{1i}\mathrm{i}_\mathrm{i}(\varrho)+\Omega_{2i}\mathrm{i}_\mathrm{i}(\varrho) &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon)h_{\mathrm{ii}_i}(\upsilon)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,\upsilon^{\varsigma_i-1})\,\mathrm{d}\upsilon \\
&\le \tfrac{\varrho^{\varsigma_i-1}}{\Gamma(\varsigma_i)}\int_{\overline{\Upsilon}}((1-\upsilon_i)^{\varsigma_i-1}+(1-\upsilon_i)^{\varsigma_i-2})\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,0)\,\mathrm{d}\upsilon \\
&= \varrho^{\varsigma_i-1}\Bigg[\tfrac{1}{\Gamma(\varsigma_i)}\int_{\overline{\Upsilon}}((1-\upsilon_i)^{\varsigma_i-1}+(1-\upsilon_i)^{\varsigma_i-2})\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,0)\,\mathrm{d}\upsilon\Bigg],
\end{aligned}$$

and

$$\begin{aligned}
\Omega_{1i}\mathrm{i}_\mathrm{i}(\varrho)+\Omega_{2i}\mathrm{i}_\mathrm{i}(\varrho) &= \int_{\overline{\Upsilon}} K_i(\varrho,\upsilon)h_{\mathrm{ii}_\mathrm{i}}(\upsilon)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,\upsilon^{\varsigma_i-1})\,\mathrm{d}\upsilon \\
&\ge \int_{\overline{\Upsilon}}\tfrac{\varrho^{\varsigma-1}\left(\varrho^{-1}(\varsigma-1)-(\varsigma-2)\right)}{\varsigma^5\Gamma(\varsigma)}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{\varrho^{\varsigma_i-1}\left(2+(\varsigma_i-1)\varrho^{-1}\right)}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon \\
&= \varrho^{\varsigma_i-1}\Bigg[\int_{\overline{\Upsilon}}\tfrac{\varrho^{-1}(\varsigma-1)-(\varsigma-2)}{\varsigma^5\Gamma(\varsigma)}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon \\
&\quad+\tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)}\int_0^\eta(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon\Bigg].
\end{aligned}$$

From (K_4) and (K_5), we have

$$V_i(\upsilon,1)+Z_i(\upsilon,0)\ge V_i(\upsilon,0)+Z_i(\upsilon,1)\ge 0.$$

Note that $\varsigma_i - 1 > 0$ and $V_i(\varrho, 0) + Z_i(\varrho, 1) \neq 0$, we can get,

$$\begin{aligned}\int_{\overline{\mathsf{Y}}}(1-\upsilon)^{\varsigma_i-1}\Phi_q\left(V_i(\upsilon,1)\right)\,\mathrm{d}\upsilon + \int_0^{\eta}(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,0)\,\mathrm{d}\upsilon \\ \geq \int_{\overline{\mathsf{Y}}}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon \\ + \int_0^{\eta}(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon > 0.\end{aligned}$$

Let

$$\begin{aligned} r_{5i} &= \int_{\overline{\mathsf{Y}}}\tfrac{\varrho^{-1}(\varsigma-1)-(\varsigma-2)}{\varsigma^5\Gamma(\varsigma)}(1-\upsilon)^{\varsigma_i-1}\Phi_q\big(V_i(\upsilon,0)\big)\,\mathrm{d}\upsilon \\ &\quad + \tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)}\int_0^{\eta}(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,1)\,\mathrm{d}\upsilon \\ r_{6i} &= \tfrac{1}{\Gamma(\varsigma_i)}\int_{\overline{\mathsf{Y}}}((1-\upsilon_i)^{\varsigma_i-1}+(1-\upsilon_i)^{\varsigma_i-2})\Phi_q\big(V_i(\upsilon,1)\big)\,\mathrm{d}\upsilon \\ &\quad + \tfrac{2+(\varsigma_i-1)\varrho^{-1}}{\Gamma(\varsigma_i)}\int_0^{\eta}(\eta-\upsilon)^{\alpha_i-1}Z_i(\upsilon,0)\,\mathrm{d}\upsilon. \end{aligned} \tag{11}$$

