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Generalized Volterra-Type Operators from Hardy Space into Iterated Weighted-Type Spaces

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Abstract. Let $H(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$ and for $1 \leq p \leq \infty$, H^p be the Hardy space. For $m \in \mathbb{N}$ suppose that I^m be mth iteration. Let $\overrightarrow{g} = (g_0, \cdots, g_{m-1})$ where $\{g_i\}_{i=0}^{m-1} \subset H(\mathbb{D})$ and $I(f) = \int_0^z f(w)dw$. If I^m for $m \in \mathbb{N}$ be the mth iteration, then the generalized Volterra-type operators $I^m_{\overrightarrow{g}}$ on $H(\mathbb{D})$ is defined as follows

$$I^m_{\overrightarrow{g}}(f) = I^m(\sum_{i=0}^{m-1} f^{(i)} g_i).$$

In this paper, we investigate boundedness and compactness of generalized Volterra-type operators from Hardy space into iterated weighted-type spaces, $V_n = \{f \in H(\mathbb{D}) : \ \sup_{z\in\mathbb{D}}(1 - |z|^2)|f^{(n)}(z)| < \infty\}$.

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1 Introduction

Let $\mathbb{D}$ be an open unit disc in the complex $\mathbb{C}$ and $H(\mathbb{D})$ be the set of analytic functions on $\mathbb{D}$. For $1 \le p < \infty$, the Hardy space H^p consists of all analytic functions $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} \Big(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \Big)^{\frac{1}{p}} < \infty.$$

Also H^∞ is the space of bounded analytic functions on $\mathbb{D}$ with the norm $\|f\|_{H^\infty} = \sup_{z\in\mathbb{D}} |f(z)|$. More information about such spaces can be found in [9].

Another space used in this paper is nth weighted-type space. Let μ be a weight (continuous and positive function on $\mathbb{D}$) and $n \in \mathbb{N}_0$. The nth weighted-type space V_n^μ, consists of all analytic functions $f \in H(\mathbb{D})$ such that $b_{V_n^\mu}(f) = \sup_{z\in\mathbb{D}} \mu(z)|f^{(n)}(z)| < \infty$. This space is a Banach with the following norm

$$\|f\|_{V_n^\mu} = \sum_{i=0}^{n-1} \mid f^{(i)}(0) \mid + b_{V_n^\mu}(f) < \infty.$$

For $\alpha > 0$ and $\mu(z) = (1-|z|^2)^\alpha$, we use V_n^α, V_n and $\|\cdot\|_n$ instead of V_n^μ, V_n^1 and $\|\cdot\|_{V_n^1}$. The space V_n^α contains a large class of analytic functions. For example when $\alpha > 0$, $V_0^\alpha = \mathcal{A}^{-\alpha}$(growth space), $V_1^\alpha = \mathcal{B}^\alpha$ (Bloch type space), $V_2^\alpha = \mathcal{Z}^\alpha$ (Zygmund type space), $V_1 = \mathcal{B}$ (classic Bloch space) and $V_2 = \mathcal{Z}$ (classic Zygmund space). The space V_n is called iterated weighted-type space. In [7] Colonna *et al.* considered iterated weighted-type spaces and obtained some properties for these spaces, especially they showed

$$\cdots \subset V_{n+1} \subset V_n \subset \cdots \subset V_3 \subset \mathcal{Z} \subset H^\infty \subset \mathcal{B} \subset \mathcal{A}^{-1}.$$

The closed subspace of V_n^μ containing of all $f \in V_n^\mu$ such that $\lim_{|z|\to 1} \mu(z)|f^{(n)}(z)| = 0$ is denoted by $V_{n,0}^\mu$ and is called little nth weighted-type space. For more information about (little) nth weighted-type spaces, see [1, 2, 4, 7, 12, 14].

