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Original Research Paper

ϕ - δ - S -Primary Submodule

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Abstract. In this paper, we introduce and investigate some properties of ϕ - δ - S -primary submodules, which is a generalization of the ϕ - δ -primary submodules and prime submodules in general. We extend several main results about ϕ - δ -primary submodules and prime submodules into the general framework. Moreover, we show that there is a one-to-one correspondence between the set of all ϕ - δ - S -primary submodules of an R -module M containing K and the set of all ϕ - δ - S -primary submodules of $\frac{M}{K}$, where K is a proper submodule of M . In the end, we present some results concerning $\phi_{\bowtie}\delta_{\bowtie}S_{\bowtie}$ -primary submodules with the amalgamation modules.

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1 Introduction

Throughout this paper, R is a non-trivial commutative ring with an identity element, S is a multiplicatively closed subset (m.c.s) of R with $1 \in S$ and M is a unitary R -module. The main purpose of this article is to introduce and investigate some properties of the concept of ϕ - δ - S -primary submodules, which is a generalization of the ϕ - δ -primary submodules and prime submodules [5, 7]. Also, this concept is a generalization of the concept of ϕ - δ - S -primary ideals in [6]. We first state the necessary definitions and preliminaries. Suppose that $\mathcal{L}(R)$ is the lattice of all ideals of R and $\mathcal{L}(M)$ is the lattice of all submodules of M . Let the function $\delta : \mathcal{L}(R) \rightarrow \mathcal{L}(R)$ be an expansion function on $\mathcal{L}(R)$ [12] and $\phi : \mathcal{L}(M) \rightarrow \mathcal{L}(M) \cup \{\emptyset\}$ be a reduction function on $\mathcal{L}(M)$ [11]. For example the functions $\delta_{id}(I) = I$ and $\delta_{rad}(I) = \sqrt{I}$ are expansion functions for every $I \in \mathcal{L}(R)$ and $\phi(N) = \emptyset$ and $\phi(N) = \{0\}$ for every $N \in \mathcal{L}(M)$ are reduction functions. Let $N \in \mathcal{L}(M)$, $J \in \mathcal{L}(R)$ and K be a submodule of M . The ideal quotient $(N :_R K)$ and the submodule $(N :_M J)$ are defined as follows. $(N :_R K) = \{r \in R \mid rK \subseteq N\}$ and $(N :_M J) = \{m \in M \mid mJ \subseteq N\}$. Particularly, we use $(N : M)$ instead of $(N :_R M)$ for simplicity. The concepts of prime ideal and prime submodule play a fundamental role in commutative algebra. They play the same role as prime numbers in number theory. As such, we intend to present a new generalization of them in module theory. In [7, 8], a proper submodule N of M is called prime (Primary) submodule, whenever $am \in N$ for $m \in M$ and $a \in R$, then $m \in N$ or $a \in (N : M)$ ($a \in \sqrt{(N : M)}$) (1984). In [11], Zamani extended the concept of prime submodules to the concept of ϕ -prime submodules, where ϕ is a reduction function (2010). A proper submodule N of M is called the ϕ -prime submodule, whenever $am \in N \setminus \phi(N)$ for $m \in M$ and $a \in R$, then $m \in N$ or $a \in (N : M)$. Also in [5], Ersoy extended the concept of ϕ -prime submodules to the concept of ϕ - δ -primary submodules, where ϕ is a reduction function and δ is an expansion function (2022). A proper submodule N of M is called ϕ - δ -primary submodule, whenever $am \in N \setminus \phi(N)$ for $m \in M$ and $a \in R$, then $m \in N$ or $a \in \delta(N : M)$. In this paper, we introduce the concept of ϕ - δ - S -primary submodule, where S is a (m.c.s) subset of R with $1 \in S$. The new proposed concept is a generalization of ϕ - δ -primary submodules. Let N be a proper submodule of M , such that

$\delta(N : M) \cap S = \emptyset$. We say N is a ϕ - δ - S -primary submodule associated to $s \in S$, whenever $am \in N \setminus \phi(N)$ for $m \in M$ and $a \in R$, then $sm \in N$ or $sa \in \delta(N : M)$. In particular, if $\phi(N) = \emptyset$, $\delta(I) = I$ and $S = \{1\}$, then the concepts of prime submodule and ϕ - δ - S -primary submodule coincide. According to Example 2.6, ϕ - δ - S -primary submodule will be a generalization of the ϕ -prime, S -primary and ϕ - δ -primary submodules, but the converse is not true in general (Example 2.11). In a recent study [6], Jaber defined the concept of ϕ - δ - S -primary ideals. A proper ideal I , with $I \cap S = \emptyset$, is called ϕ - δ - S -primary ideal of R associated to $s \in S$, whenever $ab \in I \setminus \phi(I)$ then $sa \in I$ or $sb \in \delta(I)$ (2023). Thus, the concept of ϕ - δ - S -primary submodules in module theory can be considered as the equivalent definition of ϕ - δ - S -primary ideals in ring theory.

This paper is organized as follows. In section 2, we propose some well-known properties, results and illustrative examples about ϕ - δ - S -primary submodules. One of the important theorems of this section is Theorem 2.15, which provides equivalent conditions to the main definition. Moreover, we present and analyse some results to examine the behavior of ϕ - δ - S -primary submodules in the fractions modules, multiplication modules and finitely generated modules.

Section 3 concerns the images and the inverse images of ϕ - δ - S -primary submodules. We primarily show that the images and the inverse images of ϕ - δ - S -primary submodules are also preserved under epimorphisms (Theorem 3.2). The most significant result of this section is Theorem 3.4, which demonstrates the existence of a one-to-one correspondence between the set of all ϕ - δ - S -primary submodules of M containing K and the set of all ϕ - δ - S -primary submodules of $\frac{M}{K}$. In the last section, we investigate the ϕ - δ - S -primary submodules in the direct product of modules and amalgamation modules. It is noteworthy that, if N_i is ϕ_i - δ_i - S_i -primary submodule of M_i , for $i = 1, 2$, then $N_1 \times N_2$ is not necessarily a $\phi_{\times}$ - $\delta_{\times}$ - S -primary submodule. See Example 4.6 for more details.

2 Properties of ϕ - δ - S -Primary Submodules

Let R be a non-trivial commutative ring with unity $1 \neq 0$ and $\mathcal{L}(R)$ be the lattice of all ideals of R . The ideal expansion function on $\mathcal{L}(R)$ is

defined as follows.

Definition 2.1. A function $\delta : \mathcal{L}(R) \rightarrow \mathcal{L}(R)$ is called an ideal expansion of $\mathcal{L}(R)$ if whenever I, J are ideals of R with $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ and $I \subseteq \delta(I)$ [12].

In the following, we give examples of the ideal expansion on $\mathcal{L}(R)$.

Example 2.2. Consider $I \in \mathcal{L}(R)$.

- (1) $\delta_{id}(I) = I$, which is called the identity expansion.
- (2) $\delta_{rad}(I) = \sqrt{I}$, which is called the radical expansion.
- (3) For an arbitrary fixed $J \in \mathcal{L}(R)$, consider $\delta_{res}(I) = (I : J)$.

It is clear that $\mathcal{L}(M)$ is a complete lattice concerning set inclusion. Now, in the following, we recall the definition of the submodule reduction function on $\mathcal{L}(M)$.

Definition 2.3. Let M be an R -module and $\mathcal{L}(M)$ be the lattice of all submodules of M . A function $\phi : \mathcal{L}(M) \rightarrow \mathcal{L}(M) \cup \{\emptyset\}$ is called a submodule reduction of M if whenever N, K are submodules of M with $N \subseteq K$, then $\phi(N) \subseteq \phi(K)$ and $\phi(N) \subseteq N$ [11].

Now we give examples of the submodule reduction on $\mathcal{L}(M)$.

Example 2.4. Let $N \in \mathcal{L}(M)$.

- (1) $\phi_{\emptyset}(N) = \emptyset$
- (2) $\phi_0(N) = \{0\}$
- (3) $\phi_{id}(N) = N$

In the following, we present the ϕ - δ - S -primary submodules as an extension of ϕ - δ - S -primary ideals, based on definition introduced by Jaber in [6].

Definition 2.5. (1) Let N be a proper submodule of M , and S be a multiplicatively closed subset of R such that $S \cap \delta(N : M) = \emptyset$. The submodule N is said to be δ - S -primary submodule of M associated to $s \in S$, if whenever $am \in N$, then $sa \in \delta(N : M)$ or $sm \in N$ for all $a \in R, m \in M$.

(2) Let N be a proper submodule of M such that $S \cap \delta(N : M) = \emptyset$. The submodule N is said to be ϕ - δ - S -primary submodule of M associated to $s \in S$, if whenever $am \in N \setminus \phi(N)$, then $sa \in \delta(N : M)$ or $sm \in N$ for all $a \in R, m \in M$.

Notice that, if there exists $s \in \delta(N : M) \cap S$, then $sa \in \delta(N : M)$ for every $a \in R$, which implies that N is always a ϕ - δ - S -primary submodule. Therefore, we assume that $\delta(N : M) \cap S = \emptyset$ to avoid this case.

Example 2.6. Let M be an R -module and N be a proper submodule of M . Then the following statements hold.

- (1) N is prime $\Leftrightarrow$ it is $\phi_{\emptyset}\delta_{id}\{1\}$ -primary [7].
- (2) N is primary $\Leftrightarrow$ it is $\phi_{\emptyset}\delta_{rad}\{1\}$ -primary [8].
- (3) N is ϕ -prime $\Leftrightarrow$ it is $\phi\delta_{id}\{1\}$ primary [11].
- (4) N is ϕ - δ -primary $\Leftrightarrow$ it is $\phi\delta\{1\}$ -primary [5].
- (5) N is S -prime $\Leftrightarrow$ it is $\phi_{\emptyset}\delta_{id}S$ -primary [9].
- (6) N is S -primary $\Leftrightarrow$ it is $\phi_{\emptyset}\delta_{rad}S$ -primary [1].

In the following, we give an example of a ϕ - δ - S -primary submodule.

Example 2.7. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_{18}$ and $N = \langle 3 \rangle = 3\mathbb{Z}_{18}$. It is obvious that N is a proper submodule of M . Let $S = \{1\}$, $J = 2\mathbb{Z}$, $\delta = \delta_{rad}$ and $\phi(K) = JK$. We have $\phi(\langle 3 \rangle) = 6\mathbb{Z}_{18}$ and $\delta(N : M) = 3\mathbb{Z}$. Now suppose that $a \in R$ and $m \in M$ such that $am \in N \setminus \phi(N)$. Then $am \in \{3, 9, 12\}$. Hence $sa \in 3\mathbb{Z}$ or $sm \in 3\mathbb{Z}_{18}$ and so N is a ϕ - δ - S -primary submodule.

Next, we provide two examples of submodules that are not ϕ - δ - S -primary.

Example 2.8. Let $R = \mathbb{Z}$, $M = \mathbb{Q} \times \mathbb{Q}$ be an R -module and $N = 2\mathbb{Z} \times 0$ be a submodule of M , $S = \mathbb{Z} \setminus \{0\}$. Consider $\delta(I) = \delta_{rad}(I)$, for any ideal I of R and $\phi(K) = \{(0, 0)\}$ for every $K \in \mathcal{L}(M)$. It is clear that $(N : M) = \{0\}$. Suppose that $2 \neq p \in \mathbb{Z}$ and $s \in S$ such that $\gcd(p, s) = 1$. Then $p(\frac{2}{p}, 0) \in (2\mathbb{Z} \times \{0\}) \setminus \phi(2\mathbb{Z} \times \{0\})$. Since $sp \neq 0$, so $sp \notin \delta(N : M)$ and also $s(\frac{2}{p}, 0) \notin N$. Therefore, N is not a $\phi_{\emptyset}\delta_{rad}S$ -primary submodule of M .

