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Original Research Paper

Fermatean Fuzzy Q-Ideals in Q-algebras

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Abstract. This paper introduces the notions of Q-algebras with associative law (ALQ-algebras) and commutative law (CL-algebras) and shows that ALQ-algebras and CLQ-algebras are isomorphic. It is presented in the the notions of Fermatean fuzzy Q-algebras and (3-regular) Fermatean fuzzy Q-ideals as a generalization of fuzzy Q-algebras and fuzzy Q-ideals. It investigates the connection between Fermatean fuzzy Q-algebras and Fermatean fuzzy Q-ideals and is shown that Fermatean fuzzy Q-ideals are a subclass of Fermatean fuzzy Q-subalgebras and 3-regular Fermatean fuzzy Q-subalgebras are Fermatean fuzzy Q-ideals. In this study, the concepts of nil radical Fermatean fuzzy Q-subsets are introduced, and it is stabilized that nil radical Fermatean fuzzy Q-subsets are Fermatean fuzzy Q-subalgebra. Also, the Fermatean fuzzy Q-algebras(ideals) are extended by, intersection, union, combination, and isomorphisms and $\mathbb{J}$-coverd conditions in any given Q-algebra.

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1 Introduction

An abstract set lacks practical utility; however, when it is structured as a $U.A$ (universal algebra) and forearmed with various operations, it transforms into a system and gains significant applications. One of the foundational algebras in the realm of $U.A$ is logical algebra. Logical algebra represents a crucial annex of mathematics with extensive applications across interdisciplinary sciences. For instance, the field of computer engineering and computational processes is fundamentally grounded in two-valued logic. The foundation of this algebra lies in axioms derived from two-valued logic, constructed upon these core principles. Logical algebra remains among the most critical branches of algebra, with diverse applications in interdisciplinary fields such as computer science. To explore the fundamentals of logical algebra within the theoretical context of residuated lattices, Imai and Iseki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras (for further details, see references [5, 6]). Later, J. Neggres, introduced a new notion, as Q-algebra, which is a generalization of BCI/BCK-algebras and generalized some theorems from the theory of BCI-algebras [9]. Fuzzy set theory, conceived by Zadeh to address uncertainty [11], represents a generalization of classical set theory. An Intuitionistic Fuzzy Set (IFS), introduced by K. Atanassov (1986), generalizes Zadeh's fuzzy sets by incorporating independent membership and non-membership degrees with a hesitation margin [1]. Fermatean fuzzy subsets (FFSs) are an extension of intuitionistic fuzzy sets (IFSs), introduced to model uncertainty with weaker constraints on membership and non-membership degrees. They are named after Pierre de Fermat (due to their connection with cubic equations in fuzzy logic) and were formally proposed by Yager et. al (2019) [10]. Fermatean Fuzzy Subsets (FFSs) offer greater flexibility in modeling uncertainty than Intuitionistic Fuzzy Sets. Their relaxed condition (allows for wider ranges of membership/non-membership values, making them useful in complex decision-making scenarios such as multi-criteria decision-making, medical diagnosis and healthcare, artificial intelligence and machine learning, pattern recognition and image processing, risk assessment and financial modeling, robotics and autonomous systems, blockchain and cybersecurity, environmental and climate science. See sources for further research too [**?**, 2, 4, 8, 12].