Let $m \in \mathbb{N}$, $I^m(f) = \int_0^z \int_0^{z_1} \cdots \int_0^{z_{m-1}} f(z)dzdz_1 \cdots dz_{m-1}$, and $\overrightarrow{g} = (g_0, g_1, \cdots, g_{m-1})$ where $\{g_i\}_{i=0}^{m-1} \subset H(\mathbb{D})$. The generalized Volterra-type operator on $H(\mathbb{D})$ defined as follows

$$I^m_{\overrightarrow{g}}(f) = I^m(\sum_{i=0}^{m-1} f^{(i)} g_i).$$

For $m = 1$ and $g_0 = g'$, we get Volterra type operator $(J_g f)(z) = \int_0^z f(w)g'(w)dw$ and when $m = 1$ and $g_0 = 1$, we obtain the classic Volterra operator $(If)(z) = \int_0^z f(w)dw$. Also if we set $g_i = a_{m-i-1} g^{(m-i)}$ $(0 \leq i \leq m-1)$, where $g \in H(\mathbb{D})$ and $\overrightarrow{a} = (a_0, \cdots, a_{m-1}) \in \mathbb{C}^m$, we have generalized integration operator $I^m_{g,\overrightarrow{a}}$ defined by Chalmoukis in [6].

Since

$$I^m_{\overrightarrow{g}}(f) = I^m(\sum_{i=0}^{m-1} f^{(i)} g_i) = \sum_{i=0}^{m-1} I^m(f^{(i)} g_i) := \sum_{i=0}^{m-1} I^{m,i}_{g_i}(f),$$

so for considering properties of operator $I^m_{\overrightarrow{g}}$, firstly we investigate properties of operators $I^{m,i}_{g_i}$ where $0 \leq i \leq m-1$.

Chalmoukis in [6] considered boundedness and compactness of $I^m_{g,\overrightarrow{a}} : H^p \to H^q$, where $(0 < q < p < \infty)$ and posed a conjecture that g must be in $H^{\frac{pq}{q-p}}$. Yang *et al.* provided a positive answer to the aforementioned conjecture in [13]. Arroussi *et al.* investigated boundedness and compactness of $I^m_{\overrightarrow{g}} : A^p \to A^q$, where A^p is Bergman space. They extended Chalmoukis result to Bergman spaces and showed that the Bergman space version of Chalmoukis conjecture is true (see [5]). Also some authors characterized boundedness and compactness of generalized integration operators among some other analytic function spaces [8, 10]. In [14], Zhu investigated Bloch-type spaces and uncovered numerous properties associated with these spaces. Later, Stevi expanded on this concept by generalizing Bloch-type spaces and introducing the nth weighted-type spaces, as detailed in [11, 12]. In recent years, extensive research has been conducted on such spaces, with one of the most notable references in this area being [7].

In this paper, firstly we investigate boundedness and compactness of the operators $I^{m,i}_{g_i}(0 \leq i \leq m)$ from Hardy space into iterated weighted-type spaces and we find some characterizations for boundedness and compactness of such operators. Then we consider boundedness and compactness of the operator $I^{m}_{\vec{g}} : H^p \to V_n$ and we show that the operator $I^{m}_{\vec{g}} : H^p \to V_n$ is bounded (compact) if and only if each operator $I^{m,i}_{g_i} : H^p \to V_n (0 \le i \le m-1)$ is bounded (compact).

In this work, we shall use the notation $A \preceq B$ to mean that for some $c > 0$, $A \le c\,B$, whereas $A \asymp B$ means $A \preceq B$ and $B \preceq A$.

2 Boundedness and Compactness of Operator $I^{m,i}_{g_i} : H^p \to V_n$

In this section, we investigate boundedness and compactness of the operators $I^{m,i}_{g_i}$ from Hardy space $H^p (1 \le p \le \infty)$ into iterated weighted-type spaces and we obtain some characterizations for boundedness and compactness of such operators. We begin with the following lemma.