Example 2.9. Let $M = \mathbb{Z}_{p^\infty}$, $R = \mathbb{Z}$ and $G_t = \langle \frac{1}{p^t} + \mathbb{Z} \rangle$, $t \geq 0$. Also suppose that, $S = \{q^n \mid n \in \mathbb{N}_0\}$ such that $p \neq q$. We know that $p^k(\frac{1}{p^{t+k}} + \mathbb{Z}) \in G_t$. Since $\mathbb{Z}$ is an integral domain, so $(G_t :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$ and $\delta_{rad}(G_t :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) \cap S = \emptyset$. But there is no $s \in S$ such that $s(\frac{1}{p^{t+1}} + \mathbb{Z}) \in G_t$ and $sp^k \in \delta_{rad}(G_t :_{\mathbb{Z}} \mathbb{Z}_{p^\infty}) = 0$. Therefore for every t , G_t is not $\phi_{\emptyset}\delta_{rad}S$ -primary submodule of M .

Remark 2.10. Let $\delta_1, \delta_2 : \mathcal{L}(R) \rightarrow \mathcal{L}(R)$ be two ideal expansions of R . We denote $\delta_1 \leq \delta_2$, if $\delta_1(I) \subseteq \delta_2(I)$ for every $I \in \mathcal{L}(R)$. Suppose that δ_1, δ_2 are two ideal expansions, such that $\delta_1 \leq \delta_2$.

(1) It is clear that every ϕ - δ_1 - S -primary submodule of M is a ϕ - δ_2 - S -primary submodule.

(2) Also since $\delta_{id}(I) = I \subseteq \delta(I)$, so every ϕ - S -primary submodule of M is a ϕ - δ - S -primary submodule of M .

Let $\phi_1, \phi_2 : \mathcal{L}(M) \rightarrow \mathcal{L}(M) \cup \{\emptyset\}$ be two submodule reductions of M . We also denote that $\phi_1 \leq \phi_2$, if $\phi_1(N) \subseteq \phi_2(N)$ for every $N \in \mathcal{L}(M)$. Moreover, if ϕ_1, ϕ_2 are two submodule reductions of M ,

(1) $\phi_1 \leq \phi_2$, then every ϕ_1 - δ - S -primary submodule of M is a ϕ_2 - δ - S -primary submodule.

(2) It is obvious that every δ - S -primary submodule of M is a ϕ - δ - S -primary submodule.

One can show that, every ϕ - δ -primary submodule of M is a ϕ - δ - S -primary submodule of M associated to $s \in S$ with $\delta(N : M) \cap S = \emptyset$. But in the next example we show that the converse is not true in general.

Example 2.11. Consider $\mathbb{Z}$ -module $M = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}$. Let $S = \mathbb{Z} \setminus \{0\}$ and $N = (0, 0, 0)$. Suppose that $a \in R$ and $(m, n, r) \in M$, such that $a(m, n, r) \in N$. We can consider the two following cases. Case 1 $r \neq 0$, then $ar = 0$ and since $\mathbb{Z}$ is an integral domain, so $a = 0$. Hence for every $s \in S$, $as \in \delta(N : M) = 0$. Case 2 $r = 0$, then $6(m, n, 0) \in N$. Therefore N is a $\phi_{\emptyset}$ - δ_{id} - S -primary submodule of M associated to $s = 6$. On the other hand $6(1, 1, 0) \in N \setminus \{\emptyset\}$, but neither $6 \in \delta(N : M)$ nor $(1, 1, 0) \in N$. So, N is not $\phi_{\emptyset}$ - δ_{id} -primary submodule.

In the following, we give some properties of ϕ - δ - S -primary submodules.

Proposition 2.12. Let $\{N_i : i \in I\}$ be a directed family of ϕ - δ - S -primary submodules of M associated to $s \in S$, such that $\delta(\cup N_i : M) \cap S = \emptyset$. Then the submodule $N = \cup_{i \in I} N_i$ is ϕ - δ - S -primary submodule of M associated to $s \in S$.

Proof. Let $am \in N \setminus \phi(N)$, for $a \in R, m \in M$. Suppose that $sm \notin N$ for all $s \in S$. We show that $sa \in \delta(N : M)$. Since $am \notin \phi(N)$, we have $am \notin \phi(N_i)$ for all $i \in I$. Let $j \in I$ such that $am \in N_j \setminus \phi(N_j)$. Since N_j

is a ϕ - δ - S -primary submodule of M associated to $s \in S$ and $sm \notin N_j$, we have $sa \in \delta(N_j : M) \subseteq \delta(N : M)$. Hence N is a ϕ - δ - S -primary submodule of M associated to $s \in S$. $\square$

Suppose that $\delta : \mathcal{L}(R) \rightarrow \mathcal{L}(R)$ is an ideal expansion on $\mathcal{L}(R)$ and $\phi : \mathcal{L}(M) \rightarrow \mathcal{L}(M) \cup \{\emptyset\}$ is a submodule reduction on $\mathcal{L}(M)$. We say that δ and ϕ have intersection property if for any two ideals I, J of R and any two submodules N, K of M , $\delta(I \cap J) = \delta(I) \cap \delta(J)$ and $\phi(N \cap K) = \phi(N) \cap \phi(K)$.

In Proposition 2.12, we showed that for any directed family of ϕ - δ - S -primary submodules, their union is also ϕ - δ - S -primary submodule. We also prove that the statement is true for the intersection of a finite number of ϕ - δ - S -primary submodules under suitable conditions.

Proposition 2.13. *Let submodule reduction ϕ and ideal expansion δ have the intersection property. Suppose that $N_1, \dots, N_r$ are ϕ - δ - S -primary submodules of M associated to $s \in S$. Let $\phi(N_i) = \phi(N_j)$ and $\delta(N_i : M) = \delta(N_j : M)$ for every $i, j \in \{1, \dots, r\}$. Then $N = \bigcap_{i \in \{1, 2, \dots, r\}} N_i$ is ϕ - δ - S -primary submodule of M associated to $s \in S$.*

Proof. First we note that $\delta(\bigcap N_i : M) \cap S = \emptyset$. Let $am \in N \setminus \phi(N)$, where $a \in R, m \in M$. Suppose that $sa \notin \delta(N : M)$ for all $s \in S$. So $sa \notin \delta(\bigcap N_i : M) = \delta(\bigcap (N_i : M)) = \bigcap \delta(N_i : M)$. Hence there is $j \in \{1, \dots, r\}$ such that $sa \notin \delta(N_j : M)$. Since $\delta(N_i : M) = \delta(N_j : M)$ for every $i, j \in \{1, \dots, r\}$, then for every $i \in \{1, \dots, r\}$ we have $sa \notin \delta(N_i : M)$. On the other hand, $\phi(N_i) = \phi(N_j)$ and ϕ has the property intersection, hence for every $i \in \{1, \dots, r\}$, $am \in N_i \setminus \phi(N_i)$ and $sa \notin \delta(N_i : M)$. Since all of N_i are ϕ - δ - S -primary submodules of M associated to $s \in S$, then for every i , $sm \in N_i$. So $sm \in \bigcap N_i$ and N is ϕ - δ - S -primary submodules of M associated to $s \in S$. $\square$

In the following example, we illustrate that the condition $\delta(N_i : M) = \delta(N_j : M)$ in the previous proposition, is not superfluous.

Example 2.14. Consider $\mathbb{Z}$ -module $M = 3\mathbb{Z}$. Let $S = \{1\}$ be a (m.c.s) subset and $N_1 = 6\mathbb{Z}$ and $N_2 = 15\mathbb{Z}$. Then $\delta_{rad}(N_1 : M) = 2\mathbb{Z} \neq 5\mathbb{Z} = \delta_{rad}(N_2 : M)$ and $\delta_{rad}(N_i : M) \cap S = \emptyset$. If $am \in 6\mathbb{Z}$, for $a \in \mathbb{Z}$ and $m \in 3\mathbb{Z}$, then $2 \mid am$ and $3 \mid am$. Hence $(2 \mid a \vee 2 \mid m)$ and $(3 \mid a \vee 3 \mid m)$. Considering all cases we conclude that, $sm \in 6\mathbb{Z}$ or $sa \in 2\mathbb{Z}$, for every $s \in S$. So N_1 and similarly N_2 are $\phi_{\emptyset}$ - δ_{rad} - S -primary

submodules of M . Note further that, $90 = 6 \times 15 \in N_1 \cap N_2 = 30\mathbb{Z}$, but $6 \notin 10\mathbb{Z} = \delta_{rad}(N_1 \cap N_2)$ and also $15 \notin 30\mathbb{Z}$. So $N_1 \cap N_2$ is not $\phi_{\emptyset}\delta_{rad}$ - S -primary submodule of M .

In the next theorem, we present an equivalent definition of ϕ - δ - S -primary submodules based on ideals of R and submodules of M .

Theorem 2.15. *Let M be an R -module and N be a proper submodule of M with $\delta(N : M) \cap S = \emptyset$. The following statements are equivalent:*

- (1) *N is a ϕ - δ - S -primary submodule of M associated to $s \in S$.*
- (2) *For every $a \in R \setminus (\delta(N : M) : s)$, $(N :_M a) \subseteq (N :_M s)$ or $(N :_M a) = (\phi(N) :_M a)$.*
- (3) *For every ideal I of R and every submodule K of M , if $IK \subseteq N$ and $IK \not\subseteq \phi(N)$ then $sI \subseteq \delta(N : M)$ or $sK \subseteq N$.*

Proof. (1) $\Rightarrow$ (2) Let $a \in R \setminus (\delta(N : M) : s)$ and $m \in (N :_M a)$. So $am \in N$. We consider the following cases: (i) $am \in \phi(N)$. Then $m \in (\phi(N) :_M a)$. Thus $(N :_M a) \subseteq (\phi(N) :_M a)$. Since $(\phi(N) :_M a) \subseteq (N :_M a)$, therefore $(N :_M a) = (\phi(N) :_M a)$. (ii) $am \notin \phi(N)$. Then $sm \in N$ or $sa \in \delta(N : M)$ which implies $m \in (N :_M s)$ or $sa \in \delta(N : M)$. But since $a \notin (\delta(N : M) : s)$, so $(N :_M a) \subseteq (N :_M s)$. (2) $\Rightarrow$ (3) Let I be an ideal of R and K be a submodule of M such that $IK \subseteq N$. Assume that $sI \not\subseteq \delta(N : M)$ and $sK \not\subseteq N$. We show that $IK \subseteq \phi(N)$. Let $b \in I \setminus (\delta(N : M) : s)$. Then by (2), $(N :_M b) \subseteq (N :_M s)$ or $(N :_M b) = (\phi(N) :_M b)$. Since $K \subseteq (N :_M b)$ and $K \not\subseteq (N :_M s)$, so we can conclude that $(N :_M b) = (\phi(N) :_M b)$. Hence $bK \subseteq \phi(N)$. Now let $b \in I \cap \delta(N : M)$. Since $sI \not\subseteq \delta(N : M)$, so there is $a_0 \in I$ such that $a_0 \notin (\delta(N : M) : s)$. It is easy to show that, $b + a_0 \notin (\delta(N : M) : s)$. Due to (2) and similar above argument we conclude that $bK \subseteq (b + a_0)K \subseteq \phi(N)$. Since it holds for every $b \in I$, we can conclude that $IK \subseteq \phi(N)$.