Then, $r_{6i} \geq r_{5i} > 0$ and thus

$$r_{5i}\mathrm{i_i}(\varrho) \leq \Omega_{1i}\mathrm{i_i}(\varrho) + \Omega_{2i}\mathrm{i_i}(\varrho) \leq r_{6i}\mathrm{i_i}(\varrho), \qquad \varrho \in \overline{\mathsf{Y}}.$$

So we have $\Omega_{1i}\mathrm{i_i} + \Omega_{2i}\mathrm{i_i} \in \mathcal{Q}_j$. Finally, according to Theorem 2.3, we know that operator $\Omega_{1i}(\mathrm{w_i}) + \Omega_{2i}(\mathrm{w_i}) = \mathrm{w_i}$ has a unique solution

$$\big(\mathrm{w_1}^*, \mathrm{w_2}^*, \dots, \mathrm{w_n}^*\big) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j.$$

For any initial values

$$\big((\mathrm{w}_{01}, \mathrm{w}_{02}, \dots, \mathrm{w}_{0n}), (\acute{\mathrm{w}}_{01}, \acute{\mathrm{w}}_{02}, \dots, \acute{\mathrm{w}}_{0n})\big) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j,$$

constructing successively, for the sequences

$$\begin{aligned} \mathrm{w}_{im} &= \Omega_{1i}\left(\mathrm{w}_{i(m-1)}\right) + \Omega_{2i}\left(\acute{\mathrm{w}}_{i(m-1)}\right), \\ \acute{\mathrm{w}}_{im} &= \Omega_{1i}\left(\acute{\mathrm{w}}_{i(m-1)}\right) + \Omega_{2i}\left(\mathrm{w}_{i(m-1)}\right), \end{aligned}$$

for $m \in \mathbb{N}$, we have $\mathrm{w}_{im} \to \mathrm{w_i}^*$, $\acute{\mathrm{w}}_{im} \to \acute{\mathrm{w}}_{\mathrm{i}}^*$ as $m \to \infty$. That is, $\mathbb{FDE}$s (1) has a unique positive solution $\acute{\mathrm{w}}_{\mathrm{i}}^*$ in $\mathcal{Q}_j$, where $\mathrm{i_i}(\varrho) = \varrho^{\varsigma_i - 1}$, $\varrho \in \overline{\mathsf{Y}}$. Also, for each initial value of

$$\big((\mathrm{w}_{01}, \mathrm{w}_{02}, \dots, \mathrm{w}_{0n}), (\acute{\mathrm{w}}_{01}, \acute{\mathrm{w}}_{02}, \dots, \acute{\mathrm{w}}_{0n})\big) \in \mathcal{Q}_j \times \mathcal{Q}_j \times \cdots \times \mathcal{Q}_j,$$

we can make the following sequences

$$\begin{aligned}
\mathrm{w}_{i(m+1)}(\varrho) = & \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) h_{i\mathrm{w}_{im}}(\upsilon)\, \mathrm{d}\upsilon \\
& + \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \\
& \times \int_0^\eta \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, Z_i(\upsilon, \mathrm{w}_{\mathrm{i}m}(\upsilon))\, \mathrm{d}\upsilon, \\
\acute{\mathrm{w}}_{i(m+1)}(\varrho) = & \int_{\overline{\mathsf{Y}}} K_i(\varrho, \upsilon) \acute{h}_{i\mathrm{w}_{im}}(\upsilon)\, \mathrm{d}\upsilon \\
& + \varrho^{\varsigma_i - 1}\left(2 + (\varsigma_i - 1)\varrho^{-1}\right) \\
& \times \int_0^\eta \tfrac{(\eta - \upsilon)^{\alpha_i - 1}}{\Gamma(\alpha_i)}\, Z_i(\upsilon, \acute{\mathrm{w}}_{\mathrm{i}m}(\upsilon))\, \mathrm{d}\upsilon,
\end{aligned}$$

for $m \in \mathbb{N}$, we have $\mathrm{w}_{im}(\varrho) \to \acute{\mathrm{w}}_{\mathrm{i}}^*(\varrho)$, $\acute{\mathrm{w}}_{im}(\varrho) \to \acute{\mathrm{w}}_{\mathrm{i}}^*(\varrho)$ as $m \to \infty$. □

4 Numerical Examples

It is clear that there are many functions that apply in the conditions of Theorems 3.2 and 3.4. We present two examples for the convenience of the readers.