Lemma 2.1. *Let $n, k \in \mathbb{N}$. For any $f \in V_n$, we have*

$$\|f\|_n \asymp \sum_{i=0}^{n+k-2} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^k |f^{(n+k-1)}(z)|.$$

Proof. For $n = 1$, $V_1 = \mathcal{B}$. So by using Proposition 8 in[14], we get

$$\|f\|_1 = \|f\|_{\mathcal{B}} \asymp \sum_{i=0}^{k-1} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^k |f^{(k)}(z)|.$$

For any $n \in \mathbb{N}$, $V_n \subset V_{n-1}$ [6, Proposition 2.1]. Hence, for any $f \in V_n$, $f^{(n-1)} \in \mathcal{B}$. By replacing f with $f^{(n-1)}$ in the above equation, we obtain

$$|f^{(n-1)}(0)| + \sup(1-|z|^2)|f^{(n)}(z)| \asymp \sum_{i=n-1}^{n+k-2} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}} (1-|z|^2)^k |f^{(n+k-1)}(z)|.$$

Hence,

$$\begin{aligned}
\|f\|_n &= \sum_{i=0}^{n-2} |f^{(i)}(0)| + |f^{(n-1)}(0)| + \sup_{z\in\mathbb{D}}(1-\mid z\mid^2)\mid f^{(n)}(z)\mid \\
&\asymp \sum_{i=0}^{n-2} |f^{(i)}(0)| + \sum_{i=n-1}^{n+k-2} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}}(1-\mid z\mid^2)^k\mid f^{(n+k-1)}(z)\mid \\
&= \sum_{i=0}^{n+k-2} |f^{(i)}(0)| + \sup_{z\in\mathbb{D}}(1-\mid z\mid^2)^k|f^{(n+k-1)}(z)|.
\end{aligned}$$

The proof is complete. $\square$

Let $m \geq n$. It is clear $\left(I_{\overline{g}}^m(f)\right)^{(i)}(0) = \left(I_g^{m,i}(f)\right)^{(i)}(0) = 0$, when $0 \leq i \leq m-1$, so for any $f \in H(\mathbb{D})$, by using Lemma 2.1, we have

$$\begin{aligned}
\|I_g^{m,i}(f)\|_n &\asymp \sum_{k=0}^{m-1} \left|\left(I_g^{m,i}(f)\right)^{(k)}(0)\right| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}\left|f^{(i)}g(z)\right| \quad (1)\\
&= \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}\left|f^{(i)}g(z)\right|.
\end{aligned}$$

Lemma 2.2. *Let $1 \leq p \leq \infty$. Then for any $f \in H^p$ and $k \in \mathbb{N}_0$,*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^{k+\frac{1}{p}}|f^{(k)}(z)| \preceq \|f\|_{H^p}.$$

Proof. From Proposition 5.1.2 of [14], we have $H^\infty \subset \mathcal{B}$ and for any $f \in H^\infty$

$$\sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| \leq \|f\|_{H^\infty}.$$

Applying Lemma 2.1, for each $k \in \mathbb{N}_0$ and $f \in H^\infty$, we obtain

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^k|f^{(k)}(z)| \leq \|f\|_1 \leq 2\|f\|_{H^\infty} \preceq \|f\|_{H^\infty}.$$

Similar results for $1 \leq p < \infty$ follow easily using results from [9]. $\square$

Theorem 2.3. *Let $1 \leq p \leq \infty$, $m,n \in \mathbb{N}$ such that $m \geq n$ and $g_i \in H(\mathbb{D})$. Then for each $0 \leq i \leq m$, the operator $I_{g_i}^{m,i} : H^p \to V_n$ is bounded* *if and only if* $\sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)|<\infty$. *Moreover, in this case*

$$\|I_{g_i}^{m,i}\| \asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)|.$$

Proof. Let the operator $I_{g_i}^{m,i}: H^p \to V_n$ be bounded and $f_{i,w}(z) = \frac{(1-|w|^2)^i}{(1-\overline{w}z)^{i+\frac{1}{p}}}$, where $0\le i\le m$ and $w\in\mathbb{D}-\{0\}$. For $p=\infty$,

$$\sup_{z\in\mathbb{D}}|f_{i,w}(z)| = \sup_{z\in\mathbb{D}}\left|\frac{1-|w|^2}{1-\overline{w}z}\right|^i \le \sup_{z\in\mathbb{D}}\frac{(1-|w|^2)^i}{(1-|\overline{w}|)^i} = 2^i,$$