(3) $\Rightarrow$ (1) Let $a \in R$ and $m \in M$, such that $am \in N \setminus \phi(N)$. Then $\langle a \rangle \langle m \rangle \subseteq N$ and $\langle a \rangle \langle m \rangle \not\subseteq \phi(N)$ implies that $s \langle a \rangle \subseteq \delta((N : M))$ or $s \langle m \rangle \subseteq N$. Thus, $sa \in \delta(N : M)$ or $sm \in N$. Thus, N is a ϕ - δ - S -primary submodule of M associated to $s \in S$. $\square$

Proposition 2.16. *Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$. Suppose that for every $r \in R \setminus \delta(N : M)$, $(\phi(N) :_M$*

$r) \subseteq \phi(N :_M r)$ and $\delta((N :_M r) : M) \cap S = \emptyset$. Then $(N :_M r)$ is a ϕ - δ - S -primary submodule of M associated to $s \in S$.

Proof. Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$. Suppose that $r \in R, m \in M$ and $am \in (N :_M r) \setminus \phi(N :_M r)$. Hence, $ram \in N$. Because $(\phi(N) :_M r) \subseteq \phi(N :_M r)$, $ram \notin \phi(N)$. Since N is ϕ - δ - S -primary submodule of M associated to $s \in S$, then $sa \in \delta(N : M)$ or $sr m \in N$. So $sa \in \delta((N :_M r) : M)$ or $sm \in (N :_M r)$. Therefore, $(N :_M r)$ is ϕ - δ - S -primary submodule of M associated to $s \in S$. $\square$

In proposition 2.16, if we replace $r \in R$ by ideal I of R , then the statement is still valid. One can prove it as follows.

Proposition 2.17. *Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$ and I be an ideal of R such that $I \not\subseteq \delta(N : M)$. Suppose that $(\phi(N) :_M I) \subseteq \phi(N :_M I)$, $(\delta(N : M) : I) \subseteq \delta((N : M) : I)$ and $\delta((N :_M I) : M) \cap S = \emptyset$. Then $(N :_M I)$ is a ϕ - δ - S -primary submodule of M associated to $s \in S$.*

Proof. Let $am \in (N :_M I) \setminus \phi((N :_M I))$. So $am \in (N :_M I)$ and $am \notin \phi((N :_M I))$. Thus, $amI \subseteq N$ but $amI \not\subseteq \phi(N)$. Therefore, $\langle a \rangle \langle m \rangle I \subseteq N$ and $\langle a \rangle \langle m \rangle I \not\subseteq \phi(N)$. Since N is a ϕ - δ - S -primary submodule of M associated to $s \in S$, by Theorem 2.15, $s \langle m \rangle \subseteq N$ (consequently $s \langle m \rangle I \subseteq N$) or $s \langle a \rangle I \subseteq \delta(N : M)$ which implies $s \langle m \rangle \in (N :_M I)$ or $s \langle a \rangle \in (\delta(N : M) : I) \subseteq \delta((N : M) : I)$. Therefore $sm \in (N :_M I)$ or $sa \in \delta((N :_M I) : M)$. $\square$

We recall that an R -module M is a multiplication module, if for every submodule N of M there exists an ideal I of R , such that $N = IM$ [3]. Let M be the multiplication module and N, K be two submodules of M . So there are ideals I and J of R such that $N = IM$ and $K = JM$. Hence the product of N and K is denoted by $NK = IJM$. Suppose that δ is an expansion function on $\mathcal{L}(M)$ and δ_R is an expansion function on $\mathcal{L}(R)$ such that $\delta_R(N : M) = (\delta(N) : M)$. Now based on these assumptions, we can give the following theorem.

Theorem 2.18. *Let M be a multiplication module and P be a submodule of M with $\delta(P : M) \cap S = \emptyset$, where S is a (m.c.s) subset of R . Then the following are equivalent:*

- (1) P is a ϕ - δ_R - S -primary submodule of M associated to $s \in S$.

(2) For every submodule L, N of M with $LN \subseteq P$ and $LN \not\subseteq \phi(P)$, then there is a $s \in S$ such that $sN \subseteq \delta(P)$ or $sL \subseteq P$.

Proof. (1) $\Rightarrow$ (2) Suppose that P is a ϕ - δ_R - S -primary submodule of M associated to $s \in S$. Let N and L be submodule of M with $LN \subseteq P$ and $LN \not\subseteq \phi(P)$. Since M is a multiplication module, so there are ideals I and J of R such that $L = IM$, $N = JM$ and $LN = IJM$. Hence $I(JM) \subseteq P$ and $I(JM) \not\subseteq \phi(P)$. Since P is a ϕ - δ_R - S -primary submodule of M associated to $s \in S$, so $sI \subseteq \delta_R(P : M) = (\delta(P) : M)$ or $s(JM) \subseteq P$. Thus either $sL = sIM \subseteq (\delta(P) : M)M = \delta(P)$ or $sN \subseteq P$.

(2) $\Rightarrow$ (1) Let $IK \subseteq P$ and $IK \not\subseteq \phi(P)$, which I is an ideal of R and K is a submodule of M . Since M is a multiplication module, then there is an ideal J of R such that $K = JM$. Thus $IM.JM = IJM \subseteq P$ and $IM.JM = IJM \not\subseteq \phi(P)$. Hence either $s(IM) \subseteq \delta(P)$ or $s(JM) \subseteq P$. Therefore either $sI \subseteq (\delta(P) : M) = \delta_R(P : M)$ or $sK \subseteq P$. $\square$

Theorem 2.19. Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$. Suppose that $a \in R \setminus (\delta(N : M) : s^2)$. Then $(N :_M sa) = (\phi(N) :_M sa)$ or $(N :_M sa) = (N :_M s)$.

Proof. Let $m \in (N :_M sa)$, so $msa \in N$. Assume that $msa \in \phi(N)$. Therefore $m \in (\phi(N) :_M sa)$ and so $(N :_M sa) \subseteq (\phi(N) :_M sa)$. Also we know that $(\phi(N) :_M sa) \subseteq (N :_M sa)$, since $\phi(N) \subseteq N$. This implies that $(N :_M sa) = (\phi(N) :_M sa)$. Now suppose that $msa \notin \phi(N)$. Hence $msa \in N \setminus \phi(N)$. Since N is a ϕ - δ - S -primary submodule of M associated to $s \in S$ and $a \notin (\delta(N : M) : s^2)$, consequently $sm \in N$. Therefore $m \in (N :_M s)$. Hence $(N :_M sa) \subseteq (N :_M s)$. clearly $(N :_M s) \subseteq (N :_M sa)$, hence $(N :_M sa) = (N :_M s)$. $\square$

Proposition 2.20. Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$. For every $a \in R \setminus (\sqrt{\delta(N : M)} : s)$, we have $(N :_M sa) = (\phi(N) :_M sa) \cup (N :_M s)$.

Proof. First, suppose that $a \in \sqrt{(\delta(N : M) : s^2)}$. Then $a^n \in (\delta(N : M) : s^2)$ for some positive integer n and therefore $(as)^n \in \delta(N : M)$. Accordingly $(as) \in \sqrt{\delta(N : M)}$ and so $a \in (\sqrt{\delta(N : M)} : s)$. Consequently $\sqrt{(\delta(N : M) : s^2)} \subseteq (\sqrt{\delta(N : M)} : s)$. Next suppose that

$a \in R \setminus (\sqrt{\delta(N : M)} : s)$. Therefore $a \in R \setminus (\delta(N : M) : s^2)$. Consequently by Proposition 2.19, the proof is complete. $\square$

Proposition 2.21. *Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$ and K be a submodule of M with $K \not\subseteq N$. If $\delta(N : K) \cap S = \emptyset$ and $\phi(N \cap K) = \phi(N)$, then $N \cap K$ is a ϕ - δ - S -primary submodule of K associated to $s \in S$.*

Proof. We note that $(N : M) \subseteq (N : K) = (N \cap K : K)$. Let $ak \in N \cap K \setminus \phi(N \cap K)$ for $a \in R$ and $k \in K$. Then $ak \in N \setminus \phi(N)$. Thus $sk \in N$ or $sa \in \delta(N : M)$. Since $k \in K$, so $sk \in K$. Hence if $sk \in N$, then $sk \in N \cap K$. If $sa \in \delta(N : M)$, then $sa \in \delta(N \cap K : K)$. Therefore $N \cap K$ is a ϕ - δ - S -primary submodule of K associated to $s \in S$. $\square$

Proposition 2.22. *Let N be a proper submodule of M , $\phi : \mathcal{L}(M) \rightarrow \mathcal{L}(M) \cup \{\emptyset\}$ be a reduction function on $\mathcal{L}(M)$ and $\phi_R : \mathcal{L}(R) \rightarrow \mathcal{L}(R) \cup \{\emptyset\}$ be a reduction function on $\mathcal{L}(R)$ such that $\phi_R(N : M) := (\phi(N) : M)$. If N is a ϕ - δ - S -primary of M associated to $s \in S$, then $(N : M)$ is a ϕ_R - δ - S -primary ideal of R associated to $s \in S$.*

Proof. Let $ab \in (N : M) \setminus \phi_R(N : M) = (N : M) \setminus (\phi(N) : M)$ for $a, b \in R$. So $abM \subseteq N$ and $abM \not\subseteq \phi(N)$. Let $Rb = I$ and $aM = K$, then $IK \subseteq N$ and $IK \not\subseteq \phi(N)$. By Theorem 2.15, $sK \subseteq N$ or $sI \subseteq \delta(N : M)$. Therefore $saM \subseteq N$ or $sb \in \delta(N : M)$. Hence $sa \in (N : M) \subseteq \delta(N : M)$ or $sb \in \delta(N : M)$. We note that since $\delta(N : M) \cap S = \emptyset$, then $(N : M) \cap S = \emptyset$. $\square$

The next example shows the converse of Proposition 2.22 does not hold, in general.

Example 2.23. Consider $M = \mathbb{Q} \times \mathbb{Q}$ as $\mathbb{Z} \times \mathbb{Z}$ -module, $N = p\mathbb{Z} \times (0)$ is a proper submodule of M and $S = \{(1, 1)\}$ is a (m.c.s) subset of $\mathbb{Z} \times \mathbb{Z}$. We note that $(N :_{\mathbb{Z} \times \mathbb{Z}} M) = \{(0, 0)\}$. Therefore, $(N :_{\mathbb{Z} \times \mathbb{Z}} M) = 0$ is a ϕ_0 - δ_{rad} - S -primary ideal of $\mathbb{Z}$ by default. But N is not a ϕ_0 - δ_{rad} - S -primary submodule of M . Since $(p, 1)(1, 0) \in p\mathbb{Z} \times (0) \setminus \{0\} \times \{0\}$, but $s(p, 1) \notin p\mathbb{Z} \times (0)$ and $s(1, 0) \notin \delta_{rad}(p\mathbb{Z} \times (0) : \mathbb{Z} \times \mathbb{Z}) = (0, 0)$.

In the following we show that the converse of Proposition 2.22 holds for multiplication modules.

Proposition 2.24. *Suppose that the hypotheses of Proposition 2.22, are satisfied. Let M be a multiplication R -module and $\delta(N : M) \cap S = \emptyset$. If $(N : M)$ is a ϕ_R - δ - S -primary ideal of R associated to $s \in S$, then N is a ϕ - δ - S -primary of M associated to $s \in S$.*

Proof. Let $IK \subseteq N$ and $IK \not\subseteq \phi(N)$ such that I is an ideal of R and K be a proper submodule of M . Since M is the multiplication module, so there is an ideal J of R such that $K = JM$. Hence $IJ \subseteq (N : M)$ and $IJ \not\subseteq (\phi(N) : M) = \phi_R(N : M)$. Then by [6], Theorem 2.20, $sJ \subseteq (N : M)$ or $sI \subseteq \delta(N : M)$. Hence $sJM \subseteq N$ which implies $sK \subseteq N$ or $sI \subseteq \delta(N : M)$. So by Theorem 2.15, N is a ϕ - δ - S -primary submodule of M . $\square$

In the following, we compare the classes of δ - S -primary submodules and ϕ - δ - S -primary submodules. We know that every δ - S -primary submodule of M is a ϕ - δ - S -primary submodule. In the next example we show that the converse is not true.