Example 4.1. In the specific case, we consider the following 2D-$\mathbb{FDE}$s,

$$\begin{cases} \Phi_p\left(\mathcal{D}_{0^+}^{\varsigma_1}\mathrm{w}_1(\varrho)\right) + \left(\mathrm{w}_1(\varrho)\right)^{1/2} + \frac{\mathrm{w}_1{}^2(\varrho)}{3+\mathrm{w}_1(\varrho)}e^{2\varrho} + a = 0, \\ \Phi_p\left(\mathcal{D}_{0^+}^{11/5}\mathrm{w}_2(\varrho)\right) + \left(\mathrm{w}_2(\varrho)\right)^{1/6} + \frac{1/3\mathrm{w}_2(\varrho)}{4+\mathrm{w}_2(\varrho)}e^{3\varrho} + b = 0, \\ \mathrm{w}_i(0) = 0,\ \mathrm{w}_i'(1) = 0, \quad i = 1, 2, \\ \mathrm{w}_1(1) = \mathcal{I}_{0^+}^{1/2} Z_1\big(\upsilon, \mathrm{w}_1(\upsilon)\big)\Big|_{p=1/2} \\ \qquad = \int_0^{1/2} \frac{\left(\frac{1}{2}-\upsilon\right)^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} Z_1\big(\upsilon, \mathrm{w}_1(\upsilon)\big)\,\mathrm{d}\upsilon, \\ \mathrm{w}_2(1) = \mathcal{I}_{0^+}^{1/2} Z_2\big(\upsilon, \mathrm{w}_2(\upsilon)\big)\Big|_{p=1/3} \\ \qquad = \int_0^{1/3} \frac{\left(\frac{1}{3}-\upsilon\right)^{-2/3}}{\Gamma\left(\frac{1}{3}\right)} Z_2\big(\upsilon, \mathrm{w}_2(\upsilon)\big)\,\mathrm{d}\upsilon, \end{cases} \tag{12}$$

for three values

$$\varsigma_1 \in \left\{\frac{29}{10}, \frac{14}{5}, \frac{13}{5}\right\} \subset (2, 3].$$

So $\varsigma_2 = \frac{11}{5} \in (2, 3]$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{3} \in \mathsf{Y}$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{1}{3} \in \mathsf{Y}$ and $\delta_1 = \frac{1}{2}$, $\delta_2 = \frac{1}{3} \in \mathsf{Y}$, where $a, d > 0$ is a constant. We take $0 < b < a$, $0 < c < d$ and let $V_1(\varrho, \mathrm{w}_1) = \mathrm{w}_1{}^{1/2} + b$, $V_2(\varrho, \mathrm{w}_2) = \mathrm{w}_2{}^{1/6} + d$,

$$Z_1(\varrho, \mathrm{w}_1) = \frac{\mathrm{w}_1^2(\varrho)}{3+\mathrm{w}_1(\varrho)}e^{2\varrho} + a - b,$$
$$Z_2(\varrho, \mathrm{w}_2) = \frac{\frac{1}{3}\mathrm{w}_2(\varrho)}{4+\mathrm{w}_2(\varrho)}e^{3\varrho} + d - c.$$

