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Investigating some Properties of G-frames in Krein Spaces

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Abstract. This paper introduces a new framework for g-frames in Krein spaces, aiming to extend frame theory to settings with indefinite inner products. The definition is designed to capture the geometric and operator-theoretic features specific to Krein spaces. We show that every g-frame in a Krein space is also a g-frame in the associated Hilbert space. However, the reverse does not hold unless certain conditions are met: the span of the J-adjoint of positive operators must be uniformly J-positive, and the span of the J-adjoint of negative operators must be uniformly J-negative. We define the J-g-frame operator for g-frames in Krein spaces and study its properties, including invertibility and boundedness. A J-g-dual sequence is also introduced, which is always a g-frame in the associated Hilbert space, but not necessarily in the Krein space itself. Finally, the relationship between a g-frame in a Krein space and the sequence induced by it in the associated Hilbert space is examined.

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1 Introduction

The idea of frames in Hilbert spaces was first introduced by Duffin and Schaeffer in 1952, while studying problems related to nonharmonic Fourier series. Later, in 1986, Daubechies, Grossman, and Meyer reintroduced frames as a generalization of orthonormal bases. This new definition made frames widely used in many areas, such as signal processing, image and data compression, and sampling theory in applied mathematics, computer science, and engineering (see [5, 6, 9, 12, 14, 16, 26]).

With the rise of computational and data-driven demands, several extensions of frame theory have been developed, including fusion frames, controlled frames, weaving frames, and g-frames. Sun introduced g-frames as a unified framework that brings together classical frames, oblique frames, pseudo-frames, and fusion frames—often called operator-valued frames. Controlled frames, on the other hand, were designed to improve the efficiency of algorithms used to invert frame operators.

Further operator-theoretic methods, especially those developed by Kaftal and collaborators, have advanced the study of multiwavelets and multiframes. For more details on g-frames, see [25, 19, 18, 4]. In recent years, many generalizations of frames and g-frames have been proposed (see [10, 1, 13, 8, 3, 23]).

In 1969, Mark Grigovich Krein, a Soviet mathematician and a key figure in the Soviet Union School studying functional analysis, introduced a new space by placing a special feature on Hilbert space, examining its properties, and naming it Krein Space. A Krein space is a generalization of a Hilbert space. While Hilbert spaces are more commonly studied in mathematical analysis, Krein spaces have significant applications in physics and engineering. For more applications of Krein space, refer to references [15, 24]. The theory of frames in Krein space is given in [7, 11, 21].

The arrangement of the paper is as follows: In Section 2, Krein spaces and basic definitions are introduced. Notions of g-frames in Hilbert spaces and frames in Krein spaces are also recalled. In Section 3, g-frames are introduced in a Krein space, and it is shown that although a g-frame in a Krein space is a g-frame in the associated Hilbert space, the converse is not necessarily true. It is also proved that under the conditions of Proposition 3.6, the converse is true. In Section 4, the concept of the J-g-frame operator for a g-frame in a Krein space is introduced, and invertibility, boundedness, and some other properties of this operator are discussed. In Section 5, a J-g-dual sequence for a g-frame in a Krein space is introduced. This sequence is a generalization of the dual frame sequence in Hilbert space and is used for studying the completeness of g-frames in Krein spaces. It is demonstrated that a J-g-dual sequence of a g-frame in a Krein space is a g-frame in the associated Hilbert space, but not necessarily a g-frame in the Krein space. In Section 6, a combination of g-frames in Krein spaces with operators is investigated. It is proved that the combination of a g-frame with a bounded operator is still a g-frame. Moreover, the relationship between a g-frame in a Krein space and the sequence induced by the g-frame in the associated Hilbert space is studied.

2 Preliminaries

In this section, Krein spaces and basic definitions are presented. Additionally, the concepts of g-frames in Hilbert spaces and frames in Krein spaces are restated.

We follow the terminology and notation in [17]. Let K be a vector space over the field of complex numbers $\mathbb{C}$. The mapping $[\cdot,\cdot] : K \times K \longrightarrow \mathbb{C}$ is called a sesquilinear Hermitian form if the following conditions hold:

i) $[\lambda_1 x_1 + \lambda_2 x_2, y] = [\lambda_1 x_1, y] + [\lambda_2 x_2, y]$, $(x_1, x_2, y \in K, \quad \lambda_1, \lambda_2 \in \mathbb{C})$,

ii) $[x, y] = \overline{[y, x]}, \quad (x, y \in K)$.

Since $[x, x]$ is a real number for all $x \in K$, the mapping $[\cdot,\cdot]$ is called an indefinite metric. Also if $[x, x] \geq 0$ for all $x \in K$, then $[\cdot,\cdot]$ is inner product. A vector $x \in K$ is positive, negative, or neutral whether $[x, x] > 0, [x, x] < 0$, or $[x, x] = 0$, respectively. The sets of all positive, negative, and neutral vectors of K are denoted respectively by $\mathcal{P}^{++}(K) \equiv \mathcal{P}^{++}, \mathcal{P}^{--}(K) \equiv \mathcal{P}^{--}, \, and \, \mathcal{P}^0(K) \equiv \mathcal{P}^0$. It is also denoted by $\mathcal{P}^+(K) \equiv \mathcal{P}^+ \equiv \mathcal{P}^{++} \cup \mathcal{P}^0$ and $\mathcal{P}^-(K) \equiv \mathcal{P}^- \equiv \mathcal{P}^{--} \cup \mathcal{P}^0$ the sets of all non-negative and non-positive vectors in K, respectively. A subspace $\mathcal{L} \subseteq K$ is non-negative, non-positive, or neutral if $\mathcal{L} \subseteq \mathcal{P}^+, \mathcal{L} \subseteq \mathcal{P}^-, or \mathcal{L} \subseteq \mathcal{P}^0$, respectively. A subspace $\mathcal{L} \subseteq K$ is a positive (or negative) subspace if $\mathcal{L} \subseteq \mathcal{P}^{++} \cup \{0\}$ $(or \mathcal{L} \subseteq \mathcal{P}^{--} \cup \{0\})$. Vectors $x, y (\in K)$

are [.,.]-orthogonal vectors if $[x,y]=0$, and denoted by $x[\perp]y$. Sets $M,N(\subseteq K)$ are said to be [.,.]-orthogonal whenever $x[\perp]y$ for all $x\in M$ and $y\in N$ and denoted by $M[\perp]N(\Leftrightarrow N[\perp]M)$.