so, $\sup_{w\in\mathbb{D}}\|f_{i,w}\|_{H^\infty}\le 2^i$ and when $1\le p<\infty$, from Lemma 2 [11], there exists positive constant $C_{i,p}$ such that $\sup_{w\in\mathbb{D}}\|f_{i,w}\|_{H^p}<C_{i,p}$. Applying (1) for $f_{i,w}$, we have

$$\begin{aligned}\|I_{g_i}^{m,i}(f_{i,w})\|_n &\asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}|f_{i,w}^{(i)}(z)g_i(z)| \\ &\ge (1-|w|^2)^{m-n+1}|f_{i,w}^{(i)}(w)||g_i(w)| \\ &= |\overline{w}|^i\prod_{l=0}^{i-1}(i+\frac{1}{p}+l)(1-|w|^2)^{m-n-i-\frac{1}{p}+1}|g_i(w)|.\end{aligned}$$

Therefore,

$$\begin{aligned}\sup_{w\in\mathbb{D}}(1-|w|^2)^{m-n-i-\frac{1}{p}+1}|g_i(w)| \preceq \|I_{g_i}^{m,i}(f_{i,w})\|_n &\le \|I_{g_i}^{m,i}\|\sup_{w\in\mathbb{D}}\|f_{i,w}\|_{H^p} \\ &\le C_{i,p}\|I_{g_i}^{m,i}\|.\end{aligned}$$

Conversely, we assume that $\sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)|<\infty$, by using (1), Lemmas 2.1 and 2.2, for any $f\in H^p$, we obtain

$$\begin{aligned}\|I_{g_i}^{m,i}(f)\|_n &\asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}|f^{(i)}(z)g_i(z)| \\ &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^{i+\frac{1}{p}}|f^{(i)}(z)|\sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| \\ &\preceq \|f\|_{H^p}\sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)|.\end{aligned}$$

Therefore, the operator $I_{g_i}^{m,i} : H^p \to V_n$ is bounded and $\|I_{g_i}^{m,i}\| \preceq \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)|$. The proof is complete. $\square$

To investigate compactness of operators $I_g^{m,i} : H^p \to V_n (0 \le i \le m)$, we need the following lemma, since the proof of it is similar to the proof of [7, proposition 3.11], so it is omitted.

Lemma 2.4. *Let $1 \le p \le \infty$, $m, n \in \mathbb{N}$, $\{g_i\}_{i=0}^{m-1} \subset H(\mathbb{D})$ and $T = I_{\overrightarrow{g}}^m$ or $I_{g_i}^{m,i}(0 \le i \le m)$. Then the bounded operator $T : H^p \to V_n$ is compact if and only if for any bounded sequence $\{f_k\}$ in H^p which converges to zero uniformly on compact subsets of $\mathbb{D}$, $\lim_{k\to\infty} \|T(f_k)\|_n = 0$.*

Theorem 2.5. *Let $1 \le p \le \infty$, $m, n \in \mathbb{N}$ such that $m \ge n$ and $g_i \in H(\mathbb{D})(0 \le i \le m)$. Then for each $0 \le i \le m$, the operator $I_{g_i}^{m,i} : H^p \to V_n$ is compact if and only if $\lim_{|z|\to 1}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| = 0$.*

Proof. Let the operator $I_{g_i}^{m,i} : H^p \to V_n$ be compact. For any $0 \le i \le m$ and $w \in \mathbb{D} - \{0\}$ the functions $f_{i,w}(z) = \frac{(1-|w|^2)^i}{(1-\overline{w}z)^{i+\frac{1}{p}}}$ are bounded and converge uniformly to zero on compact subsets of $\mathbb{D}$ when $|w|$ tends to 1, so by applying Lemma 2.4 $\lim_{|w|\to 1} \|I_{g_i}^{m,i}(f_{i,w})\|_n = 0$. Now by using (1), we obtain

$$\begin{aligned}\|I_{g_i}^{m,i}(f_{i,w})\|_n &\asymp (1-|z|^2)^{m-n+1}|f_{i,w}^{(i)}(z)||g_i(z)| \\ &\succeq (1-|w|^2)^{m-n-i-\frac{1}{p}+1}|\overline{w}|^i|g_i(w)|.\end{aligned}$$

In the above inequality, let $|w| \to 1$. Then, we obtain $\lim_{|w|\to 1}(1-|w|^2)^{m-n-i-\frac{1}{p}+1}|g_i(w)| = 0$.