Example 2.25. Consider $M = \mathbb{Z}_{12}$ as a $\mathbb{Z}$ -module. Suppose that $S = \{1\}$, $I = (N :_{\mathbb{Z}} M)$ and $\phi(N) = IN$ and $N = \langle 4 \rangle$ as a proper submodule of M . We note that $S \cap \delta_{id}(N : M) = \emptyset$. Since $(N :_{\mathbb{Z}} M) = 4\mathbb{Z}$ and $\phi(N) = (N : M)N = 4\mathbb{Z}N = \{0, 4, 8\}$. So for $a \in R$ and $m \in M$ such that $am \in N \setminus \phi(N)$, then $am \in \emptyset$. Hence N is a ϕ - δ_{id} - S -primary submodule of M by default. It is clear that $2.2 \in N$, but neither $2 \in N$, nor $2 \in \delta_{id}(N : M) = 4\mathbb{Z}$, so N is not a δ - S -primary submodule of M .

In the following proposition, we put a condition on the ϕ - δ - S -primary submodule N of an R -module M to be δ - S -primary submodule of M .

Proposition 2.26. *Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$. Suppose that $(N : M)N \not\subseteq \phi(N)$. Then N is a δ - S -primary submodule of M associated to $s \in S$.*

Proof. Let $am \in N$ for $a \in R, m \in M$. We consider the following two cases: Case i $am \notin \phi(N)$, we have $sa \in \delta(N : M)$ or $sm \in N$ for $s \in S$. Case ii $am \in \phi(N)$. If $aN \not\subseteq \phi(N)$, then $n \in N$ exists such that $an \notin \phi(N)$. Thus, $a(m + n) \in N \setminus \phi(N)$, which implies $sa \in \delta(N : M)$ or $s(m + n) \in N$ for $s \in S$, which completes the proof. Now suppose that $aN \subseteq \phi(N)$. Similarly, if $(N : M)m \not\subseteq \phi(N)$, then the proof is complete. So assume that $(N : M)m \subseteq \phi(N)$. Since $(N : M)N \not\subseteq \phi(N)$,

there exists $b \in (N : M)$ and $m' \in N$ such that $bm' \notin \phi(N)$, hence $(a+b)(m+m') \in N \setminus \phi(N)$. Since N is a ϕ - δ - S -primary submodule, either $s(a+b) \in \delta(N : M)$ or $s(m+m') \in N$, which implies $sa \in \delta(N : M)$ or $sm \in N$. Therefore, N is a δ - S -primary submodule of M associated to $s \in S$. $\square$

Also, Example 2.25 shows that $(N : M)N \not\subseteq \phi(N)$ in above proposition is not superfluous. In other words, Proposition 2.26 states that if N is a ϕ - δ - S -primary submodule of M associated to $s \in S$, but is not a δ - S -primary submodule of M associated to $s \in S$, then $(N : M)N \subseteq \phi(N)$. For a multiplication module, Proposition 2.26 can be expressed as follows.

Proposition 2.27. *Let M be a multiplication module, N be a proper submodule of M and the situation be as described in Proposition 2.22. Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$, but is not a δ - S -primary submodule of M associated to $s \in S$, then $N^2 \subseteq \phi(N)$.*

Proof. Due to Propositions 2.22 and 2.24, we have N as a ϕ - δ - S -primary submodule of M associated to $s \in S$ if and only if $(N : M)$ is a ϕ_R - δ - S -primary ideal of R associated to $s \in S$. So, N is not a δ - S -primary submodule of M associated to $s \in S$ if and only if $(N : M)$ is not a δ - S -primary ideal of R associated to $s \in S$. Therefore $(N : M)$ is a ϕ_R - δ - S -primary ideal of R associated to $s \in S$, but is not a δ - S -primary ideal of R associated to $s \in S$. Hence by [6], Proposition 2.17. $(N : M)^2 \subseteq \phi_R(N : M)$. So, $(N : M)^2 M \subseteq \phi_R(N : M)M = (\phi(N) : M)M = \phi(N)$ which implies that $N^2 \subseteq \phi(N)$. $\square$

Corollary 2.28. *Let M be a multiplication module. Suppose that N is a ϕ - δ - S -primary submodule of M but is not a δ - S -primary submodule of M associated to $s \in S$, then $\sqrt{N} = \sqrt{\phi(N)}$.*

Proof. Suppose that N is not δ - S -primary submodule of M . Hence by Proposition 2.27, $N^2 \subseteq \phi(N) \subseteq N$. So $\sqrt{N^2} \subseteq \sqrt{\phi(N)} \subseteq \sqrt{N}$. On the other hand $N = IM$ for some ideal I of R and by Theorem 2.12. [3], $\sqrt{N} = \sqrt{IM}$. So $\sqrt{N^2} = \sqrt{I^2 M} = \sqrt{IM} = \sqrt{N}$ which implies $\sqrt{N} = \sqrt{\phi(N)}$. $\square$

Proposition 2.29. *Let $\phi \neq \phi_0$ be a submodule reduction of M such that $\phi(N) = \phi^2(N)$ for each submodule N of M . The following statements*

are equivalent:

- (1) Every ϕ - δ - S -primary submodule of M is a δ -primary submodule.
- (2) For any submodule N of M , $\phi(N)$ is a δ -primary submodule of M and every δ - S -primary submodule of M is δ -primary.

Proof. (1) $\Rightarrow$ (2) Suppose that $am \in \phi(N) \setminus \phi(\phi(N))$ for $a \in R, m \in M$. Hence, $am \in \phi(N) \setminus \phi^2(N) = \emptyset$. Therefore $\phi(N)$ is ϕ - δ - S -primary submodule of M associated to $s \in S$ by default. So by (1) the proof is complete. First we note that every δ - S -primary submodule of M is ϕ - δ - S -primary submodule of M , so by (1) the proof is complete.
 (2) $\Rightarrow$ (1) Let N be ϕ - δ - S -primary submodule of M associated to $s \in S$. We show that N is a δ -primary submodule of M . Let $a \in R, m \in M$ such that $am \in N$. We have two cases. Case i $am \notin \phi(N)$, then $am \in N \setminus \phi(N)$. Hence $sa \in \delta(N : M)$ or $sm \in N$ and so N is δ - S -primary submodule of M . Therefore by (2) the proof is complete. Case ii $am \in \phi(N)$. Since $\phi(N)$ is a δ -primary submodule of M then $m \in \phi(N)$ or $a \in \delta(\phi(N) : M)$ which implies $sm \in \phi(N) \subseteq N$ or $sa \in \delta(\phi(N) : M) \subseteq \delta(N : M)$. Thus, N is a δ - S -primary submodule of M associated to $s \in S$. So by (2), N is a δ -primary submodule. $\square$

Theorem 2.30. Let N be a submodule of M . Then N is a ϕ - δ - S -primary submodule of M associated to $s \in S$ if and only if $\frac{N}{\phi(N)}$ is a ϕ_0 - δ - S -primary submodule of $\frac{M}{\phi(N)}$.

Proof. Assume that N be ϕ - δ - S -primary submodule of M associated to $s \in S$. Let $a \in R, m + \phi(N) \in \frac{M}{\phi(N)}$ and $0 \neq a(m + \phi(N)) \in \frac{N}{\phi(N)}$. Hence, $am + \phi(N) \in \frac{N}{\phi(N)}$ and $am \notin \phi(N)$. Since N is ϕ - δ - S -primary submodule of M associated to $s \in S$, $sm \in N$ or $sa \in \delta(N : M)$. Suppose that $sm \in N$, so $sm + \phi(N) \in \frac{N}{\phi(N)}$. But if $sa \in \delta(N : M)$, since $\delta(N : M) = \delta(\frac{N}{\phi(N)} : \frac{M}{\phi(N)})$, thus $sa \in \delta(\frac{N}{\phi(N)} : \frac{M}{\phi(N)})$. Therefore, $\frac{N}{\phi(N)}$ is a ϕ_0 - δ - S -primary submodule of $\frac{M}{\phi(N)}$.
 Conversely, suppose that for $a \in R$ and $m \in M$, $am \in N \setminus \phi(N)$. So $0 \neq am + \phi(N) \in \frac{N}{\phi(N)}$. Since $\frac{N}{\phi(N)}$ is a ϕ - δ - S -primary submodule of $\frac{M}{\phi(N)}$, so $s(m + \phi(N)) \in \frac{N}{\phi(N)}$ which implies that $sm \in N$ or $sa \in \delta(\frac{N}{\phi(N)} : \frac{M}{\phi(N)})$ which implies $sa \in \delta(N : M)$. Therefore, N is ϕ - δ - S -primary submodule of M associated to $s \in S$. $\square$

Let $S_1 \subseteq S_2$ be multiplicative subsets of R . Clearly, if N is a ϕ - δ - S_1 -primary submodule of M associated to $s \in S_1$, then N is a ϕ - δ - S_2 -primary submodule of M associated to $s \in S_2$. However, as we see in the next example, the converse does not hold, in general.

Example 2.31. Let $R = \mathbb{Z}$, $M = \mathbb{Z}[X]$ and $N = \langle 4X \rangle = (4X)\mathbb{Z}$. Suppose that $\phi = \phi_\emptyset$, $\delta = \delta_{id}$. It is shown that $(N : M) = (\langle 4X \rangle :_{\mathbb{Z}} \mathbb{Z}[X]) = 0$. Let $S_1 = \{1\}$ and $S_2 = \{2^n \mid n \in \mathbb{N} \cup \{0\}\}$. Suppose that $rf(X) \in (4X)\mathbb{Z}$ for some $r \in \mathbb{Z}$ and $f(X) \in \mathbb{Z}[X] = M$. Then $\sum_{i=0}^n (ra_i)X^i = 4mX$ for $m \in \mathbb{Z}$. So $ra_i = 0$, for $i \neq 1$ and $ra_1 = 4m$. If $r \neq 0$, then $a_i = 0$ for $i \neq 1$. So $f(X) = a_1X$. Since $2^2f(X) \in \langle 4X \rangle$, therefore $\langle 4X \rangle$ is a ϕ - δ - S_2 -primary submodule associated to $2^2 \in S_2$. On the other hand $2.2X \in \langle 4X \rangle$ but, $1.2 \notin (N : M)$ and $1.2X \notin \langle 4X \rangle$. So $\langle 4X \rangle$ is not a ϕ - δ - S_1 -primary submodule of M .

In the following proposition, we show that under some condition on the multiplicatively closed subsets S_1 and S_2 of R (with $S_1 \subseteq S_2$), every ϕ - δ - S_2 -primary submodule can be a ϕ - δ - S_1 -primary submodule.

Proposition 2.32. *Let $S_1 \subseteq S_2$ be multiplicatively closed subsets of R such that for any $s \in S_2$, there exists $t \in S_1$ with $st \in S_1$. If N is a ϕ - δ - S_2 -primary submodule of M associated to $s \in S_2$, then N is a ϕ - δ - S_1 -primary submodule of M associated to $st \in S_1$.*

Proof. Let $t \in S_2$ such that $st \in S_1$. Suppose that $a \in R$ and $m \in M$ and $am \in N \setminus \phi(N)$. Then $sm \in N$ or $sa \in \delta(N : M)$. Therefore $stm \in N$ or $sta \in \delta(N : M)$. Hence, N is a ϕ - δ - S_1 -primary submodule of M associated to $st \in S_1$. $\square$

Let S be a (m.c.s) subset of R with $1 \in S$. Set $S^* = \{x \in R : \frac{x}{1} \in U(S^{-1}R)\}$. It is clear that S^* is multiplicatively closed subset of $S^{-1}R$ which contains S .