Definition 2.1. *Let K be a linear space with the indefinite metric [.,.], and admit decomposition into the direct sum of a positive (K^+) and a negative (K^-) subspace: $K=K^+\dot{+}K^-$. Now if $K^+[\perp]K^-$, then $K=K^+[\dot{+}]K^-$ where the symbol $[\dot{+}]$ denotes [.,.]-orthogonal direct sum. This decomposition is called a canonical decomposition of the space K [17].*

In this definition the subspace K^+ with the inner product of (a positive-definite form) $[x,y](x,y\in K^+)$ and the subspace K^- with the inner product of $-[x,y](x,y\in K^-)$ are pre-Hilbert spaces.

Definition 2.2. *A space K with an indefinite-metric $[.,.]$, which admits a canonical decomposition in which K^+ and K^- are complete, i.e. Hilbert spaces relative to the norms $\|x\|=[x,x]^{\frac{1}{2}}(x\in K^+)$ and $\|x\|=(-[x,x])^{\frac{1}{2}}(x\in K^-)$ respectively, is called a Krein space.[1]*

A canonical decomposition $K=K^+[\dot{+}]K^-$ makes a inner product on Krein space $(K,[.,.])$, indeed for all $x,y\in K$

$$<x,y>=[x^+,y^+]-[x^-,y^-],\quad x=x^++x^-,\quad y=y^++y^-;$$
$$x^+,y^+\in K^+;\quad x^-,y^-\in K^-.$$

Here $K^+\perp K^-$, where the symbol $\perp$ denotes, as usual, orthogonality relative to the inner product $<.,.>$.

Proposition 2.3. *[17] The Krein space K is a Hilbert space relative to the norm*

$$\|x\|=<x,x>^{\frac{1}{2}},\quad (x\in K).$$

In this paper, all Krein spaces with respect to the norm $\|x\|=<x,x>^{\frac{1}{2}}$ $(x\in K)$ are Hilbert spaces, which are called associated Hilbert spaces. The norm of Krein spaces is $\|x\|=<x,x>^{\frac{1}{2}}$ $(x\in K)$. Two projectors $\mathcal{P}_+$ and $\mathcal{P}_-$ are defined on K projecting onto K^+ and K^-, respectively for $x=x^++x^-$, $\mathcal{P}_\pm x=x^\pm$. The projectors $\mathcal{P}_+$ and $\mathcal{P}_-$ are named canonical projectors.

The linear operator $J_K:K\to K$ defined by $J_K=\mathcal{P}_+-\mathcal{P}_-$, is called the fundamental symmetry of the Krein space K.

The fundamental symmetry J_K has the following properties:

- $J_K^* = J_K$,
- $J_K^2 = id_K$,
- $J_K^{-1} = J_K^*$,
- $< x, y >= [J_K x, y]$ *and* $[x, y] =< J_K x, y >$, $(x, y \in K)$.

Let $(K, [.,.])$ be a Krein space with fundamental symmetry J_K. A vector $x \in K$ is called J-positive if $[x, x] > 0$. A subspace $\mathcal{L} \subseteq K$ is called J-positive if every nonzero $x \in \mathcal{L}$, is a J-positive vector. A subspace $\mathcal{L}$ of K is called uniformly J-positive if there exists $\alpha > 0$ such that $[x, x] \geq \alpha \|x\|^2$, for every $x \in \mathcal{L}$ where $\|.\|$ stands for the norm of the associated Hilbert space $(K, < .,. >)$. J-nonnegative, J-neutral, J-negative, J-nonpositive, and uniformly J-negative vectors and subspaces are defined analogously. The J-orthogonal complement of a set $\mathcal{L}$ in K is the set

$$\mathcal{L}^{[\perp]} = \{x \in K : \quad x[\perp]\mathcal{L}\}.$$

A subspace $\mathcal{L}$ in a Krein space K is called projectively complete if $\mathcal{L} + \mathcal{L}^{[\perp]} = K$ [11, 17].

Remark 2.4. *Let $\mathcal{L}_+$ be a closed uniformly J-positive subspace of a Krein space $(K, [.,.])$. Then $(\mathcal{L}_+, [.,.])$ is a Hilbert space. In fact, the form $[.,.]$ and $< .,. >$ are equivalent, since $\alpha\|f\|^2 \leq [f, f] \leq \|f\|^2$, for every $f \in \mathcal{L}_+$. Analogously, if $\mathcal{L}_-$ is a closed uniformly J-negative subspace of $(K, [.,.])$, then $(\mathcal{L}_-, -[.,.])$ is a Hilbert space [11].*

Definition 2.5. *Let $(K, [.,.]_K)$ be a Krein space with fundamental symmetry J_K. Then $F = \{f_i\}_{i \in I} \subseteq K$ is called the Bessel family in K, if there exist constant $B > 0$ such that*

$$\sum_{i \in I} | [J_K f, f_i] |^2 \leq B\|f\|^2, \quad (f \in K).$$

Let $(K, [.,.]_K)$ and $(H, [.,.]_H)$ be Krein spaces and $(K, < .,. >_K)$ and $(H, < .,. >_H)$ be their associated Hilbert spaces. For $T \in L(K, H)$, the J-adjoint operator of T is defined by $T^\sharp = J_K T^* J_H$, where T^* is the adjoint operator of T and J_K, J_H are the fundamental symmetries for K and H, respectively.

Let K be a Krein space with fundamental symmetry J_K and $F = \{f_i\}_{i\in I}$ be a Bessel family in K. Also, let $T : \ell_2(I) \longrightarrow K$ be the synthesis operator with

$$T(\{c_i\}_{i\in I}) = \sum_{i\in I} c_i f_i.$$

Consider $I_+ = \{i \in I : [f_i, f_i] \geq 0\}$, $I_- = \{i \in I : [f_i, f_i] < 0\}$ and so $\ell_2(I) = \ell_2(I_+) \oplus \ell_2(I_-)$, where $\oplus$ stands for the usual Hilbert space direct orthogonal sum of them. Now, if $P_\pm$ is the orthogonal projection onto $\ell_2(I_\pm)$, $T_\pm = TP_\pm$ and $M_\pm = \overline{span\{f_i : i \in I_\pm\}}$, then $span\{f_i : i \in I_\pm\} \subseteq R(T_\pm) \subseteq M_\pm$ and $R(T) = R(T_+) + R(T_-)$[11].

Definition 2.6. *According to the above assumptions, the Bessel family $F = \{f_i\}_{i\in I}$ is called a J-frame for K if $R(T_+)$ is a maximal(with respect to the inclusion) uniformly J-positive subspace of K and $R(T_-)$ is a maximal uniformly J-negative subspace of K[11].*

Definition 2.7. *Let E and W be two Hilbert spaces and let $\{W_i\}_{i\in I}$ be a sequence of closed subspaces of W, where I is a countable index set. A sequence of bounded linear operators $\{\Lambda_i : E \to W_i\}_{i\in I}$ is called a g-frame for E with respect to the subspaces $\{W_i\}_{i\in I}$, if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in E). \tag{1}$$

The numbers A and B in this definition are named g-frame bounds [5]. While the right-hand inequality of (1) holds, the sequence $\{\Lambda_i\}_{i\in I}$ is termed a g-Bessel sequence for E with respect to $\{W_i\}_{i\in I}$ with bound B.