Now, $V_i, Z_i : \overline{\mathsf{Y}} \times \mathsf{Y}_0 \to \mathsf{Y}_0, i = 1, 2$ are continuous and increasing with respect to the second argument, $V_1(\varrho, 0) = a - b > 0$. In addition for $\lambda_1 \in \mathsf{Y}$, $\varrho \in \overline{\mathsf{Y}}$, $\mathrm{w}_1 \in \mathsf{Y}_0$. We have,

$$\begin{aligned} Z_1(\varrho, \lambda_1\mathrm{w}_1) &= \frac{\lambda_1\mathrm{w}_1{}^2}{3+\lambda_1\mathrm{w}_1}\, e^{2\varrho} + a - b \\ &\geq \frac{\lambda_1\mathrm{w}_1{}^2}{3+\mathrm{w}_1}\, e^{2\varrho} + \lambda_1(a - b) = \lambda_1 Z_1(\varrho, \mathrm{w}_1), \end{aligned}$$

and

$$\begin{aligned} Z_2(\varrho, \lambda_2\mathrm{w}_2) &= \frac{\frac{1}{3}\lambda_2\mathrm{w}_2}{4+\lambda_2\mathrm{w}_2}\, e^{3\varrho} + d - c \\ &\geq \frac{\lambda_2\left(\frac{1}{3}\mathrm{w}_2\right)}{4+\mathrm{w}_2}\, e^{3\varrho} + \lambda_2(d - c) = \lambda_2 Z_2(\varrho, \mathrm{w}_2). \end{aligned}$$

So, we get Z_1, Z_2 are sub-homogenous.

$$\begin{aligned}
V_1(\varrho, \lambda_1 \mathrm{w}_1) &= (\lambda_1 \mathrm{w}_1)^{1/2} + b = {\lambda_1}^{1/2}\, {\mathrm{w}_1}^{1/2} + b \\
&\geq {\lambda_1}^{1/2}\, {\mathrm{w}_1}^{1/2} + {\lambda_1}^{1/2} b = {\lambda_1}^{1/2} V_1(\varrho, \mathrm{w}_1), \\
V_2(\varrho, \lambda_2 \mathrm{w}_2) &= (\lambda_2 \mathrm{w}_2)^{1/6} + c = {\lambda_2}^{1/6}\, {\mathrm{w}_2}^{1/6} + c \\
&\geq {\lambda_2}^{1/6}\, {\mathrm{w}_2}^{1/6} + {\lambda_2}^{1/6} c = {\lambda_2}^{1/6} V_2(\varrho, \mathrm{w}_2).
\end{aligned}$$

Hence, V_1, V_2 are $\delta -$ concave. Moreover, if we take $0 < \delta_1 < \frac{b}{e+a-b}$, $0 < \delta_2 < \frac{c}{e+d-c}$, then we obtain,

$$\begin{aligned}
V_1(\varrho, \mathrm{w}_1) &= {\mathrm{w}_1}^{1/2} + b \geq b = \tfrac{b}{e+a-b}(e + a - b) \\
&\geq \delta_1 \left(\tfrac{{\mathrm{w}_1}^2}{3+\mathrm{w}_1}\, e^{2\varrho} + a - b \right), \\
V_2(\varrho, \mathrm{w}_2) &= {\mathrm{w}_2}^{1/6} + c \geq c = \tfrac{c}{e+d-c}(e + d - c) \\
&\geq \delta_2 \left(\tfrac{\frac{1}{3}\mathrm{w}_2}{4+\mathrm{w}_2}\, e^{3\varrho} + d - c \right),
\end{aligned}$$

and $V_1(\varrho, \mathrm{w}_1) \geq \delta_1 Z_1(\varrho, \mathrm{w}_1)$, $V_2(\varrho, \mathrm{w}_2) \geq \delta_2 Z_2(\varrho, \mathrm{w}_2)$. By taking $a = 2.35$, $b = 1.63$, $c = 2.14$, $d = 3.18$, one can choose $0 < \delta_1 < \frac{1.63}{e+0.72}$, $0 < \delta_2 < \frac{2.14}{e+1.03}$ and from Eq. (4), we found that Ineq. (5) holds as shown in Table 1 for $\varrho \in \overline{\Upsilon}$ when $\upsilon = 0.29$. In fact,