Conversely, suppose that $\lim_{|z|\to 1}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| = 0$. Hence, for any $\varepsilon > 0$ there exists $0 < \delta < 1$ such that for each $\delta < |z| < 1$,

$$(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| < \varepsilon. \tag{2}$$

Now by using (1) and Lemma 2.1, for any bounded sequence $\{f_k\} \subset H^p$

which converges uniformly to zero on compact subsets of $\mathbb{D}$, we obtain

$$\begin{aligned}
\|I_{g_i}^{m,i}(f_k)\|_n &\asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}|f_k^{(i)}(z)||g_i(z)| \\
&\le \sup_{|z|\le\delta}(1-|z|^2)^{m-n+1}|f_k^{(i)}(z)||g_i(z)| \\
&\quad + \sup_{\delta<|z|<1}(1-|z|^2)^{m-n+1}|f_k^{(i)}(z)||g_i(z)| \\
&\le \sup_{|z|\le\delta}(1-|z|^2)^{i+\frac{1}{p}}|f_k^{(i)}(z)| \sup_{|z|\le\delta}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| \\
&\quad + \sup_{\delta<|z|<1}(1-|z|^2)^{i+\frac{1}{p}}|f_k^{(i)}| \sup_{\delta<|z|<1}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| \\
&=: X_1 + X_2.
\end{aligned}$$

By using Cauchy's estimates, for any $i\in\mathbb{N}_0$, the sequence $\{f_k^{(i)}\}$ converges uniformly to zero on compact subsets of $\mathbb{D}$, therefore

$$\lim_{k\to\infty} X_1 \le \sup_{|z|\le\delta}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| \lim_{k\to\infty}\sup_{|z|\le\delta}|f_k^{(i)}(z)| = 0.$$

Also applying Lemma 2.2 and (2), we get

$$\begin{aligned}
\lim_{k\to\infty} X_2 &\preceq \lim_{k\to\infty}\|f_k\|_{H^p} \sup_{\delta<|z|<1}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| \\
&\le \varepsilon \sup_{k\in\mathbb{N}}\|f_k\|_{H^p}.
\end{aligned}$$

So, $\lim_{k\to\infty}\|I_{g_i}^{m,i}(f_k)\|_n = 0$. By using Lemmas 2.4, the operator $I_{g_i}^{m,i}: H^p\to V_n$ is compact. The proof is complete. □

Putting $n=1$ and $n=2$ in Theorems 2.3 and 2.5, we get the following corollaries.

Corollary 2.6. *Let $1\le p\le\infty$, $m\in\mathbb{N}$ and $g_i\in H(\mathbb{D})$. Then for each $0\le i\le m$, the operator $I_{g_i}^{m,i}: H^p\to\mathcal{B}$ is bounded (compact) if and only if $g_i\in V_0^{m-i-\frac{1}{p}}$ $(g_i\in V_{0,0}^{m-i-\frac{1}{p}})$.*

Corollary 2.7. *Let $1\le p\le\infty$, $m\in\mathbb{N}$ such that $m\ge 2$ and $g_i\in H(\mathbb{D})$. Then for each $0\le i\le m$, the operator $I_{g_i}^{m,i}: H^p\to\mathcal{Z}$ is bounded (compact) if and only if $g_i\in V_0^{m-i-\frac{1}{p}-1}$ $(g_i\in V_{0,0}^{m-i-\frac{1}{p}-1})$.*

3 Boundedness and Compactness of Operator $I_{\vec{g}}^{m} : H^p \to V_n$

In this section, we will consider boundedness and compactness of the operator $I_{\vec{g}}^{m}$ from Hardy spaces into iterated weighted-type spaces. Especially we show that if the operator $I_{\vec{g}}^{m} : H^p \to V_n$ is bounded (compact) then for each $0 \le i \le m-1$, the operator $I_{g_i}^{m,i} : H^p \to V_n$ is bounded (compact). For this purpose, we need the following lemma which comes from [2, Lemma 2.5] and [3, Lemma 2.3].