3 g-frames in krein spaces

In this section, the concept of g-frames is extended to Krein spaces. It is shown that a g-frame in a Krein space is also a g-frame in the associated Hilbert space, but the converse is not necessarily true. This means that a g-frame in a Krein space and in the associated Hilbert space are related but not identical. Under conditions of Proposition 3.6, a g-frame in a Hilbert space is a g-frame in the corresponding Krein space as well.

In the rest of this paper, $(K, [.,.]_K)$ and $(H, [.,.]_H)$ are two Krein spaces and $\{H_i\}_{i\in I}$ is a sequence of projectively complete subspaces of H. $B(K, H_i)$ is the collection of all bounded linear operators from K into H_i. The sequence space

$$\ell^2(\{H_i\}_{i\in I})=\{\{f_i\}_{i\in I} : f_i \in H_i \text{ for all } i \in I \text{ and } \sum_{i\in I} \|f_i\|^2 < \infty\},$$

with the inner product

$$< \{f_i\}_{i\in I}, \{g_i\}_{i\in I} >= \sum_{i\in I} < f_i, g_i >, \quad \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I}),$$

is a complex Hilbert space.

Definition 3.1. *A sequence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is called a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$ if there is a positive constant B such that*

$$\sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in K).$$

As an immediate consequence, any J-g-Bessel sequence is a g-Bessel sequence.

Let P_H^{++} be the set of all positive subspaces of H, P_H^{--} the set of all negative subspaces of H, and $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$, where $H_i \in P_H^{++} \cup P_H^{--}$ is projectively complete, for all $i \in I$. Also, let $I_+ = \{i \in I : [f_i, f_i]_H > 0$ for all $f_i \in H_i\backslash\{0\}\}$ and $I_- = \{i \in I : [f_i, f_i]_H < 0$ for all $f_i \in H_i\backslash\{0\}\}$. For $i \in I_+ (i \in I_-)$, the fundamental symmetry J_{H_i} is (the negative of) the identity operator. Now, consider the orthogonal decomposition of $\ell^2(\{H_i\}_{i\in I})$ by

$$\ell^2(\{H_i\}_{i\in I}) = \ell^2(\{H_i\}_{i\in I_+}) \oplus \ell^2(\{H_i\}_{i\in I_-}),$$

and denoted by $P_\pm$ the orthogonal projection onto $\ell^2(\{H_i\}_{i\in I_\pm})$. Also, let $Q : \ell^2(\{H_i\}_{i\in I}) \to K$, defined by

$$Q(f) = \sum_{i\in I} \Lambda_i^{\sharp} f_i, \quad f = \{f_i\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I}),$$

be the J-synthesis operator for the J-g-Bessel sequence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, and let $Q_\pm = QP_\pm$. If $M_\pm = \overline{span\{\Lambda_i^{\sharp} H_i : \quad i \in I_\pm\}}$, then

$$span\{\Lambda_i^{\sharp} H_i : \quad i \in I_\pm\} \subseteq R(Q_\pm) \subseteq \overline{span\{\Lambda_i^{\sharp} H_i : \quad i \in I_\pm\}}.$$

Notice that

$$R(Q) = R(Q_+) + R(Q_-). \tag{2}$$

Definition 3.2. *The J-g-Bessel sequence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ for K with respect to $\{H_i\}_{i\in I}$, is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$, if $R(Q_+)$ is a maximal uniformly J-positive subspace of K and $R(Q_-)$ is a maximal uniformly J-negative subspace of K.*

Example 3.3. Let $(K, [\ ,\ ])$ be a Krein space, M_+(*resp.* M_-) be a maximal uniformly J-positive (resp. maximal uniformly J-negative) subspace of K, and $F_\pm = \{f_i\}_{i\in I_\pm}$ be a frame for the Hilbert space $(M_\pm, \pm[\ ,\ ])$. Consider the Krein space $H = \mathbb{C} \oplus \mathbb{C}$ with sesquilinear form

$$[(x_1, x_2), (y_1, y_2)] = x_1\overline{y_1} - x_2\overline{y_2}.$$

For $i \in I_+$, consider the operator $\Lambda_i : K \longrightarrow \mathbb{C} \oplus \{0\}$, defined by

$$\Lambda_i(f) = ([f, f_i], 0),$$

and for $i \in I_-$, consider the operator $\Lambda_i : K \longrightarrow \{0\} \oplus \mathbb{C}$, defined by

$$\Lambda_i(f) = (0, -[f, f_i]).$$

Since $F_+ \cup F_- = \{f_i\}_{i\in I}$ is a Bessel sequence in K [11, Example 3.4], there exists $B > 0$ such that

$$\sum_{i\in I} |[J_K f, f_i]|^2 \leq B\|f\|^2, \quad (f \in K).$$

So

$$\sum_{i\in I} |[J_K^2 f, f_i]|^2 \leq B\|J_K f\|^2 \leq B\|f\|^2, \quad (f \in K),$$

and therefore,

$$\sum_{i\in I} \|\Lambda_i(f)\|^2 \leq B\|f\|^2, \quad (f \in K).$$

So $\{\Lambda_i\}_{i\in I}$ is a g-Bessel sequence for K with respect to $\{\mathbb{C}\oplus\{0\}\}_{i\in I_+} \cup \{\{0\}\oplus\mathbb{C}\}_{i\in I_-}$. Furthermore, $\mathbb{C}\oplus\{0\} \in P_H^+$ and $\{0\}\oplus\mathbb{C} \in P_H^{--}$ are projectively complete, and $R(Q_+) = M_+$ and $R(Q_-) = M_-$. So $\{\Lambda_i \in B(K, \mathbb{C}\oplus\{0\})\}_{i\in I_+} \cup \{\Lambda_i \in B(K, \{0\}\oplus\mathbb{C})\}_{i\in I_-}$ is a J-g-frame for K with respect to $\{\mathbb{C}\oplus\{0\}\}_{i\in I_+} \cup \{\{0\}\oplus\mathbb{C}\}_{i\in I_-}$.