Proposition 2.33. *Let M be an R -module and N be a proper submodule of M . Then N is a ϕ - δ - S -primary submodule of M if and only if N is a ϕ - δ - S^* -primary submodule of M .*

Proof. Since $S \subseteq S^*$ and N is a ϕ - δ - S -primary submodule of M associated to $s \in S$, it follows that N is a ϕ - δ - S^* -primary submodule of

M associated to $s \in S^*$. Also, we note that $S^* \cap \delta(NM) = \emptyset$. Indeed, suppose that $r \in S^* \cap \delta(NM)$. Then there exist $a \in R$ and $x \in S$ such that $\frac{r}{1} \frac{a}{x} = 1$. Therefore, there exists $t \in S$ such that $tra = tx$. Since $tra \in \delta(NM) \cap S = \emptyset$, we obtain a contradiction.

Now suppose that N is a ϕ - δ - S^* -primary submodule of M associated to $s \in S^*$. Since $S^* \cap \delta(N : M) = \emptyset$, it is clear that $S \cap \delta(N : M) = \emptyset$. Suppose that $r \in S^*$. Therefore, there exists $\frac{a}{x} \in S^{-1}R$ such that $\frac{r}{1} \frac{a}{x} = \frac{1}{1}$. Consequently, there exists $t \in S$ such that $tra = tx \in S$. Now we have $ta \in S^*$, since $\frac{ta}{1} \frac{r}{xt} = 1$. Therefore, by Theorem 2.32, we conclude that N is a ϕ - δ - S -primary submodule of M . $\square$

Let M be an R -module and $S \subseteq R$ be a multiplicatively closed subset of R and suppose that $S^{-1}M$ is fraction module of M . Let N be a submodule of M with $\delta(N : M) \cap S = \emptyset$. Suppose that, $\phi_S : \mathcal{L}(S^{-1}M) \rightarrow \mathcal{L}(S^{-1}M) \cup \{\emptyset\}$ be a submodule reduction on $\mathcal{L}(S^{-1}M)$ such that $S^{-1}\phi(N) = \phi_S(S^{-1}N)$ for every $N \in \mathcal{L}(M)$, $\delta_S : \mathcal{L}(S^{-1}R) \rightarrow \mathcal{L}(S^{-1}R)$ be an ideal expansion on $S^{-1}R$ such that $\delta_S(S^{-1}I) = S^{-1}\delta(I)$ for every $I \in \mathcal{L}(R)$.

Proposition 2.34. *Let the situation be as described above. Suppose that N is a ϕ - δ - S -primary submodule of finitely generated R -module M associated to $s \in S$. Then $S^{-1}N$ is a ϕ_S - δ_S - S -primary submodule of $S^{-1}M$ associated to $s \in S$.*

Proof. Let $s = \frac{s}{1} \in \delta_S(S^{-1}N : S^{-1}M) \cap S$. Therefore $s \in \delta_S(S^{-1}(N : M)) = S^{-1}(\delta(N : M))$. So $\frac{s}{1} = \frac{a}{t}$ for some $a \in \delta(N : M)$ and $t \in S$. Therefore $ust = ua$ for some $u \in S$. Hence $ua \in S \cap \delta(N : M)$, which is a contradiction. So $\delta_S(S^{-1}N : S^{-1}M) \cap S = \emptyset$. Now let $t_1, t_2 \in S$, $a \in R$ and $m \in M$ such that $\frac{a}{t_1} \frac{m}{t_2} \in S^{-1}N \setminus \phi_S(S^{-1}N)$. So there are $n \in N$ and $s' \in S$ such that $\frac{am}{t_1 t_2} = \frac{n}{s'}$. Therefore $uams' = ut_1 t_2 n \in N$ for some $u \in S$. On the other hand it is clear that $uams' \notin \phi(N)$. But since N is a ϕ - δ - S -primary submodule of M , hence $sa \in \delta(N : M)$ and so $\frac{sa}{t_1} \in S^{-1}\delta(N : M) = \delta_S(S^{-1}(N : M)) \subseteq \delta_S(S^{-1}N : S^{-1}M)$ or $sums' \in N$, which implies that $\frac{sums'}{us't_2} = \frac{sm}{t_2} \in N$. Therefore, $S^{-1}N$ is a ϕ_S - δ_S - S -primary submodule of $S^{-1}M$ associated to $s \in S$. $\square$

Remark 2.35. *Suppose that N be a proper submodule of R -module M , then*

- (1) $N = M$ if and only if $(N : M) = R$.
- (2) $S^{-1}N = S^{-1}M$ if and only if $(S^{-1}N : S^{-1}M) = S^{-1}R$

Let the situation be as described before Proposition 2.34. Suppose that N be a proper submodule of a finitely generated R -module M such that $\delta((N : M) : a) = (\delta(N : M) : a)$ and $\phi(N :_M a) = (\phi(N) :_M a)$ for every $a \in R$. Also assume that $S^{-1}(\delta(N : M)) \cap R = \delta(S^{-1}N \cap R)$. Then the following theorem holds under two mentioned conditions.

Theorem 2.36. *Let N be a proper submodule of finitely generated R -module M . Let $\delta(N : M) \cap S = \emptyset$ and $\delta_S(S^{-1}(N : M)) \neq S^{-1}R$. In addition, assume that $\phi(N) = (\phi(N) :_M s)$ for some $s \in S$ and $(\phi(N) :_M t) \subseteq (\phi(N) :_M s)$ for all $t \in S$. Then the following statements are equivalent:*

- (1) N is ϕ - δ - S -primary submodule of M associated to $s \in S$.
- (2) $(N :_M s)$ is ϕ - δ -primary submodule of M .
- (3) $S^{-1}N$ is ϕ_S - δ_S -primary submodule of $S^{-1}M$. And also for all $t \in S$, we have $(N :_M t) \subseteq (N :_M s)$ and $((N : M) :_M t) \subseteq ((N : M) :_M s)$.
- (4) $S^{-1}N$ is ϕ_S - δ_S -primary submodule of $S^{-1}M$ and $S^{-1}N \cap M = (N :_M s)$ and $S^{-1}(N : M) \cap R = ((N : M) :_M s)$.

Proof. (1) $\Rightarrow$ (2) Since $\delta(N : M) \cap S = \emptyset$, $(N :_M s) \neq M$. Let $am \in (N :_M s) \setminus \phi(N :_M s)$ for $a \in R$ and $m \in M$. By assumption, $am \in (N :_M s) \setminus (\phi(N) :_M s)$. Therefore, $sam \in N \setminus \phi(N)$ but, N is ϕ - δ - S -primary submodule of M associated to $s \in S$, so $s^2m \in N$ or $sa \in \delta(N : M)$. Assume that $s^2m \in \phi(N)$, then $m \in (\phi(N) :_M s^2) \subseteq (\phi(N) :_M s)$. So $sm \in \phi(N)$ and $sam \in \phi(N)$ which is a contradiction. Therefore $s^2m \in N \setminus \phi(N)$ or $sa \in \delta(N : M)$. Case i If $s^2m \in N \setminus \phi(N)$. Then $s^3 \in \delta(N : M)$ which is a contradiction or $sm \in N$ and $m \in (N : s)$. Case ii If $sa \in \delta(N : M)$, then $a \in (\delta(N : M) :_M s) = \delta((N : M) : s) = \delta((N :_M s) : M)$.

(2) $\Rightarrow$ (3) Since $\delta(N : M) \cap S = \emptyset$, $(N : M) \cap S = \emptyset$. Hence $S^{-1}(N : M) \neq S^{-1}R$, so $(S^{-1}N : S^{-1}M) \neq S^{-1}R$ since M is finitely generated R -module. Therefore by Remark 2.35, $S^{-1}N \neq S^{-1}M$. Now let $\frac{a}{s_1} \frac{m}{s_2} \in S^{-1}N \setminus \phi_S(S^{-1}N)$ for $\frac{a}{s_1} \in S^{-1}R$ and $\frac{m}{s_2} \in S^{-1}M$. So there is $t \in S$ such that $tam \in N \setminus \phi(N)$, which means that $tam \in (N :_M s) \setminus \phi(N :_M s)$. $(N :_M s)$ is ϕ - δ -primary submodule of M , so $m \in (N :_M s)$ which implies $\frac{m}{s_2} \in S^{-1}N$ or $ta \in \delta((N :_M s) : M)$ which implies $\frac{a}{s_1} \in S^{-1}\delta(N :_M s)$.

$M) = \delta_S(S^{-1}(N : M)) \subseteq \delta_S(S^{-1}N : S^{-1}M)$. Therefore $S^{-1}N$ is ϕ_S - δ_S -primary submodule of $S^{-1}M$.

Now let $t \in S$ and $m \in (N :_M t)$. If $m \in (\phi(N) :_M t)$, then $m \in (\phi(N) :_M s) \subseteq (N :_M s)$. If $m \notin (\phi(N) :_M t)$, then $mt \notin \phi(N) = (\phi(N) :_M s) = \phi(N : s)$. Also $mt \in N \subseteq (N :_M s)$, so $mt \in (N :_M s) \setminus \phi(N :_M s)$. By (2) $m \in (N :_M s)$ which completes the proof or $t \in \delta((N :_M s) : M) = (\delta(N : M) :_M s)$. Hence $ts \in \delta(N : M)$ which is a contradiction.

Let $r \in ((N : M) :_M t)$, then $rM \subseteq (N :_M t) \subseteq (N :_M s)$ and $r \in ((N : M) :_M s)$.

(3) $\Rightarrow$ (4) The first part is obvious. We note that $(N :_M s) \subseteq S^{-1}N \cap M$. Now let $\frac{m}{1} \in S^{-1}N \cap M$, then there is $u \in S$ such that $um \in N$. By assumption $m \in (N :_M u) \subseteq (N :_M s)$. Therefore, $S^{-1}N \cap M = (N :_M s)$.

Let $r \in ((N : M) :_M s)$, then $rs \in (N : M)$. Hence $r = \frac{rs}{s} \in S^{-1}(N : M) \cap R$. Conversely if $r = \frac{r}{1} \in S^{-1}(N : M) \cap R$, then there is $t \in S$ such that $tr \in (N : M)$. Hence $r \in ((N : M) :_M t) \subseteq ((N : M) :_M s)$.