$$\begin{aligned}
&\tfrac{\varrho^{\varsigma_1 - 1}}{\varsigma_1^5 \Gamma(\varsigma_1)} \left(\varrho^{-1}(\varsigma_1 - 1) - (\varsigma_1 - 2) \right) (1 - \upsilon)^{\varsigma_1 - 1} \\
&\simeq \begin{cases} 0.001, & \varsigma_1 = 29/10, \\ 0.002, & \varsigma_1 = 14/5, \\ 0.003, & \varsigma_1 = 13/5, \end{cases} \qquad (13) \\
&\leq K_1(\varrho, \upsilon) \simeq \begin{cases} 0.236, & \varsigma_1 = 29/10, \\ 0.218, & \varsigma_1 = 14/5, \\ 0.174, & \varsigma_1 = 13/5, \end{cases} \qquad (14) \\
&\leq \left\{ \begin{array}{ll} 0.688, & \varsigma_1 = 29/10, \\ 0.776, & \varsigma_1 = 14/5, \\ 0.974, & \varsigma_1 = 13/5, \end{array} \right\} \simeq \tfrac{\varrho^{\varsigma_1 - 1}\left((1-\upsilon)^{\varsigma_1 - 1} + (1-\upsilon)^{\varsigma_1 - 2} \right)}{\Gamma(\varsigma_1)}, \qquad (15)
\end{aligned}$$

and

$$\frac{\varrho^{\varsigma_2-1}(1-\upsilon)^{\varsigma_1-1}}{\varsigma_2^5\Gamma(\varsigma_2)}\left(\varrho^{-1}(\varsigma_2-1)-(\varsigma_2-2)\right)\simeq 0.0116 \tag{16}$$
$$\leq K_2(\varrho,\upsilon)\simeq 0.061$$
$$\leq 1.449 \simeq \frac{\varrho^{\varsigma_2-1}\left((1-\upsilon)^{\varsigma_2-1}+(1-\upsilon)^{\varsigma_2-2}\right)}{\Gamma(\varsigma_2)}. \tag{17}$$

So, all the conditions of Theorem (3.2) are satisfied. Therefore, 2D-

Table 1: The results of Ineq. (13) for 2D-$\mathbb{FDE}$s (12) with three values of derivative order ς_1 in Example 4.1 for $\varrho \in \overline{\mathsf{Y}}$ when $\upsilon = 0.29$.

ϱ	$\varsigma_1 = 29/10$			$\varsigma_1 = 14/5$			$\varsigma_1 = 13/5$		
	$K(\varrho,\upsilon)$	(13)	(15)	$K(\varrho,\upsilon)$	(13)	(15)	$K(\varrho,\upsilon)$	(13)	(15)
0.05	0.013	0.000	0.002	0.019	0.000	0.004	0.038	0.001	0.008
0.10	0.022	0.000	0.009	0.029	0.001	0.012	0.050	0.001	0.024
0.15	0.026	0.000	0.019	0.034	0.001	0.026	0.053	0.002	0.047
0.20	0.028	0.001	0.032	0.034	0.001	0.043	0.050	0.002	0.074
0.25	0.027	0.001	0.049	0.032	0.001	0.064	0.042	0.002	0.106
0.30	0.024	0.001	0.070	0.027	0.001	0.089	0.032	0.002	0.142
0.35	0.022	0.001	0.094	0.024	0.001	0.117	0.027	0.003	0.182
0.40	0.023	0.001	0.121	0.025	0.001	0.149	0.027	0.003	0.225
0.45	0.028	0.001	0.151	0.029	0.001	0.184	0.031	0.003	0.271
0.50	0.034	0.001	0.184	0.036	0.002	0.223	0.036	0.003	0.321
0.55	0.044	0.001	0.221	0.045	0.002	0.264	0.044	0.003	0.374
0.60	0.056	0.001	0.260	0.056	0.002	0.309	0.053	0.003	0.430
0.65	0.070	0.001	0.303	0.069	0.002	0.357	0.063	0.003	0.489
0.70	0.087	0.001	0.349	0.085	0.002	0.408	0.076	0.003	0.550
0.75	0.106	0.001	0.398	0.102	0.002	0.462	0.089	0.003	0.615
0.80	0.128	0.001	0.450	0.122	0.002	0.519	0.104	0.003	0.682
0.85	0.151	0.001	0.505	0.143	0.002	0.579	0.120	0.003	0.751
0.90	0.177	0.001	0.563	0.166	0.002	0.642	0.137	0.003	0.823
0.95	0.206	0.001	0.624	0.191	0.002	0.707	0.155	0.003	0.897
1.00	0.236	0.001	0.688	0.218	0.002	0.776	0.174	0.003	0.974

$\mathbb{FDE}$ (12) has a unique positive solution in $\mathcal{Q}_{\mathrm{j}} \times \mathcal{Q}_{\mathrm{j}}$, where $\mathrm{i}_1(\varrho) = \varrho^{9/5}$, $\mathrm{i}_2(\varrho) = \varrho^{6/5}$, $\varrho \in \overline{\mathsf{Y}}$.