Proposition 3.4. *If $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$, then $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-frame for K with respect to $\{H_i\}_{i\in I}$.*

Proof. By(2) and [2, Corollary1.5.2], Q is surjective. Let $\Theta : \ell^2(\{H_i\}_{i\in I}) \to K$ be the operator given by

$$\Theta(f) = \sum_{i\in I} \Lambda_i^*(f_i), \quad f = \{f_i\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I}).$$

If $g \in K$, since Q is surjective, there exists $\{f_i\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I})$ such that

$$\sum_{i\in I} \Lambda_i^\sharp(f_i) = g.$$

On the other hand, $\Lambda_i^\sharp = J_K \Lambda_i^* J_{H_i}$. So

$$g = \sum_{i\in I} J_K \Lambda_i^* J_{H_i}(f_i) = J_K \sum_{i\in I} \Lambda_i^* J_{H_i}(f_i) = J_K \Theta\{J_{H_i} f_i\}_{i\in I}.$$

Therefore

$$J_K(g) = \Theta\{J_{H_i} f_i\}_{i\in I}.$$

Now, since J_K is surjective and $\|J_{H_i}\| = 1$ for all $i \in I$, then Θ is surjective. So by [20, Proposition 2.6], $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-frame for K with respect to $\{H_i\}_{i\in I}$. $\square$

The following example shows that the converse of the above proposition is not true in general.

Example 3.5. Consider the Krein space $\mathbb{C}^3$ with the sesquilinear form

$$[(x_1, x_2, x_3), (y_1, y_2, y_3)] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3},$$

and fundamental symmetry $J_{\mathbb{C}^3} : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$(a, b, c) \mapsto (a, b, -c), \quad a, b, c \in \mathbb{C}.$$

Let $\mathbb{C}$ be the Krien space with the sesquilinear form $[x, y] = x\overline{y}$ and let $J_{\mathbb{C}} = id$. Now, if $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$ and $f_3 = (0, 0, 1)$, then $\{\Lambda_{f_1}, \Lambda_{f_2}, \Lambda_{f_3}\}$ is a g-frame for $\mathbb{C}^3$ with respect to $\mathbb{C}$, where $\Lambda_{f_i} : \mathbb{C}^3 \to \mathbb{C}$ is the operator given by

$$\Lambda_{f_i}(f) = [J_{\mathbb{C}^3} f, f_i],$$

for $i = 1, 2, 3$. On the other hand, $R(Q_+) = \mathbb{C}^3$ and $[f_3, f_3] < 0$. So $R(Q_+)$ is not uniformly J-positive and it follows that $\{\Lambda_{f_1}, \Lambda_{f_2}, \Lambda_{f_3}\}$ is not a J-g-frame for $\mathbb{C}^3$ with respect to $\mathbb{C}$.

But the important point is that, by placing a condition on $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, every g-frame becomes a J-g-frame.

Proposition 3.6. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a g-frame, where $H_i (i \in I)$ are projectively complete subspaces of H. If $span\{\Lambda_i^\sharp H_i : \ i \in I_+\}$ be uniformly J-positive and $span\{\Lambda_i^\sharp H_i : \ i \in I_-\}$ be uniformly J-negative, then $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$.*

Proof. By the assumption, $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-frame. Hence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$. Also, $R(\Theta) = K$ where Θ is the synthesis operator for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, so $R(Q) = K$ and therefore $R(Q_+) + R(Q_-) = K$. Since $span\{\Lambda_i^\sharp H_i : i \in I_+\}$ is uniformly J-positive, $\overline{span\{\Lambda_i^\sharp H_i : \ i \in I_+\}}$ and consequently $R(Q_+)$, is uniformly J-positive. Analogously $R(Q_-)$ is uniformly J-negative. On the other hand, $R(Q_+) + R(Q_-) = K$ and $R(Q_+)$ is positive and $R(Q_-)$ is negative. Therefore, by [17, 1.25], $R(Q_+)\big(R(Q_-)\big)$ is maximal uniformly J-positive (maximal uniformly J-negative.)

□

4 Relation of the operators of g-frames and J-g-frames

In this section, we define the concept of the J-g-frame operator for a g-frame in a Krein space. The operator's invertibility, boundedness, and other key properties are examined. Moreover, the relationship between g-frame operators in Krein spaces and their counterparts in Hilbert spaces is investigated.

Let $J_2 : \ell^2(\{H_i\}_{i\in I}) \to \ell^2(\{H_i\}_{i\in I})$ be defined by

$$J_2(\{f_i\}_{i\in I}) = \{J_{H_i} f_i\}_{i\in I},$$

where $J_{H_i} = id$, for $i \in I_+$ and $J_{H_i} = -id$, for $i \in I_-$. Since $\|J_{H_i}\| = 1$, J_2 is well-defined. Therefore $\ell^2(\{H_i\}_{i\in I})$ with the fundamental symmetry J_2 forms a Krein space.

Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$. Then the operator $\Theta : \ell^2(\{H_i\}_{i\in I}) \to K$ given by

$$\Theta(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i,$$

is the synthesis operator for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, and the operator $\Theta^* : K \to \ell^2(\{H_i\}_{i\in I})$ given by

$$\Theta^*(f) = \{\Lambda_i f\}_{i\in I},$$

is the analysis operator of $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$.

Now, let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$. Then the operator $Q : \ell^2(\{H_i\}_{i\in I}) \to K$ defined by

$$Q(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^\sharp f_i,$$

is called the J-synthesis operator for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ and it's J-adjoint $Q^\sharp : K \to \ell^2(\{H_i\}_{i\in I})$ satisfying

$$Q^\sharp(f) = \{\Lambda_i f\}_{i\in I},$$

is called the J-analysis operator of $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, that is the same as the analysis operator, i.e., $Q^\sharp = \Theta^*$.

Lemma 4.1. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$. Then, according to the above assumptions, $Q = J_k \Theta J_2$.*

Proof. If $\{f_i\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I})$, then

$$\begin{aligned} Q(\{f_i\}_{i\in I}) &= \sum_{i\in I} \Lambda_i^\sharp f_i \\ &= \sum_{i\in I} J_K \Lambda_i^* J_{H_i} f_i \\ &= J_K \sum_{i\in I} \Lambda_i^* J_{H_i} f_i \\ &= J_K \Theta(\{J_{H_i} f_i\}_{i\in I}) \\ &= J_K \Theta J_2(\{f_i\}_{i\in I}). \end{aligned}$$

□

If $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-frame for K with respect to $\{H_i\}_{i\in I}$, then the g-frame operator $S : K \to K$ for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is defined by

$$S(f) = \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

Definition 4.2. *If $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$, then the J-g-frame operator $S_J : K \to K$ for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is defined by*

$$S_J(f) = \sum_{i\in I} \Lambda_i^\sharp \Lambda_i f.$$

We set $Q_+^\sharp = P_+ Q^\sharp$ and $Q_-^\sharp = P_- Q^\sharp$.