Lemma 3.1. *Let $1 \le p \le \infty$. For any $0 \ne \xi \in \mathbb{D}$ and $i \in \{0, 1, \cdots, n\}$, there exists a function $v_{i,\xi} \in H^p$ with the following conditions:*

a) $v_{i,\xi}(z) = \sum_{j=1}^{n+1} c_j^i f_{i,\xi}(z)$, where $f_{i,\xi}(z) = \frac{(1-|\xi|^2)^i}{(1-\bar{\xi}z)^{i+\frac{1}{p}}}$ and c_j^i is independent of choice ξ.

b) $\sup_{\xi \in \mathbb{D}} \|v_{i,\xi}\|_{H^p} < \infty$ and

$$v_{i,\xi}^{(k)}(\xi) = \begin{cases} \frac{\bar{\xi}^i}{(1-|\xi|^2)^{i+\frac{1}{p}}}, & k = i, \\ 0, & k \ne i. \end{cases}$$

c) For any sequence $\{\xi_k\} \subset \mathbb{D}$ such that $\lim_{k\to\infty} |\xi_k| = 1$, the sequence $\{v_{i,\xi_k}\}$ converges to zero uniformly on compact subsets of $\mathbb{D}$.

Let $m \ge n$. For any $f \in H(\mathbb{D})$, applying Lemma 2.1, we get

$$\begin{aligned} \|I_{\vec{g}}^{m}(f)\|_n &\asymp \sum_{k=0}^{m-1} \left| \left(I_{\vec{g}}^{m}(f)\right)^{(k)}(0) \right| + \sup_{z\in\mathbb{D}} (1-|z|^2)^{m-n+1} \left| \left(I_{\vec{g}}^{m}(f)\right)^{(m)} \right| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)^{m-n+1} \left| \sum_{j=0}^{m-1} f^{(j)}(z) g_j(z) \right|. \end{aligned} \tag{3}$$

Theorem 3.2. *Let $1 \le p \le \infty$ and $m, n \in \mathbb{N}$ such that $m \ge n$. Then the following conditions are equivalent:*

a) The operator $I_{\vec{g}}^m : H^p \to V_n$ *is bounded.*

b) For each $0 \le i \le m-1$*, the operator* $I_{g_i}^{m,i} : H^p \to V_n$ *is bounded.*

c) For each $0 \le i \le m-1$*,* $\sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| < \infty$*.*

Proof. (b)⇒ (a) Since $I_{\vec{g}}^m = \sum_{i=0}^{m-1} I_{g_i}^{m,i}$ and all operators $I_{g_i}^{m,i} : H^p \to V_n (0 \le i \le m-1)$ are bounded, hence the operator $I_{\vec{g}}^m : H^p \to V_n$ is bounded.

(b)⇔ (c) Theorem 2.3.

(a)⇒ (c) Let the operator $I_{\vec{g}}^m : H^p \to V_n$ be bounded. For each $i \in \{0, \cdots, m-1\}$ and $\xi \in \mathbb{D} - \{0\}$, let $v_{i,\xi}$ be function found in Lemma 3.1, by using (3), we have

$$\begin{aligned} \|I_{\vec{g}}^m(v_{i,\xi})\|_{V_n} &\asymp \sup_{z\in\mathbb{D}}(1-|z|^2)^{m-n+1}|\sum_{j=0}^{m-1} v_{i,\xi}^{(j)}(z)g_j(z)| \qquad (4)\\ &\ge (1-|\xi|^2)^{m-n+1}|\sum_{j=0}^{m-1} v_{i,\xi}^{(j)}(\xi)g_j(\xi)| \\ &\ge (1-|\xi|^2)^{m-n+1}|v_{i,\xi}^{(s)}(\xi)||g_i(\xi)| \\ &= (1-|\xi|^2)^{m-n+1}\frac{|\bar{\xi}|^i}{(1-|\xi|^2)^{i+\frac{1}{p}}}|g_i(\xi)| \\ &= |\bar{\xi}|^i(1-|\xi|^2)^{m-n-i-\frac{1}{p}+1}|g_i(\xi)|. \end{aligned}$$