Example 4.2. Consider the 2D-𝔽𝔻𝔼s of the form,

$$\begin{cases} \Phi_p\left(\mathcal{D}_{0^+}^{23/10}\mathrm{w}_1(\varrho)\right) + \left(\mathrm{w}_1(\varrho)\right)^{1/4} + \frac{2\varrho+1}{1+2{\mathrm{w}_1}^{2/3}(\varrho)} + a' = 0, \\ \Phi_p\left(\mathcal{D}_{0^+}^{\varsigma_2}\mathrm{w}_2(\varrho)\right) + \left(\mathrm{w}_2(\varrho)\right)^{1/7} + \frac{5\varrho}{1+{\mathrm{w}_2}^{2/7}(\varrho)} + b' = 0, \\ \mathrm{w}_i(0) = 0,\ \mathrm{w}_i'(1) = 0, \quad i = 1, 2, \\ \mathrm{w}_1(1) = \mathcal{I}_{0^+}^{1/2} Z_1\big(\upsilon, \mathrm{w}_1(\upsilon)\big)\Big|_{p=1/2} \\ \qquad = \int_0^{1/2} \frac{\left(\frac{1}{2}-\upsilon\right)^{-1/2}}{\Gamma\left(\frac{1}{2}\right)} Z_1\big(\upsilon, \mathrm{w}_1(\upsilon)\big)\,\mathrm{d}\upsilon, \\ \mathrm{w}_2(1) = \mathcal{I}_{0^+}^{1/4} Z_2\big(\upsilon, \mathrm{w}_2(\upsilon)\big)\Big|_{p=1/3} \\ \qquad = \int_0^{1/3} \frac{\left(\frac{1}{3}-\upsilon\right)^{-3/4}}{\Gamma\left(\frac{1}{4}\right)} Z_2\big(\upsilon, \mathrm{w}_2(\upsilon)\big)\,\mathrm{d}\upsilon, \end{cases} \tag{18}$$

for

$$\varsigma_2 \in \left\{\tfrac{28}{13}, \tfrac{5}{4}, \tfrac{12}{5}\right\} \subset (2, 3].$$

Clearly, $\varsigma_1 = \frac{23}{10} \in (2,3]$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4} \in \mathsf{Y}$, $\eta_1 = \frac{1}{2}$, $\eta_2 = \frac{1}{3} \in \mathsf{Y}$, where $a', b' > 0$ is a constant, $V_1(\varrho, \mathrm{w}_1) = {\mathrm{w}_1}^{1/4} + a'$, $V_2(\varrho, \mathrm{w}_2) = {\mathrm{w}_2}^{1/7} + b'$, and

$$Z_1(\varrho, \mathrm{w}_1) = \tfrac{2\varrho+1}{1+2{\mathrm{w}_1}^{2/3}}, \qquad Z_2(\varrho, \mathrm{w}_2) = \tfrac{5\varrho}{1+{\mathrm{w}_1}^{2/7}}.$$