Proposition 4.3. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-frame for K with respect to $\{H_i\}_{i\in I}$ with the J-synthesis operator $Q \in B(\ell^2(\{H_i\}_{i\in I}), K)$. Then for a J-g-frame operator S_J the following conditions hold:*

1. *$S_J = QQ^\sharp$ and $S_J = S_{J+} - S_{J-}$, where $S_{J+} = Q_+ Q_+^\sharp$ and $S_{J-} = -Q_- Q_-^\sharp$.*
2. *S_{J+} and S_{J-} are J-positive.*
3. *$R(S_{J\pm}) = M_\pm$ and $N(S_{J\pm}) = M_\pm^{[\perp]}$ where $M_+ = \overline{span\{\Lambda_i^\sharp H_i : \ i \in I_+\}}$ and $M_- = \overline{span\{\Lambda_i^\sharp H_i : \ i \in I_-\}}$.*

4. *S_J is J-self-adjoint and invertible.*
5. *S_J is bounded.*
6. *S_J^{-1} is bounded.*

Proof.

1. is obvious.

2. $$[S_{J+}f,f]=[Q_+Q_+^{\sharp}f,f]=[\sum_{i\in I_+}\Lambda_i^{\sharp}\Lambda_i f,f]=\sum_{i\in I_+}[\Lambda_i f,\Lambda_i f]\geq 0,$$

   and

   $$[S_{J-}f,f]=[-Q_-Q_-^{\sharp}f,f]=[-\sum_{i\in I_-}\Lambda_i^{\sharp}\Lambda_i f,f]=-\sum_{i\in I_-}[\Lambda_i f,\Lambda_i f]\geq 0.$$

3. Since

   $$S_{J+}=Q_+Q_+^{\sharp}=Q_+J_2Q_+^{*}J_K=Q_+Q_+^{*}J_K,$$

   then
   $$R(S_{J+})=R(Q_+Q_+^{*}J_K)=R(Q_+Q_+^{*})=R(Q_+). \tag{3}$$
   On the other hand $\{\Lambda_i\in B(K,H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. Therefore $R(Q_+)$ is closed. Then by (3), $R(S_{J+})=R(Q_+)=M_+$. It is clear that S_{J+} is J-self-adjoint and so $N(S_{J+})=R(S_{J+})^{[\perp]}=M_+^{[\perp]}$. Analogously, $R(S_{J-})=M_-$ and $N(S_{J-})=M_-^{[\perp]}$.

4. $$S_J^{\sharp}=(QQ^{\sharp})^{\sharp}=Q^{\sharp\sharp}Q^{\sharp}=QQ^{\sharp}=S_J.$$

   Now we prove that S_J is injective. If for $f\in K, S_J(f)=0$, then by part (1), $S_{J+}f=S_{J-}f$. On the other hand,

   $$R(S_{J+})\cap R(S_{J-})\subseteq R(Q_+)\cap R(Q_-)=\{0\},$$

so $S_{J+}f = 0$ *and* $S_{J-}f = 0$, which implies that $\sum_{i\in I_+} \Lambda_i^\sharp \Lambda_i f = 0$ *and* $\sum_{i\in I_-} \Lambda_i^\sharp \Lambda_i f = 0$. Therefore

$$0 = [\sum_{i\in I_+} \Lambda_i^\sharp \Lambda_i f, f] = \sum_{i\in I_+} [\Lambda_i f, \Lambda_i f],$$

and similarly,

$$0 = [\sum_{i\in I_-} \Lambda_i^\sharp \Lambda_i f, f] = \sum_{i\in I_-} [\Lambda_i f, \Lambda_i f].$$

It follows that $\Lambda_i f = 0$ $(i \in I)$, since H_i is positive for all $i \in I_+$ and H_i is negative for all $i \in I_-$. Thus $A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 = 0$ and so $f = 0$. Next we show that S_J is surjective. Since $R(S_J) = S_J(K) = S_J(M_+^{[\perp]} + M_-^{[\perp]})$, according to part (3), $M_\pm^{[\perp]} = N(S_{J\pm})$, therefore $S_J(M_\pm^{[\perp]}) = S_{J\mp}(M_\pm^{[\perp]})$. Again according to part (3), $R(S_J) = S_{J-}(M_+^{[\perp]}) + S_{J+}(M_-^{[\perp]}) = R(S_{J-}) + R(S_{J+}) = M_- + M_+ = K$, therefore S_J is surjective and so invertible.

5. Since

$$\begin{aligned}\|Q\| &= \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} |< Q\{f_i\}_{i\in I}, g > | \\ &= \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} |[Q\{f_i\}_{i\in I}, J_K g]| \\ &= \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} |[\sum_{i\in I} \Lambda_i^\sharp f_i, J_K g]| \\ &= \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} |\sum_{i\in I} [f_i, \Lambda_i J_K g]| \\ &\leq \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} \sum_{i\in I} |[f_i, \Lambda_i J_K g]| \\ &\leq \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} \sum_{i\in I} \|f_i\| \|\Lambda_i J_K g\| \\ &\leq \sup_{\|\{f_i\}_{i\in I}\|=1,\|g\|=1} (\sum_{i\in I} \|f_i\|^2 + \sum_{i\in I} \|\Lambda_i J_K g\|^2) \\ &\leq \sup_{\|g\|=1} (1 + B\|J_K g\|^2) \\ &\leq 1 + B,\end{aligned}$$

where B is a bound for the J-g-Bessel sequence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$. Consequently, S_J is bounded.

6. According to statements in[22, 2,12 (a)], S_J is open and by [22, 2,12 (b)], S_J^{-1} is bounded.

□

5 The dual of J-g-frame

In this section, J-g-dual sequences for g-frames in Krein spaces are introduced. It is shown that any of these sequences in a Krein space is a g-frame in the associate Hilbert space, but it is not necessarily a g-frame in the Krein space.

First, note that for any $f \in K$,

$$f = S_J^{-1} S_J f = S^{-1} \sum_{i\in I} \Lambda_i^\sharp \Lambda_i f = \sum_{i\in I} S^{-1} \Lambda_i^\sharp \Lambda_i f$$

and

$$f = S_J S_J^{-1} f = \sum_{i\in I} \Lambda_i^\sharp \Lambda_i S_J^{-1} f.$$

Therefore, $f = \sum_{i\in I} \tilde{\Lambda}_i^\sharp \Lambda_i f = \sum_{i\in I} \Lambda_i^\sharp \tilde{\Lambda}_i f$, where $\tilde{\Lambda}_i = \Lambda_i S_J^{-1}$ $(i \in I)$.