(4) $\Rightarrow$ (1) Suppose that $am \in N \setminus \phi(N)$ where $a \in R$ and $m \in M$. First we show that $\frac{a}{1}\frac{m}{1} \notin \phi_S(S^{-1}N)$. Let $\frac{a}{1}\frac{m}{1} \in \phi_S(S^{-1}N) = S^{-1}\phi(N)$. Thus $t \in S$ exists such that $tam \in \phi(N)$, so $am \in (\phi(N) :_M t) \subseteq \phi(N :_M s) = \phi(N)$. Hence $am \in \phi(N)$ which is a contradiction. So $\frac{a}{1}\frac{m}{1} \in S^{-1}N \setminus \phi_S(S^{-1}N)$, but N_S is ϕ_S - δ_S -primary submodule of M_S . Then $\frac{m}{1} \in S^{-1}N$ or $\frac{a}{1} \in \delta_S(S^{-1}N : S^{-1}M)$. If $\frac{m}{1} \in S^{-1}N$, then $m \in S^{-1}N \cap M = (N :_M s)$ and $sm \in N$. If $\frac{a}{1} \in \delta_S(S^{-1}N : S^{-1}M) = \delta_S(S^{-1}(N : M)) = S^{-1}\delta(N : M)$. Thus $\frac{a}{1} \in S^{-1}\delta(N : M) \cap R = \delta(S^{-1}(N : M) \cap R) = \delta((N : M) :_M s) = (\delta(N : M) : s)$. $\square$

3 Image and Inverse Image of ϕ - δ - S -Primary Submodules

Let M and M' be two R -modules and $f : M \rightarrow M'$ be an R -module homomorphism. Suppose that ϕ is a reduction function on $\mathcal{L}(M)$ and ϕ' is a reduction on $\mathcal{L}(M')$. Then f is called (ϕ, ϕ') -homomorphism if $\phi(f^{-1}(N')) = f^{-1}(\phi'(N'))$ for all $N' \in \mathcal{L}(M')$.

For any (ϕ, ϕ') -epimorphism we can give the following remark.

Remark 3.1. Let $f : M \rightarrow M'$ be a nonzero $(\phi-\phi')$ -epimorphism and N be a proper submodule of M containing $\ker(f)$, then

- (1) $(N : M) = (f(N) : M')$.
- (2) $f(\phi(N)) = \phi'(f(N))$.

Proof. (1) Let $r \in (N : M)$, so $rM \subseteq N$. Since f is epimorphism, $rM' = f(rM) \subseteq f(N)$ so $r \in (f(N) : M')$. Conversely let $r \in (f(N) : M')$, so $rM' = rf(M) \subseteq f(N)$. Then $rM \subseteq rM + \ker(f) \subseteq f^{-1}f(N) = N + \ker(f) = N$. Then $rM \subseteq N$ and $r \in (N : M)$ which implies $(N : M) = (f(N) : M')$.

(2) We know that $f^{-1}f(N) = N + \ker(f) = N$, Hence $f(\phi(N)) = f(\phi(f^{-1}f(N))) = f(f^{-1}(\phi'(f(N)))) = \phi'(f(N))$. $\square$

Theorem 3.2. Let $f : M \rightarrow M'$ be a nonzero $(\phi-\phi')$ -epimorphism and N be a proper submodule of M containing $\ker(f)$. Then N is ϕ - δ - S -primary submodule of M associated to $s \in S$ if and only if $f(N)$ is a ϕ' - δ - S -primary submodule of M' associated to $s \in S$.

Proof. By Remark 3.1, $\delta(f(N) : M') \cap S = \emptyset$ if and only if $\delta(N : M) \cap S = \emptyset$ and $f(N) \neq M'$. Let $a \in R$, $m' \in M'$ and $am' \in f(N) \setminus \phi'(f(N)) = f(N) \setminus f(\phi(N))$, (By Remark 3.1). Since f is epimorphism, so there is $m \in M$ such that $m' = f(m)$. Hence $f(am) \in f(N) \setminus f(\phi(N))$. Since $\ker(f) \subseteq N$, so $f(am) \in f(N)$ implies that $am \in N$. On the other hand $f(am) \notin f(\phi(N))$ which implies that $am \notin \phi(N)$. So $am \in N \setminus \phi(N)$. As N is a ϕ - δ - S -primary submodule of M associated to $s \in S$, we conclude that $sm \in N$ or $sa \in \delta(N : M)$. Therefore $sm' = sf(m) = f(sm) \in f(N)$ or $sa \in \delta(f(N) : M')$.

Now let $f(N)$ be a ϕ' - δ - S -primary submodule of M' associated to $s \in S$. We show that N is a ϕ - δ - S -primary submodule of M associated to $s \in S$. Let $a \in R$, $m \in M$ and $am \in N \setminus \phi(N)$. So $af(m) \in f(N)$ and $af(m) \notin f(\phi(N)) = \phi'(f(N))$. But $f(N)$ is a ϕ' - δ - S -primary submodule of M' associated to $s \in S$. Therefore $sa \in \delta(f(N) : M') = \delta(N : M)$ or $f(sm) = sf(m) \in f(N)$, consequently $sm \in f^{-1}f(N) = N$. $\square$

Corollary 3.3. Let $f : M \rightarrow M'$ be a nonzero $(\phi-\phi')$ -epimorphism. Suppose that N' is ϕ' - δ - S -primary submodule of M' . Then $f^{-1}(N')$ is a ϕ - δ - S -primary submodule of M .

Proof. We know that $\ker(f) = f^{-1}(\{0\}) \subseteq f^{-1}(\phi'(N')) = \phi(f^{-1}(N')) \subseteq f^{-1}(N')$. Also we know that, f is an epimorphism, so $f(f^{-1}(N')) = N'$. Since N' is a ϕ' - δ - S -primary submodule of M' , so by Theorem 3.2, $f^{-1}(N')$ is a ϕ - δ - S -primary submodule of M . Since $(f^{-1}(N') : M) \subseteq (N' : M')$, hence $\delta(f^{-1}(N') : M) \cap S = \emptyset$. $\square$

The next theorem is an immediate consequence of Theorem 3.2.

Theorem 3.4. *Let M be an R -module and K be a proper submodule of M . Suppose that $f : M \rightarrow \frac{M}{K}$ is a nonzero (ϕ, ϕ') -natural epimorphism. Then, there is a one-to-one correspondence between the set of all ϕ - δ - S -primary submodules of M containing K and the set of all ϕ - δ - S -primary submodules of $\frac{M}{K}$.*

In Example 2.25, we showed that the class of δ - S -primary submodules is a proper subclass of ϕ - δ - S -primary submodules. Motivated by the above result, we now introduce the following definitions.

Definition 3.5. *Let N be a ϕ - δ - S -primary submodule of M associated to $s \in S$, which is not δ - S -primary submodule. Then there is $a \in R$ and $m \in M$ such that $am \in N$ but $am \in \phi(N)$, $sm \notin N$ and $sa \notin \delta(N : M)$. In this case, we say that (a, m) , is a ϕ - δ - S -twin zero of N .*

Definition 3.6. *Let I be a proper ideal of R and K a proper submodule of M such that $IK \subseteq N$ and $IK \not\subseteq \phi(N)$, and N a ϕ - δ - S -primary submodule of M . Then N is said to be a ϕ - δ - S -free twin zero submodule of M with respect to IK , if (a, k) is not a ϕ - δ - S -twin zero of N for every $a \in I$ and $k \in K$.*

In particular, we say that a submodule N of R -module M is a ϕ - δ - S -free twin zero if, for an ideal I of R and a submodule K of M , whenever $IK \subseteq N$ and $IK \not\subseteq \phi(N)$, then N is a ϕ - δ - S -free twin zero with respect to IK .

Theorem 3.7. *Suppose that N is a ϕ - δ - S -primary submodule of M associated to $s \in S$. Then N is ϕ - δ - S -free twin zero if and only if for every ideal I of R and every proper submodule K of M with $IK \subseteq N$ and $IK \not\subseteq \phi(N)$, either $sK \subseteq N$ or $sI \subseteq \delta(N : M)$.*

Proof. Suppose that N is a ϕ - δ - S -free twin zero submodule of M associated to $s \in S$. Let $IK \subseteq N$ and $IK \not\subseteq \phi(N)$, for some ideal I of

R and proper submodule K of M . Suppose that $sI \not\subseteq \delta(N : M)$. Then there is $a \in I$ such that $sa \in sI$ but $sa \notin \delta(N : M)$. Let $m \in K$, then (a, m) is not ϕ - δ - S -twin zero of N . If $am \notin \phi(N)$, then $am \in N \setminus \phi(N)$. Since N is ϕ - δ - S -primary submodule of M associated to $s \in S$ and $sa \notin \delta(N : M)$, so $sm \in N$. Therefore $sK \subseteq N$, which completes the proof. Now let $am \in \phi(N)$. Since $sa \notin \delta(N : M)$ and N is ϕ - δ - S -free twin zero, thus $sm \in N$ which implies $sK \subseteq N$.

Conversely, let K be a submodule of M and I be an ideal of R . And suppose that $IK \subseteq N$ and $IK \not\subseteq \phi(N)$ implies that either $sK \subseteq N$ or $sI \subseteq \delta(N : M)$. We prove that N is a ϕ - δ - S -free twin zero with respect to IK . Let $a \in I$ and $k \in K$. If (a, k) is a ϕ - δ - S -twin zero of N , then $ak \in \phi(N)$, $sk \notin N$ and $sa \notin \delta(N : M)$, which is a contradiction. So (a, k) is not a ϕ - δ - S -twin zero of N . So N is ϕ - δ - S -free twin zero with respect to IK . Thus N is ϕ - δ - S -free twin zero submodule of M . $\square$

4 Direct Product of ϕ - δ - S -Primary Submodules

Suppose that $R = R_1 \times R_2$ is a direct product of commutative rings. Let M_i be an R_i -module and S_i be a (m.c.s) of R_i , for $i = 1, 2$. Then $M = M_1 \times M_2$ can be viewed as an R -module using scalar multiplication $(r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2)$ and $S = S_1 \times S_2$ can be viewed as a (m.c.s) of R . For $i = 1, 2$ let δ_i be an expansion function of ideals of R_i and ϕ_i be a reduction function of submodules of M_i . We define the following two functions: $\delta_\times(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ and $\phi_\times(N_1 \times N_2) = \phi_1(N_1) \times \phi_2(N_2)$. It is easy to see that $\delta_\times$ is an ideal expansion of R and $\phi_\times$ is a submodule reduction of M [10]. With the assumptions stated above, the following statements can be formulated.

Theorem 4.1. *Let $M = M_1 \times M_2$ and N_1 be a proper submodule of M_1 , the following statements are equivalent:*

- (1) $N_1 \times M_2$ is a $\phi_\times$ - $\delta_\times$ - S -primary submodule of M associated to $(s_1, s_2) \in S$.
- (2) N_1 is a δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$ with $\phi_2(M_2) \neq M_2$ and also $N_1 \times M_2$ is a $\delta_\times$ - S -primary submodule of M associated to $(s_1, s_2) \in S$.

Proof. We show that $\delta_{\times}(N_1 \times M_2 : M_1 \times M_2) \cap (S_1 \times S_2) = \emptyset$, if and only if $\delta_1(N_1 : M_1) \cap S_1 = \emptyset$. If $r \in \delta_1(N_1 : M_1) \cap S_1$, then $(r, 1) \in S_1 \times S_2$ and also $(r, 1) \in \delta_1(N_1 : M_1) \times R_2 = \delta_1(N_1 : M_1) \times \delta_2(M_2 : M_2)$, which is a contradiction. Conversely if $(r_1, r_2) \in \delta_{\times}(N_1 \times M_2 : M_1 \times M_2) \cap (S_1 \times S_2) = \delta_1(N_1 : M_1) \times \delta_2(M_2 : M_2) \cap (S_1 \times S_2)$, then $r_1 \in \delta_1(N_1 : M_1) \cap S_1$, which is a contradiction. We now prove that the statements are equivalent.