Applying Lemma 3.1(b), we get

$$\sup_{\xi\in\mathbb{D}}(1-|\xi|^2)^{m-n-i-\frac{1}{p}+1}|g_i(\xi)| \le \|I_{\vec{g}}^m(v_{i,\xi})\|_{V_n} \le \|I_{\vec{g}}^m\| \sup_{\xi\in\mathbb{D}} \|v_{i,\xi}\|_{H^p} < \infty.$$

The proof is complete. □

Theorem 3.3. *Let* $1 \le p \le \infty$ *and* $m, n \in \mathbb{N}$ *such that* $m \ge n$*. Then the following conditions are equivalent:*

a) The operator $I_{\vec{g}}^m : H^p \to V_n$ *is compact.*

b) For each $0 \le i \le m-1$*, the operator* $I_{g_i}^{m,i} : H^p \to V_n$ *is compact.*

c) For each $0 \leq i \leq m-1$, $\lim_{|z|\to 1}(1-|z|^2)^{m-n-i-\frac{1}{p}+1}|g_i(z)| = 0$.

Proof. (b)⇒ (a) It is clear when for each $0 \leq i \leq m-1$, the operator $I_{g_i}^{m,i} : H^p \to V_n$ is compact, then $I_{\overrightarrow{g}}^m = \sum_{i=0}^{m-1} I_{g_i}^{m,i}$ is compact.

(b)⇔ (c) Theorem 2.5.

(a)⇒ (c) Assume that the operator $I_{\overrightarrow{g}}^m : H^p \to V_n$ is compact. For each $i \in \{0, \cdots, m-1\}$ and $\xi \in \mathbb{D} - \{0\}$, the sequence $\{v_{i,\xi}\}$ is bounded and converges to zero uniformly on compact subsets of $\mathbb{D}$ when $|\xi|$ tends to 1 (Lemma 3.1), so by using Lemma 2.4, $\lim_{|\xi|\to 1} \|I_{\overrightarrow{g}}^m(v_{i,\xi})\|_n = 0$. Now it is enough to let $|\xi|$ tends to 1 in the inequality (4), therefore $\lim_{|\xi|\to 1}(1-|\xi|^2)^{m-n-i-\frac{1}{p}+1}|g_i(\xi)| = 0$. The proof is complete. □

Let $m \in \mathbb{N}$, $\overline{a} = (a_0, \cdots, a_{m-1}) \in \mathbb{C}^m$ and $g \in H(\mathbb{D})$. Applying Theorems 3.2 and 3.3, we obtain similar results for the Chalmoukis operator

$$I_{g,\overline{a}}^m f(z) = I^m \left(f g^{(n)} + a_1 f' g^{(n-1)} + \cdots + a_{n-1} f^{(n-1)} g' \right)$$

acting from the Hardy space into iterated weighted-type spaces.

Corollary 3.4. *Let* $1 \leq p \leq \infty$, $m \in \mathbb{N}$, $\overline{a} = (a_0, \cdots, a_{m-1}) \in \mathbb{C}^m$ *and* $g \in H(\mathbb{D})$. *Then*

a) the operator $I_{g,\overline{a}}^m : H^p \to V_n$ *is bounded if and only if*

$$g \in \bigcap_{i=0}^{m-1} V_{m-i}^{m-n-i-\frac{1}{p}+1}.$$

b) the operator $I_{g,\overline{a}}^m : H^p \to V_n$ *is compact if and only if*

$$g \in \bigcap_{i=0}^{m-1} V_{m-i,0}^{m-n-i-\frac{1}{p}+1}.$$

Remark 3.5. By choosing suitable parameters m, p, n and $\overrightarrow{g}$, the results obtained in this paper, can be stated for some well-known operators and spaces.

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