Proposition 5.1. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. Then $\{\tilde{\Lambda}_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$, where $\tilde{\Lambda}_i = \Lambda_i S_J^{-1}$ $(i \in I)$.*

Proof. First, we show that $\{\tilde{\Lambda}_i \in B(K, H_i)\}_{i\in I}$ is a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$. Since $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-Bessel sequence for K with respect to $\{H_i\}_{i\in I}$, there exists $B > 0$ such that

$$\sum_{i\in I} ||\tilde{\Lambda}_i f\|^2 \le B\|S_J^{-1} f\|^2 \le B\|S_J^{-1}\|^2\|f\|^2 \le BM^2\|f\|^2, \quad (f \in K),$$

where M is a bound for S_J^{-1}. Now, assume that $\tilde{Q}$ is the J-synthesis operator for $\{\tilde{\Lambda}_i \in B(K, H_i)\}_{i\in I}$. By Proposition 4.3, $S_J(M_\mp^{[\perp]}) = M_\pm$ and so $S_J^{-1}(M_+) = M_-^{[\perp]}$. Therefore $R(\tilde{Q}_+) = R(S_J^{-1} Q_+) = M_-^{[\perp]}$. It follows that $R(\tilde{Q}_+)$ is maximal uniformly J-positive. Analogously, $R(\tilde{Q}_-)$ is maximal uniformly J-negative. Finally, $\{\tilde{\Lambda}_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. $\square$

Definition 5.2. *Suppose that $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. A J-g-Bessel sequence $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ for K with respect to $\{H_i\}_{i\in I}$ is a J-g-dual sequence for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$, if $f = \sum_{i\in I} \Gamma_i^\sharp \Lambda_i f = \sum_{i\in I} \Lambda_i^\sharp \Gamma_i f$ $(f \in K)$.*

Every J-g-frame $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ for K with respect to $\{H_i\}_{i\in I}$ has a J-g-dual sequence as $\{\Lambda_i S_J^{-1} \in B(K, H_i)\}_{i\in I}$ for K with respect to $\{H_i\}_{i\in I}$. In the next proposition, it is shown that every J-g-dual sequence is a g-frame.

Proposition 5.3. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-frame for K with respect to $\{H_i\}_{i\in I}$ and $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ be a J-g-dual sequence for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$. Then $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ is a g-frame for K with respect to $\{H_i\}_{i\in I}$.*

Proof. Let Θ be the synthesis operator of a g-Bessel sequence $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$. For $f \in K$, since J_K is surjective, there exists $g \in K$ such that $f = J_K g$. On the other hand,

$$g = \sum_{i\in I} \Gamma_i^\sharp \Lambda_i g = \sum_{i\in I} J_K \Gamma_i^* J_{H_i} \Lambda_i g = J_K \sum_{i\in I} \Gamma_i^* J_{H_i} \Lambda_i g,$$

since $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ is a J-g-dual sequence for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$. So

$$f = J_K g = \sum_{i\in I} \Gamma_i^* J_{H_i} \Lambda_i g.$$

Also $\{J_{H_i} \Lambda_i g\}_{i\in I} \in \ell^2(\{H_i\}_{i\in I})$, Since $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a g-Bessel sequence and $\|J_{H_i}\| = 1$. Therefore Θ is surjective and $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ is a g-frame for K with respect to $\{H_i\}_{i\in I}$. $\square$

The next example shows that $\{\Gamma_i \in B(K, H_i)\}_{i\in I}$ is not necessarily a J-g-frame.

Example 5.4. Let $\{e_1, e_2, e_3\}$ be the orthonormal basis of $\mathbb{C}^3$ equipped with the indefinite inner-product $[x, y] = x_1\overline{y_1} + x_2\overline{y_2} - x_3\overline{y_3}$ where $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in Krein space $\mathbb{C}^3$. Consider the operator $\Lambda_i : \mathbb{C}^3 \longrightarrow \mathbb{C} \oplus \{0\}$ defined by

$$\Lambda_i(f) = ([f, f_i], 0),$$

for $i = 1, 2, 4, 5$, and the operator $\Lambda_i : \mathbb{C}^3 \longrightarrow \{0\} \oplus \mathbb{C}$ defined by

$$\Lambda_i(f) = (0, -[f, f_i]),$$

for $i = 3, 6$ where $\{f_i\}_{i=1}^6 = \{f_1 = e_1, f_2 = e_2, f_3 = e_3, f_4 = e_1, f_5 = e_2, f_6 = e_3\}$. By Example 3.3, $\{\Lambda_i \in B(\mathbb{C}^3, \mathbb{C} \oplus \{0\})\}_{i=1,2,4,5} \cup \{\Lambda_i \in L(\mathbb{C}^3, \{0\} \oplus \mathbb{C})\}_{i=3,6}$ is a J-g-frame for $\mathbb{C}^3$ with respect to $\{\mathbb{C} \oplus \{0\}\}_{i=1,2,4,5} \cup \{\{0\} \oplus \mathbb{C}\}_{i=3,6}$. Also, consider the operator $\Gamma_i : \mathbb{C}^3 \longrightarrow \mathbb{C} \oplus \{0\}$ given by

$$\Gamma_i(f) = ([f, g_i], 0),$$

for $i = 1, 2, 4, 5$ and consider the operator $\Gamma_i : \mathbb{C}^3 \longrightarrow \{0\} \oplus \mathbb{C}$ given by

$$\Gamma_i(f) = (0, -[f, g_i]),$$

for $i = 3,6$ where $\{g_i\}_{i=1}^{6} = \{g_1 = (\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}), g_2 = (1, \frac{1}{2}, -1), g_3 = (0, 0, \frac{1}{2}), g_4 = (\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}), g_5 = (-1, \frac{1}{2}, 1), g_6 = (0, 0, \frac{1}{2})\}$. Since $\{g_1, g_2, g_3, g_4, g_5, g_6\}$ is a Bessel sequence for $\mathbb{C}^3$, there exists $B > 0$ such that for any $f \in \mathbb{C}^3$,

$$\sum_{i=1}^{6} |[J_{\mathbb{C}^3} f, g_i]|^2 \leq B\|f\|^2,$$

so

$$\sum_{i=1}^{6} |[J_{\mathbb{C}^3}^2 f, g_i]|^2 \leq B\|J_{\mathbb{C}^3} f\|^2 \leq B\|f\|^2,$$

and therefore

$$\sum_{i=1}^{6} \|\Gamma_i(f)\|^2 \leq B\|f\|^2.$$

Also

$$\begin{aligned}\sum_{i=1}^{6} \Gamma_i^{\sharp}\Lambda_i f &= \sum_{i=1,2,4,5} \Gamma_i^{\sharp}\Lambda_i f + \sum_{i=3,6} \Gamma_i^{\sharp}\Lambda_i f \\ &= \sum_{i=1,2,4,5} \Gamma_i^{\sharp}([f, f_i], 0) + \sum_{i=3,6} \Gamma_i^{\sharp}(0, -[f, f_i]) \\ &= \sum_{i=1,2,4,5} [f, f_i]g_i + \sum_{i=3,6} -[f, f_i]g_i = f,\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{6} \Lambda_i^{\sharp}\Gamma_i f &= \sum_{i=1,2,4,5} \Lambda_i^{\sharp}\Gamma_i f + \sum_{i=3,6} \Lambda_i^{\sharp}\Gamma_i f \\ &= \sum_{i=1,2,4,5} \Lambda_i^{\sharp}([f, g_i], 0) + \sum_{i=3,6} \Lambda_i^{\sharp}(0, -[f, g_i]) \\ &= \sum_{i=1,2,4,5} [f, g_i]f_i + \sum_{i=3,6} -[f, g_i]f_i = f.\end{aligned}$$