$V_1, V_2 : \overline{\mathsf{Y}} \times \mathsf{Y}_0 \to \mathsf{Y}_0$ is continuous and increasing with respect to the second argument, $V_1(\varrho, 0) = a' > 0, V_2(\varrho, 0) = b' > 0$. $Z_1, Z_2 : \overline{\mathsf{Y}} \times \mathsf{Y}_0 \to \mathsf{Y}_0$ is continuous and decreasing with respect to the second argument, $Z_1(\varrho, 1) = \frac{2\varrho+1}{3} \neq 0$, $Z_2(\varrho, 1) = \frac{5\varrho}{2} \neq 0$. In addition, let $\varphi_{11}(\lambda_1) = {\lambda_1}^{1/4}$, $\varphi_{12}(\lambda_1) = {\lambda_1}^{2/3}$, $\varphi_{21}(\lambda_2) = {\lambda_2}^{1/7}$, $\varphi_{22}(\lambda_2) = {\lambda_2}^{2/7}$. Then $\varphi_{1i}(\lambda_1), \varphi_{2i}(\lambda_2) \in (\lambda_i, 1)$ for $\lambda_i \in \mathsf{Y}$, $i = 1, 2$, we have,

$$\begin{aligned} V_1(\varrho, \lambda_1 \mathrm{w}_1) &= (\lambda_1 \mathrm{w}_1)^{1/4} + a' = {\lambda_1}^{1/4}{\mathrm{w}_1}^{1/4} + a' \\ &\geq {\lambda_1}^{1/4}{\mathrm{w}_1}^{1/4} + {\lambda_1}^{1/4}a' \\ &= {\lambda_1}^{1/4} V_1(\varrho, \mathrm{w}_1) = \varphi_{11}(\lambda_1) V_1(\varrho, \mathrm{w}_1), \\ Z_1(\varrho, \lambda_1 \mathrm{w}_1) &= \tfrac{2\varrho+1}{1+2\lambda_1{\mathrm{w}_1}^{2/3}} \leq \tfrac{2\varrho+1}{{\lambda_1}^{2/3}(1+2{\mathrm{w}_1}^{2/3})} = \tfrac{Z_1(\varrho, \mathrm{w}_1)}{\varphi_{12}(\lambda_1)}, \\ V_2(\varrho, \lambda_2 \mathrm{w}_2) &= (\lambda_2 \mathrm{w}_2)^{1/7} + b' = {\lambda_2}^{1/7}{\mathrm{w}_2}^{1/7} + b' \\ &\geq {\lambda_2}^{1/7}{\mathrm{w}_2}^{1/7} + {\lambda_2}^{1/7}b' \end{aligned}$$

$$= {\lambda_2}^{1/7} V_2(\varrho, \mathrm{w}_2) = \varphi_{21}(\lambda_2) V_2(\varrho, \mathrm{w}_2),$$
$$Z_2(\varrho, \lambda_2 \mathrm{w}_2) = \frac{5\varrho}{1+\lambda_1 {\mathrm{w}_1}^{2/7}} \leq \frac{5\varrho}{{\lambda_2}^{2/7}(1+{\mathrm{w}_1}^{2/7})} = \frac{Z_2(\varrho,\mathrm{w}_2)}{\varphi_{22}(\lambda_2)}.$$

So all the condition of Theorem 3.4 are satisfied. Therefore, 2D-$\mathbb{FDE}$s (18) has a unique positive solution in $\mathcal{Q}_{\mathrm{j}} \times \mathcal{Q}_{\mathrm{j}}$, where $\mathrm{i}_1(\varrho) = \varrho^{13/10}, \mathrm{i}_2(\varrho) = \varrho^{15/13}$, $\varrho \in \overline{\Upsilon}$.

5 Conclusion

In today's world, the need to study natural phenomena and modeling for these types of phenomena has increased significantly. In this regard, fractional operators help researchers a lot for modeling. Fractional differential problems can solve complex and different models. In this paper, we investigated a $\mathbb{FDE}$ with derivative and integral boundary conditions with p-Laplacian operator. We used some tools like conic metric space and normal cone. We used the fixed point theorem in Banach and conic space. In the end, we presented two examples to express our main results so that the reader can easily communicate with the article and the results of the article become more concrete for the reader.