(1) $\Rightarrow$ (2) Suppose that $N_1 \times M_2$ is a $\phi_{\times}$ - $\delta_{\times}$ - S -primary submodule of M associated to $(s_1, s_2) \in S$. Let $a \in R_1$, $m_1 \in M_1$ and $am_1 \in N_1$. Since $M_2 \neq \phi_2(M_2)$, so there is $m_2 \in M_2 \setminus \phi(M_2)$. Hence $(a, 1)(m_1, m_2) = (am_1, m_2) \in N_1 \times M_2 \setminus \phi_{\times}(N_1 \times M_2)$. So $(s_1, s_2)(m_1, m_2) \in N_1 \times M_2$ or $(s_1, s_2)(a, 1) \in \delta_{\times}(N_1 \times M_2 : M_1 \times M_2)$. Since $\delta_{\times}(N_1 \times M_2 : M_1 \times M_2) = \delta_1(N_1 : M_1) \times \delta_2(M_2 : M_2)$, hence $s_1 m_1 \in N_1$ and $s_2 m_2 \in M_2$ or $s_1 a \in \delta_1(N_1 : M_1)$ and $s_2 \in \delta_2(M_2 : M_2)$. Therefore, $s_1 m_1 \in N_1$ or $s_1 a \in \delta_1(N_1 : M_1)$. So N_1 is a δ_1 - S_1 -primary submodule of M associated to $(s_1, s_2) \in S$. Now suppose that $N_1 \times M_2$ is not a $\delta_{\times}$ - S -primary submodule of M associated to $(s_1, s_2) \in S$. By Proposition 2.26, $(N_1 \times M_2)((N_1 \times M_2) : (M_1 \times M_2)) \subseteq \phi_{\times}(N_1 \times M_2)$, so $(N_1 \times M_2)((N_1 : M_1) \times (M_2 : M_2)) \subseteq \phi_{\times}(N_1 \times M_2)$. Thus $(N_1 \times M_2)((N_1 : M_1) \times R_2) = (N_1(N_1 : M_1) \times (M_2 \times R_2)) \subseteq \phi_1(N_1) \times \phi_2(M_2)$, which results $N_1(N_1 : M_1) \subseteq \phi_1(N_1)$ and $M_2 \subseteq \phi_2(M_2)$. Hence, we obtain a contradiction with $M_2 \neq \phi_2(M_2)$.

(2) $\Rightarrow$ (1) It is obvious, since we know that every $\delta_{\times}$ - S -primary submodule of M is a $\phi_{\times}$ - $\delta_{\times}$ - S -primary submodule of M . $\square$

Theorem 4.2. Let $M = M_1 \times M_2$ and N_1 be a proper submodule of M_1 . The following statements are equivalent:

- (1) $N_1 \times M_2$ is a $\phi_{\times}$ - $\delta_{\times}$ - S -primary submodule of M associated to $(s_1, s_2) \in S$, with $\phi_2(M_2) = M_2$.
- (2) N_1 is a ϕ_1 - δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$.

Proof. We know that $\delta_{\times}(N_1 \times M_2 : M_1 \times M_2) \cap (S_1 \times S_2) = \emptyset$ if and only if $\delta_1(N_1 : M_1) \cap S_1 = \emptyset$.

(1) $\Rightarrow$ (2) Let $a_1 m_1 \in N_1 \setminus \phi_1(N_1)$ for $a_1 \in R_1$ and $m_1 \in M_1$. Then $(a_1, 0)(m_1, 0) \in (N_1 \times M_2) \setminus \phi_{\times}(N_1 \times M_2)$. By assumption, $(s_1, s_2)(m_1, 0) \in (N_1 \times M_2)$ or $(s_1, s_2)(a_1, 0) \in \delta_{\times}(N_1 \times M_2 : M_1 \times M_2)$, which implies $s_1 m_1 \in N_1$ or $s_1 a_1 \in \delta_1(N_1 : M_1)$. Therefore, N_1 is a ϕ_1 - δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$.

(2) $\Rightarrow$ (1) Suppose that N_1 is a ϕ_1 - δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$ and $\phi_2(M_2) = M_2$. Let $(a_1, a_2)(m_1, m_2) \in N_1 \times M_2 \setminus \phi_\times(N_1 \times M_2)$. Hence $(a_1 m_1, a_2 m_2) \in N_1 \times M_2 \setminus \phi_\times(N_1 \times M_2)$. Thus $(a_1 m_1, a_2 m_2) \in N_1 \times M_2 \setminus \phi(N_1) \times M_2$. But since $a_2 m_2 \in M_2 = \phi(M_2)$, so we must have $a_1 m_1 \in N_1$ and $a_1 m_1 \notin \phi_1(N)$. Now we have $a_1 m_1 \in N_1 \setminus \phi_1(N_1)$. Then $s_1 m_1 \in N_1$ or $s_1 a_1 \in \delta_1(N_1 : M_1)$ which implies $(s_1, s_2)(m_1, m_2) = (s_1 m_1, s_2 m_2) \in N_1 \times M_2$ or $(s_1, s_2)(a_1, a_2) \in \delta_1(N_1 : M_1) \times \delta_2(M_2 : M_2) = \delta_\times(N_1 \times M_2 : M_1 \times M_2)$. Since $s_2 a_2 \in \delta_2(M_2 : M_2) = R_2$. Therefore, $N_1 \times M_2$ is a $\phi_\times$ - $\delta_\times$ - S -primary submodule of M associated to $(s_1, s_2) \in S$. $\square$

Remark 4.3. *There are similar results for any submodule $M_1 \times N_2$ of R -module $M_1 \times M_2$ where N_2 is a submodule of M_2 .*

Proposition 4.4. *Let $M = M_1 \times M_2$ be an $R = R_1 \times R_2$ module and $S = S_1 \times S_2$. Suppose that $N = N_1 \times N_2$ is $\phi_\times$ - $\delta_\times$ - S -primary submodule of M . Then N_1 is ϕ_1 - δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$ or N_2 is ϕ_2 - δ_2 - S_2 -primary submodule of M_2 associated to $s_2 \in S_2$.*

Proof. Let $N = N_1 \times N_2$ be $\phi_\times$ - $\delta_\times$ - S -primary submodule of M associated to $(s_1, s_2) \in S_1 \times S_2$. Hence $\delta_\times(N_1 \times N_2 : M_1 \times M_2) \cap S = \emptyset$, so $\delta_1(N_1 : M_1) \cap S_1 = \emptyset$ or $\delta_2(N_2 : M_2) \cap S_2 = \emptyset$. Assume that $\delta_2(N_2 : M_2) \cap S_2 = \emptyset$. Now we show that N_2 is ϕ_2 - δ_2 - S_2 -primary submodule of M_2 associated to $s_2 \in S_2$. Let $r_2 m_2 \in N_2 \setminus \phi_2(N_2)$, for $m_2 \in M_2$ and $r_2 \in R_2$. Hence, $(1, r_2)(0, m_2) = (0, r_2 m_2) \in N_1 \times N_2 \setminus \phi_\times(N_1 \times N_2)$. Based on assumptions, $(s_1, s_2)(0, m_2) \in N_1 \times N_2$ or $(s_1, s_2)(1, r_2) \in \delta_\times(N_1 \times N_2 : M_1 \times M_2) = \delta_1(N_1 : M_1) \times \delta_2(N_2 : M_2)$. Then $0 \in N_1$ and $s_2 m_2 \in N_2$ or $s_1 \in \delta_1(N_1 : M_1)$ and $s_2 r_2 \in \delta_2(N_2 : M_2)$. So $s_2 m_2 \in N_2$ or $s_2 r_2 \in \delta_2(N_2 : M_2)$. Therefore N_2 is ϕ_2 - δ_2 - S_2 -primary submodule of M_2 associated to $s_2 \in S_2$. Similarly, if assume that $\delta_1(N_1 : M_1) \cap S_1 = \emptyset$, it can be proved that N_1 is ϕ_1 - δ_1 - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$. $\square$

Corollary 4.5. *Suppose that $P = P_1 \times P_2$ is an ideal of $R = R_1 \times R_2$. If $P = P_1 \times P_2$ is a $\phi_\times$ - $\delta_\times$ - S -primary ideal of R associated to $(s_1, s_2) \in S_1 \times S_2$, then P_1 is ϕ_1 - δ_1 - S_1 -primary ideal of R_1 associated to $s_1 \in S_1$ or P_2 is ϕ_2 - δ_2 - S_2 -primary ideal of R_2 associated to $s_2 \in S_2$.*

The following example shows that the converse of the above proposition is not true.

Example 4.6. Let $M_1 \times M_2 = \mathbb{Z} \times \mathbb{Z}$ be an $R_1 \times R_2 = \mathbb{Z} \times \mathbb{Z}$ -module. Suppose that $N_1 = p\mathbb{Z}$, $N_2 = q\mathbb{Z}$, such that p and q are two distinct prime numbers and $S_1 = \{q^n \mid n \in \mathbb{N}_0\}$, $S_2 = \{p^n \mid n \in \mathbb{N}_0\}$. Let $am \in p\mathbb{Z}$, so $p \mid am$. Hence $p \mid a$ or $p \mid m$ and $m \in p\mathbb{Z}$ or $a \in \delta_{rad}(p\mathbb{Z} : \mathbb{Z}) = p\mathbb{Z}$. Therefore N_1 and similarly N_2 are $\phi_{\emptyset}$ - δ_{rad} - S_i -primary submodules of $\mathbb{Z}$. On the other hand $(p, 1)(1, q) \in p\mathbb{Z} \times q\mathbb{Z}$, but for every $(s_1, s_2) \in S_1 \times S_2$ neither $(s_1, s_2)(p, 1) \in p\mathbb{Z} \times q\mathbb{Z}$, nor $(s_1, s_2)(1, q) \in \delta_{\times}(N_1 \times N_2 : M_1 \times M_2) = N_1 \times N_2$. Therefore $p\mathbb{Z} \times q\mathbb{Z}$ is not a $\phi_{\times}$ - $\delta_{\times}$ - S -primary submodule of $M_1 \times M_2$.

5 $\phi_{\bowtie}$ - $\delta_{\bowtie}$ - $S_{\bowtie}$ -Primary Amalgamation Submodules

Let R be a ring and I an ideal of R . The amalgamated duplication of R along the ideal I is defined by $R \bowtie I = \{(a, a + i) : a \in R, i \in I\}$, which is a subring of $R \times R$. Also if $f : R \rightarrow S$ be an R -homomorphism and J is an ideal of S , then $R \bowtie^f J = \{(a, f(a) + j) : a \in R, j \in J\}$ is called the amalgamation of R and S along J with respect to f , which is a subring of $R \times S$.

Suppose that M be an R -module, N a submodule of M and I be an ideal of R . Then the duplication of the R -module M along the ideal I is defined by $M \bowtie I = \{(m, m') \in M \times M : m - m' \in IM\}$, which is an $(R \bowtie I)$ -module under the following scalar multiplication. $(a, a + i).(m, m') := (am, (a + i)m')$ for $a \in R, i \in I$ and $(m, m') \in M \bowtie I$. It is clear that $N \bowtie I = \{(n, m) \in N \times M : n - m \in IM\}$ and $\overline{N} = \{(m, n) \in M \times N : m - n \in IM\}$ are submodules of $M \bowtie I$ [2]. Let K be an ideal of R and L an ideal of S . Set $K \bowtie^f J := \{(k, f(k) + j) : k \in K, j \in J\}$ and $\overline{L}^f := \{(r, f(r) + j) : r \in R, f(r) + j \in L\}$, then one can show that $K \bowtie^f J$ and $\overline{L}^f$ are ideals of $R \bowtie^f J$. We call them amalgamation ideals of $R \bowtie^f J$.