Consequently, $\{\Gamma_i \in B(\mathbb{C}^3, \mathbb{C}\oplus\{0\})\}_{i=1,2,4,5} \cup \{\Gamma_i \in B(\mathbb{C}^3, \{0\}\oplus\mathbb{C})\}_{i=3,6}$ is a J-g-dual sequence for $\{\Lambda_i \in B(\mathbb{C}^3, \mathbb{C}\oplus\{0\})\}_{i=1,2,4,5} \cup \{\Lambda_i \in B(\mathbb{C}^3, \{0\}\oplus$

$\mathbb{C})\}_{i=3,6}$. Since $e_1 = g_1 + g_4,\ e_2 = g_2 + g_5$, and $e_3 = \dfrac{1}{2}g_5 - \dfrac{1}{2}g_2 + g_1 + g_4$, we have $span\{\Gamma_i^\sharp(\mathbb{C} \oplus \{0\}) : \quad i = 1, 2, 4, 5\} = span\{g_1, g_2, g_4, g_5\} = \mathbb{C}^3$. It follows that $\{\Gamma_i \in B(\mathbb{C}^3, \mathbb{C}\oplus\{0\})\}_{i=1,2,4,5} \cup \{\Gamma_i \in B(\mathbb{C}^3, \{0\}\oplus\mathbb{C})\}_{i=3,6}$ is not a J-g-frame.

6 Some properties of J-g-frames

This section examines the relationship between g-frames in Krein spaces and their corresponding sequences in the associated Hilbert spaces. Furthermore, the transformations of g-frames in Krein spaces under the action of bounded operators are analyzed.

Proposition 6.1. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. Then $\{\Lambda_i|_{M_+} \in B(M_+, H_i)\}_{i\in I}$ is a g-frame for $(M_+, [.,.])$ with respect to $\{H_i\}_{i\in I}$ and similarly $\{\Lambda_i|_{M_-} \in B(M_-, H_i)\}_{i\in I}$ is a g-frame for $(M_-, -[.,.])$ with respect to $\{H_i\}_{i\in I}$.*

Proof. $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame and consequently a g-frame for K with respect to $\{H_i\}_{i\in I}$. Then there exists $B \geq A \geq 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in K).$$

M_+ is uniformly J-positive. So

$$A[f, f] \leq A\|f\|^2, \quad (f \in M_+),$$

and there exists $\alpha > 0$ such that

$$B\alpha^{-1}\alpha\|f\|^2 \leq B\alpha^{-1}[f, f], \quad (f \in M_+).$$

Therefore,

$$A[f, f] \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\alpha^{-1}[f, f], \quad (f \in M_+).$$

Similarly

$$-A[f, f] \leq A\|f\|^2, \qquad (f \in M_-),$$

and there exists $\acute{\alpha} > 0$ such that

$$B\acute{\alpha}^{-1}\acute{\alpha}\|f\|^2 \leq -B\acute{\alpha}^{-1}[f,f], \quad (f \in M_-d).$$

Therefore,

$$-A[f,f] \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq -B\acute{\alpha}^{-1}[f,f], \quad (f \in M_-).$$

Now, the proof is complete. $\square$

According to [25], let $\Lambda_i \in B(K, H_i)$ and $\{e_{i,z} : z \in Z_i\}$ be an orthonormal basis for H_i, where $Z_i \subseteq \mathbb{Z}$ and $i \in I$. Then $K_{i,z} : K \to \mathbb{C}$ given by

$$K_{i,z}(f) =< \Lambda_i f, e_{i,z} >,$$

is a bounded linear functional on K. So there exists $u_{i,z} \in K$ such that

$$< f, u_{i,z} >=< \Lambda_i f, e_{i,z} >, \quad (f \in K),$$

and thus,

$$\Lambda_i f = \sum_{z\in Z_i} < \Lambda_i f, e_{i,z} > e_{i,z} = \sum_{z\in Z_i} < f, u_{i,z} > e_{i,z}, \quad (f \in K).$$

Consequently,

$$\sum_{z\in Z_i} | < f, u_{i,z} > |^2 = \|\Lambda_i f\|^2 \leq \|\Lambda_i\|^2 \|f\|^2, \quad (f \in K),$$

thus $\{u_{i,z} : \quad z \in Z_i\}$ is a Bessel sequence for K. Now, for $f \in K$ and $h \in H_i$

$$\begin{aligned} < f, \Lambda_i^* h > &=< \Lambda_i f, h > \\ &=< \sum_{z\in Z_i} < f, u_{i,z} > e_{i,z}, h > \\ &= \sum_{z\in Z_i} < f, u_{i,z} >< e_{i,z}, h > \\ &=< f, \sum_{z\in Z_i} < h, e_{i,z} > u_{i,z} > . \end{aligned}$$

Then

$$\Lambda_i^* h = \sum_{z\in Z_i} < h, e_{i,z} > u_{i,z}, \quad (h \in H_i), \tag{4}$$

and finally

$$u_{i,z} = \Lambda_i^* e_{i,z}.$$

$\{u_{i,z} : i \in I, z \in Z_i\}$ is called the sequence induced by $\{\Lambda_i : i \in I\}$ with respect to $\{e_{i,z} : i \in I, z \in Z_i\}$. In the next proposition, the relationship between $\{u_{i,z} : i \in I, z \in Z_i\}$ and J-g-frames is studied.