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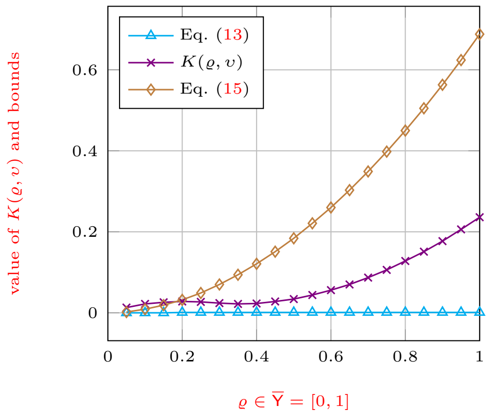
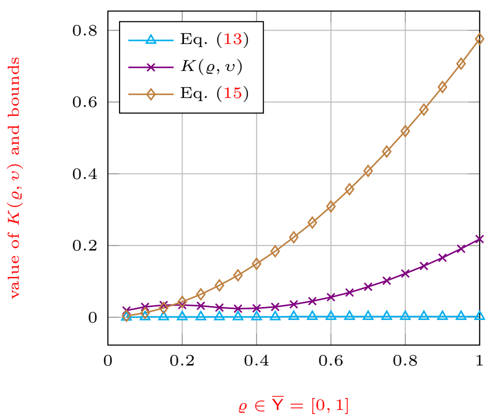
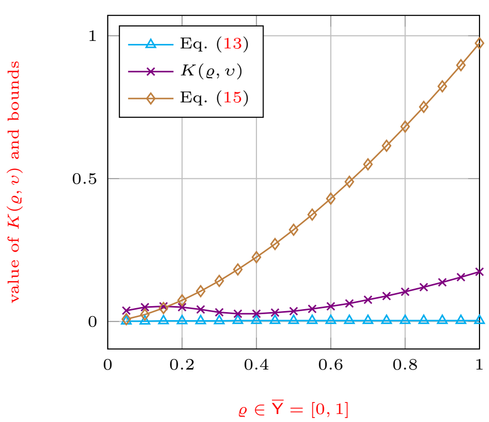


(a) $\varsigma_1 = {}^{29}/_{10}$



(b) $\varsigma_1 = {}^{14}/_{5}$



(c) $\varsigma_1 = {}^{13}/_{5}$

Figure 1: The curves of $K_1(\varrho, \upsilon)$ and Ineqs. (13), (15) for 2D-$\mathbb{FDE}$s (12) with three values of derivative order ς_1 in Example 4.1 for $\varrho \in \overline{\Upsilon}$ when $\upsilon = 0.29$.

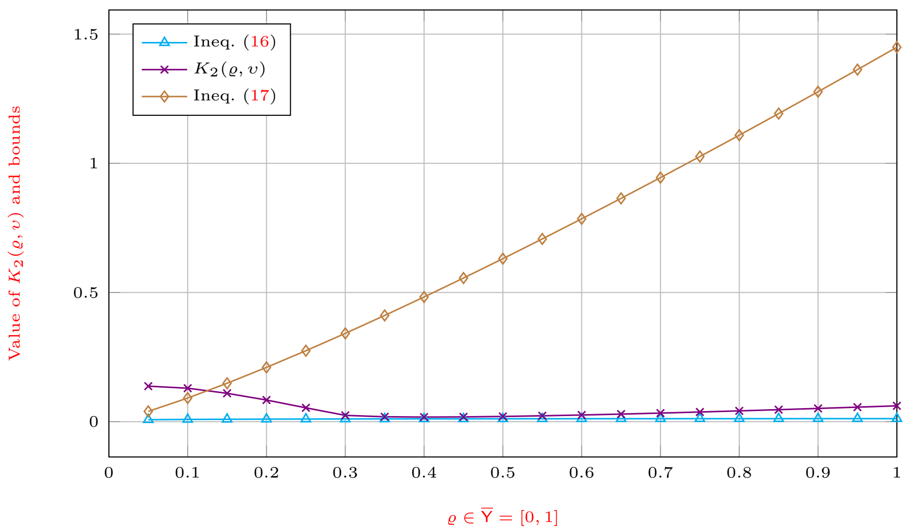


Figure 2: Representation of $K_2(\varrho, \upsilon)$ and Ineqs. (16), (17) for 2D-$\mathbb{FDE}$s (12) with derivative order $\varsigma_2 = \frac{11}{5}$ in Example 4.1 for $\varrho \in \overline{\Upsilon}$ when $\upsilon = 0.29$.