Let M_1, M_2 be two R -modules and $\varphi : M_1 \rightarrow M_2$ be R -homomorphism. Then $M_1 \bowtie^{\varphi} JM_2$ is called the amalgamation of M_1 and M_2 along J with respect to φ is defined $M_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) : m_1 \in M_1, m_2 \in JM_2\}$, which is a $(R \bowtie^f J)$ -module with the following scalar multiplication, $(a, f(a) + i)(m_1, \varphi(m_1) + m_2) = (am_1, \varphi(am_1) + f(a)m_2 + i\varphi(m_1) + im_2)$. Let N_1 and N_2 be submodules of M_1 and M_2 respectively, then $N_1 \bowtie^{\varphi} JM_2 = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^{\varphi} JM_2 : m_1 \in N_1\}$ and

$\overline{N}_2^\varphi = \{(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2 : \varphi(m_1) + m_2 \in N_2\}$ are submodules of $M_1 \times^\varphi JM_2$ [4]. We call them amalgamation submodules of $M_1 \bowtie^\varphi JM_2$.

In the following, we define the reduction and expansion functions on the above two submodules. Let M_1, M_2 be two R -modules, I an ideal of R , J and K be ideals of S , N_1 and N_2 be submodules of M_1 and M_2 respectively.

Definition 5.1. Let ϕ_1 and ϕ_2 be reduction submodules on $\mathcal{L}(M_1)$ and $\mathcal{L}(M_2)$ respectively. Set:

- (1) $\phi_{\bowtie}(N_1 \bowtie^\varphi JM_2) := \phi_1(N_1) \bowtie^\varphi JM_2$.
- (2) $\phi_{\bowtie}(\overline{N}_2^\varphi) := \overline{\phi_2(N_2)}^\varphi$.

Clearly, $\phi_{\bowtie}$ is a reduction function on submodules $N_1 \bowtie^\varphi J(M_2)$ and $\overline{N}_2^\varphi$ of $M_1 \bowtie^\varphi JM_2$.

Definition 5.2. Let δ_1 and δ_2 be expansion ideals on $\mathcal{L}(R)$ and $\mathcal{L}(S)$ respectively. Set:

- (1) $\delta_{\bowtie}(I \bowtie^f J) := \delta_1(I) \bowtie^f J$.
- (2) $\delta_{\bowtie}(\overline{K}^f) := \overline{\delta_2(K)}^f$.

Clearly, $\delta_{\bowtie}$ is an expansion function on amalgamation ideals of $I \bowtie^f J$ and $\overline{K}^f$ respectively.

Let S_1 be a multiplicatively closed subset of R and S_2 be a multiplicatively closed subset of S . Set $S_{\bowtie} := S_1 \bowtie^f J$ and $S_{\bowtie} := \overline{S_2}^f$. Clearly, $S_{\bowtie}$ is a multiplicatively closed subset of $R \bowtie^f J$ with $(1, 1) \in S_{\bowtie}$ in both cases.

Proposition 5.3. Let N_1 and N_2 be submodules of M_1 and M_2 respectively. Then :

- (1) $(N_1 \bowtie^\varphi JM_2 :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2) = (N_1 : M_1) \bowtie^f J$.
- (2) $(\overline{N}_2^\varphi :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2) = \overline{(N_2 : M_2)}^f$.

Proof. We only prove the second statement. It is clear that,

$(\overline{N}_2^\varphi :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2) \subseteq \overline{(N_2 : M_2)}^f$. Conversely, If $(r, f(r) + j) \in \overline{(N_2 : M_2)}^f$, then $f(r) + j \in (N_2 : M_2)$ and $(f(r) + j)M_2 \subseteq N_2$. Hence $(r, f(r) + j) \in (\overline{N}_2^\varphi :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2)$. $\square$

Theorem 5.4. *Let the situation be as described above definition and S_1 be a m.c.s. of R . Then N_1 is a ϕ - δ - S_1 -primary submodule of M_1 associated to $s_1 \in S_1$ if and only if $N_1 \bowtie^\varphi JM_2$ is a $\phi_{\bowtie}$ - $\delta_{\bowtie}$ - $S_{\bowtie}$ -primary submodule of $M_1 \bowtie^\varphi JM_2$ associated to $(s_1, f(s_1)) \in S_{\bowtie}$.*

Proof. First notice that N_1 is a proper submodule if and only if $N_1 \bowtie^\varphi JM_2$ is a proper submodule. Also, we show that $\delta_{\bowtie}(N_1 \bowtie^\varphi JM_2 : M_1 \bowtie^\varphi JM_2) \cap (S_1 \bowtie^f J) = \emptyset$, if and only if $\delta(N_1 : M_1) \cap S_1 = \emptyset$. We know that $0 \neq r \in \delta(N_1 : M_1) \cap S_1$ if and only if $(0, 0) \neq (r, f(r)) \in \delta(N_1 : M_1) \bowtie^f J \cap (S_1 \bowtie^f J) = \delta_{\bowtie}((N_1 : M_1) \bowtie^f J) \cap (S_1 \bowtie^f J) = \delta_{\bowtie}(N_1 \bowtie^\varphi JM_2 : M_1 \bowtie^\varphi JM_2) \cap (S_1 \bowtie^f J)$.

($\Rightarrow$) Let $(a, f(a) + j) \in R_1 \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2$ such that $(a, f(a) + j)(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^\varphi JM_2 \setminus \phi_{\bowtie}(N_1 \bowtie^\varphi JM_2)$. Then $am_1 \in N_1$ and $\varphi(am_1) + f(a)m_2 + j(\varphi(m_1) + m_2) \in JM_2$. Since $(am_1, \varphi(am_1) + f(a)m_2 + j(\varphi(m_1) + m_2)) \notin \phi_{\bowtie}(N_1 \bowtie^\varphi JM_2) = \phi(N_1) \bowtie^\varphi JM_2$, so $am_1 \notin \phi(N_1)$. Hence $am_1 \in N_1 \setminus \phi(N_1)$. By assumption $s_1 m_1 \in N_1$ or $s_1 a \in \delta(N_1 : M_1)$. Case i $s_1 m_1 \in N_1$, then $(s_1, f(s_1))(m_1, \varphi(m_1) + m_2) \in N_1 \bowtie^\varphi JM_2$. Case ii $(s_1, f(s_1))(a, f(a) + j) \in \delta(N_1 : M_1) \bowtie^f J = \delta_{\bowtie}(N_1 \bowtie^\varphi JM_2 :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2)$.

($\Leftarrow$) Let $am_1 \in N_1 \setminus \phi(N_1)$, for $a \in R$ and $m_1 \in M_1$. Then $(a, f(a)) \in R \bowtie^f J$, $(m_1, \varphi(m_1)) \in M_1 \bowtie JM_2$ and $(am_1, \varphi(am_1)) \in N_1 \bowtie^\varphi JM_2$. Since $am_1 \notin \phi(N_1)$, so $(am_1, \varphi(am_1)) \notin \phi(N_1) \bowtie^\varphi JM_2 = \phi_{\bowtie}(N_1 \bowtie^\varphi JM_2)$. Hence $(s_1, f(s_1))(m_1, \varphi(m_1)) \in N_1 \bowtie^\varphi JM_2$ or $(s_1, f(s_1))(a, f(a)) \in \delta_{\bowtie}(N_1 \bowtie^\varphi JM_2 :_{R \bowtie^f J} M_1 \bowtie^\varphi JM_2) = \delta(N_1 : M_1) \bowtie^f J$. Thus $s_1 m_1 \in N_1$ or $s_1 a \in \delta(N_1 :_R M_1)$. $\square$

Theorem 5.5. *Let the situation be as described above definition and S_2 be a m.c.s. of S . Suppose that f and φ are epimorphisms. Then N_2 is a ϕ - δ - S_2 -primary submodule of M_2 associated to $s_2 \in S_2$ if and only if $\overline{N}_2^\varphi$ is a $\phi_{\bowtie}$ - $\delta_{\bowtie}$ - $S_{\bowtie}$ -primary submodule of $M_1 \bowtie^\varphi JM_2$ associated to $(f^{-1}(s_2), s_2) \in S_{\bowtie} = \overline{S}_2^f$.*

Proof. We show that $\delta(N_2 : M_2) \cap S_2 = \emptyset$ if and only if $\delta_{\bowtie}(\overline{N}_2^\varphi : M_1 \bowtie^\varphi JM_2) \cap \overline{S}_2^f = \emptyset$. We know that $0 \neq b \in \delta(N_2 : M_2) \cap S_2$ if and only if $(0, 0) \neq (f^{-1}(b), b) \in \overline{\delta(N_2 : M_2)}^f \cap \overline{S}_2^f = \delta_{\bowtie}((\overline{N}_2 : M_2)^f) \cap \overline{S}_2^f = \delta_{\bowtie}(\overline{N}_2^\varphi : M_1 \bowtie^\varphi JM_2) \cap \overline{S}_2^f$.

($\Rightarrow$) First, $\overline{N}_2^\varphi$ is a proper submodule. If $\overline{N}_2^\varphi = M_1 \bowtie^\varphi JM_2$, then

for every $m_1 \in M_1$ and $g \in JM_2$, we have $\varphi(m_1) + g \in N_2$. So $M_2 = M_2 + JM_2 \subseteq N_2$, which is a contradiction. Let $(a, f(a) + j) \in R \bowtie^f J$ and $(m_1, \varphi(m_1) + m_2) \in M_1 \bowtie^\varphi JM_2$, for $a \in R$ and $m_2 \in JM_2$ such that $(a, f(a) + j)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi \setminus \phi_{\bowtie}(\overline{N_2}^\varphi)$. So $(f(a) + j)(\varphi(m_1) + m_2) \in N_2$. Since $(a, f(a) + j)(m_1, \varphi(m_1) + m_2) \notin \phi_{\bowtie}(\overline{N_2}^\varphi) = \overline{\phi(N_2)}^\varphi$, then $(f(a) + j)(\varphi(m_1) + m_2) \notin \phi(N_2)$. So $(f(a) + j)(\varphi(m_1) + m_2) \in N_2 \setminus \phi(N_2)$. Hence $s_2(\varphi(m_1) + m_2) \in N_2$ or $s_2(f(a) + j) \in \delta(N_2 :_S M_2)$. Case i $(f^{-1}(s_2), s_2)(m_1, \varphi(m_1) + m_2) \in \overline{N_2}^\varphi$. Case ii $(f^{-1}(s_2), s_2)(a, f(a) + j) \in \overline{\delta(N_2 : M_2)}^f = \delta_{\bowtie}(\overline{N_2 : M_2})^f = \delta_{\bowtie}(\overline{N_2}^\varphi : M_1 \bowtie^\varphi JM_2)$.
 $(\Leftarrow)$ We note that N_2 is a proper submodule of M_2 . Let $a_2 m_2 \in N_2 \setminus \phi(N_2)$ for $a_2 \in R_2$ and $m_2 \in M_2$. Since φ and f are epimorphisms, there are $a_1 \in R$ and $m_1 \in M_1$ such that $a_2 = f(a_1)$ and $m_2 = \varphi(m_1)$. So $(a_1, a_2) \in R \bowtie^f J$, $(m_1, m_2) \in M_1 \bowtie^\varphi JM_2$ and $(a_1, a_2)(m_1, m_2) = (a_1 m_1, a_2 m_2) \in \overline{N_2}^\varphi$. Since $a_2 m_2 \notin \phi(N_2)$, so $(a_1 m_1, a_2 m_2) \in \overline{N_2}^\varphi \setminus \phi_{\bowtie}(\overline{N_2}^\varphi)$, by assumption $(f^{-1}(s_2), s_2)(m_1, m_2) \in \overline{N_2}^\varphi$ or $(f^{-1}(s_2), s_2)(a_1, a_2) \in \delta_{\bowtie}(\overline{N_2}^\varphi : M_1 \bowtie^\varphi JM_2) = \overline{\delta(N_2 : M_2)}^f$. Case i $s_2 m_2 \in N_2$. Case ii $s_2 a_2 \in \delta(N_2 : M_2)$.
 $\square$

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