Proposition 6.2. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ and $u_{i,z}$ be defined as above. Then $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K if and only if $\{J_K u_{i,z}\}_{i\in I, z\in Z_i}$ is a J-frame for K.*

Proof. If $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-Bessel sequence for K with respect to $\{H_i\}$, then there exists $B > o$ such that

$$\sum_{i\in I} \|\Lambda_i f\|_H^2 \leq B\|f\|^2, \quad (f \in K).$$

Now, for every $f \in K$,

$$\begin{aligned}
\sum_{i\in I, z\in Z_i} | < f, J_k u_{i,z} > |^2 &= \sum_{i\in I, z\in Z_i} | < f, J_k \Lambda_i^* e_{i,z} > |^2 \\
&= \sum_{i\in I} \sum_{z\in Z_i} | < \Lambda_i J_k f, e_{i,z} > |^2 \\
&= \sum_{i\in I} \|\Lambda_i J_K f\|_H^2 \\
&\leq B\|J_K f\|^2 \\
&\leq B\|f\|^2.
\end{aligned}$$

So $\{J_K u_{i,z}\}_{i\in I, z\in Z_i}$ is a Bessel sequence for K. Conversely, If $\{J_K u_{i,z} : i \in I, z \in Z_i\}$ is a Bessel sequence for K, then there exists $\acute{B} > 0$ such that

$$\sum_{i\in I, z\in Z_i} | < f, J_k u_{i,z} > |^2 \leq \acute{B}\|f\|^2, \quad (f \in K).$$

Now, for every $f \in K$,

$$\begin{aligned}
\sum_{i\in I} \|\Lambda_i f\|_H^2 &= \sum_{i\in I} < \Lambda_i f, \Lambda_i f >_H \\
&= \sum_{i\in I} < f, \Lambda_i^* \Lambda_i f >_H \\
&= \sum_{i\in I} < f, \Lambda_i^* \sum_{z\in Z_i} < \Lambda_i f, e_{i,z} >_H e_{i,z} >_H \\
&= \sum_{i\in I} \sum_{z\in Z_i} \overline{< f, \Lambda_i^* e_{i,z} >_H} < f, \Lambda_i^* e_{i,z} >_H \\
&= \sum_{i\in I, z\in Z_i} | < f, u_{i,z} >_H |^2 \\
&= \sum_{i\in I, z\in Z_i} | < J_K f, J_K u_{i,z} > |^2 \\
&\leq \acute{B} \|J_K f\|^2 \\
&\leq \acute{B} \|f\|^2.
\end{aligned}$$

Hence $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-Bessel sequence for K with respect to $\{H_i\}$.
To complete the proof, let $e_{i,z} = J_{H_i} f_{i,z}$ where $f_{i,z} \in H_i$. Then

$$\begin{aligned}
[u_{i,z}, u_{i,z}] &= [J_K u_{i,z}, J_K u_{i,z}] \\
&= [J_K \Lambda_i^* e_{i,z}, J_K \Lambda_i^* e_{i,z}] \\
&= [J_K \Lambda_i^* J_{H_i} f_{i,z}, J_K \Lambda_i^* J_{H_i} f_{i,z}] \\
&= [\Lambda_i^\sharp f_{i,z}, \Lambda_i^\sharp f_{i,z}].
\end{aligned}$$

So $J_K u_{i,z}$ is J-positive if and only if $i \in I_+$, and $J_K u_{i,z}$ is J-negative if and only if $i \in I_-$. But

$$\begin{aligned}
\overline{span\{\Lambda_i^\sharp H_i :\ i \in I_+\}} &= \overline{span\{J_K \Lambda_i^* id H_i :\ i \in I_+\}} \\
&= \overline{span\{J_K \Lambda_i^* H_i :\ i \in I_+\}} \\
&= \overline{span\{J_K \Lambda_i^* \overline{span\{e_{i,z} :\quad z \in Z_i\}} :\ i \in I_+\}} \\
&= \overline{span\{\overline{span\{J_K \Lambda_i^* e_{i,z} :\quad z \in Z_i\}} :\ i \in I_+\}} \\
&= \overline{span\{J_K \Lambda_i^* e_{i,z} :\quad i \in I_+, z \in Z_i\}} \\
&= \overline{span\{J_K u_{i,z} :\quad i \in I_+, z \in Z_i\}}.
\end{aligned}$$

Therefore, $\overline{span\{\Lambda_i^\sharp H_i : i \in I_+\}}$ is maximal uniformly J-positive if and only if $\overline{span\{J_K u_{i,z} : i \in I_+, z \in Z_i\}}$ is maximal uniformly J-positive. Analogously, $\overline{span\{\Lambda_i^\sharp H_i :\ i \in I_-\}}$ is maximal uniformly J-negative if and only if $\overline{span\{J_K u_{i,z} :\quad i \in I_-, z \in Z_i\}}$ is maximal uniformly J-negative, and the proof is complete. $\square$

In the next proposition, it is shown that the transformation of a J-g-frame under a bounded operator is again a J-g-frame, but the combination with an unbounded operator may not be a J-g-frame.

Proposition 6.3. *Let $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ be a J-g-frame for K with respect to $\{H_i\}_{i\in I}$ and let $T \in B(K)$ with following conditions:*

1) $T^\sharp$ is surjective.

2) If $[x, x] > 0$, then $[T^\sharp x, T^\sharp x] \geqslant [x, x]$.

3) If $[x, x] < 0$, then $[T^\sharp x, T^\sharp x] \leqslant [x, x]$.

Then $\{\Lambda_i T \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$.

Proof. Let Θ be the synthesis operator for $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$. Thus $R(\Theta) = K$. Since $T^\sharp$ is surjective, so is T^*, and then, $R(T^*\Theta) = K$ and $\{\Lambda_i T \in B(K, H_i)\}_{i\in I}$ is a g-frame. Since $\{\Lambda_i \in B(K, H_i)\}_{i\in I}$ is a J-g-frame, $\overline{span\{\Lambda_i^\sharp H_i :\ i \in I_+\}}$ is uniformly J-positive. Then

$$\overline{T^\sharp \overline{span\{\Lambda_i^\sharp H_i :\ i \in I_+\}}} = \overline{span\{T^\sharp \Lambda_i^\sharp H_i :\ i \in I_+\}}$$

is uniformly J-positive. Analogously, $\overline{span\{T^\sharp \Lambda_i^\sharp H_i :\ i \in I_-\}}$ is uniformly J-negative. Finally, by Proposition 3.6, $\{\Lambda_i T \in B(K, H_i)\}_{i\in I}$ is a J-g-frame for K with respect to $\{H_i\}_{i\in I}$. $\square$

7 Conclusion

This paper explored key properties of g-frames in Krein spaces, showing how frame theory can be extended to settings with indefinite inner products. We described the relationship between g-frames in Krein spaces and their counterparts in associated Hilbert spaces. Then, we introduced the J-g-frame operator, examined its structural features, and explained its role in reconstruction within Krein spaces.

We also defined a J-g-dual sequence for g-frames in Krein spaces, noting that while it always forms a g-frame in the associated Hilbert space, it may not do so in the Krein space itself. Finally, we studied how frames interact with bounded operators and how this affects the preservation of the g-frame property.

Overall, these results deepen the understanding of frame theory in indefinite inner product spaces and suggest several promising directions for future research. In particular, the study of g-woven frames in Krein spaces appears to be a natural and important extension of the theory that deserves further investigation.

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