Motivation, advantage and contributions: In this paper, we first introduce Fermatean fuzzy Q-subalgebras as a generalization of Q-fuzzy subalgebras and discuss their properties. Our motivation for introducing Fermatean fuzzy Q-subalgebras is to equip Q-algebras with two membership functions and to construct a new generalization of Q-algebras based on the axioms on Q-fuzzy subalgebras. We have generalized Q-fuzzy subalgebras to Fermatean fuzzy Q-subalgebras and indeed, Fermatean fuzzy Q-subalgebras have the properties of the axioms of logic algebras and fuzzy nonlinear problems. Considering the importance of the isomorphisms in this paper, we have introduced the null radial Fermatean fuzzy Q-subalgebras, and consider the relation between of the null radial image of Fermatean fuzzy Q-subalgebras and have shown under the conditions that the null radial image of Fermatean fuzzy Q-subalgebras and the null radial image of Fermatean fuzzy Q-subalgebras are equal to each other. Considering that the characteristic function is of great importance in issues related to the membership or non-membership of a part in the whole, we have shown that the characteristic function of a subalgebra is extended to a Fermatean fuzzy Q-subalgebra. Since the characteristic function is a fuzzy subset, the motivation is to ask whether an arbitrary fuzzy Q-subalgebra can be a Fermatean fuzzy Q-subalgebra. By investigating this issue, it can be seen that if the component is a fuzzy Q-subalgebra in a Fermatean fuzzy subset, the second component (as a non-membership of the element) must be an anti-fuzzy Q-subalgebra to obtain a Fermatean fuzzy Q-subalgebra. By transforming the fuzzy Q-subalgebra into the Fermatean fuzzy Q-subalgebra, the idea, and motivation of introducing anti-fuzzy Q-subalgebras (as a non-membership component) is created and we show that if the Fermatean fuzzy subset has two components, fuzzy Q-subalgebra, and anti-fuzzy Q-subalgebra, then it forms a Fermatean fuzzy Q-subalgebra and in this research, for further confirmation, we call it a two-sided Fermatean fuzzy Q-subalgebra. The purpose of introducing Fermatean fuzzy Q-subalgebras is to be able to transform the combination of two Fermatean fuzzy Q-subalgebra into a Fermatean fuzzy Q-subalgebra in such a way that this combination includes one of these Fermatean fuzzy Q-subalgebra. In this research, considering that the combination of Fermatean fuzzy Q-subalgebras cannot necessarily be a Fermatean fuzzy Q-subalgebra (the elements of the Fermatean fuzzy subset are nonlinear), we have shown that the combination of Fermatean fuzzy Q-subalgebras can be a Fermatean fuzzy Q-subalgebra with the commutative condition. Therefore, the introduction of two-sided Fermatean fuzzy Q-subalgebras is another motivation to be able to combine Fermatean fuzzy Q-subalgebras without considering the power of Fermatean fuzzy Q-subalgebras. The Q-algebras with associate property(ALQ-algebra), play an important role in the structure and development of Fermatean fuzzy Q-subalgebras, so that if our underline set is not a ALQ-algebra, the combination of Fermatean fuzzy Q-subalgebras cannot necessarily be a Fermatean fuzzy Q-subalgebra. Finally, considering the importance of isomorphisms, which are transformations or mappings from a Fermatean fuzzy Q-subalgebra to a Fermatean fuzzy Q-subalgebra, we have been able to address the null radial connection of Fermatean fuzzy Q-subalgebras and establish the conditions for the development of a Fermatean fuzzy Q-subalgebra between Fermatean fuzzy Q-subalgebras. Also, we present the notion of Fermatean fuzzy Q-ideals as another types of extension of Q-algebras and prove that 3-regular Fermatean fuzzy Q-subalgebras are Fermatean fuzzy Q-ideals.

2 Preliminaries

This section compiles key concepts pertinent to our work.

Definition 2.1. [5] A structure $(X, *, 0_X)$ is a Q-algebra, if, $\forall\ x, y, z \in X$,

(i) $x * x = 0_X$,

(ii) $x * 0_X = x$,

(iii) $(x * y) * z = (x * z) * y$.

In $(X, *, 0_X)$, define $\leq_X$ by $x \leq_X y \Leftrightarrow x * y = 0_X$. Let X be a Q-algebra and $\emptyset \neq Y \subseteq X$. Then Y is referred to as a subalgebra of X, if for any $x, y \in Y, x * y \in Y$ and will denote by $Y \leq_{sub} X$.

Theorem 2.2. [5] *Let $(X, *, 0_X)$ be a Q-algebra. Then $\forall\ x, y \in X$,*

(i) $(x * (x * y)) * y = 0_X$,

(ii) *if* $x \leq_X 0_X$, *then* $X = \{0_X\}$,

(iii) *if* $x \leq_X y$, *then* $x * (x * (x * y)) = 0_X$.

Theorem 2.3. [5]

(i) *Any quadratic Q-algebra on finite fields is isomorphic to any other similar algebra on finite fields.*

(ii) *Any quadratic Q-algebra on finite fields generates the abelian groups.*

Definition 2.4. [5] $I \neq \emptyset$ of Q-algebra $(X, *, 0_X)$ is referred to as an ideal of X, if $(1), 0_X \in I$ and $(2), \forall\, x, y \in X, x * y \in I$ and $y \in I$, imply $y \in I$.

Definition 2.5. [7] A fuzzy subset (FS) of $\emptyset \neq X$, is a set as $A = \{(x, \mu(x)) \mid x \in X\}$, that $\mu : X \to \mathbb{I} = [0, 1]$ is membership map.

In what follow, for any $k \in \mathbb{N}$, any $x \in X$, denote $\mu^k(x) = (\mu(x))^k$

Definition 2.6. [10] A Fermatean fuzzy subset (FFS) E of $\emptyset \neq X$ is given by $E = \{(\mu_E, \nu_E) \mid x \in X\}$, that $\mu_E, \nu_E : X \to \mathbb{I}$ (as membership and non-membership, respectively) and for any $x \in X, 0 \leq \mu_E^3(x) + \nu_E^3(x) \leq 1$.

From now on, will denote any $FS, A = \{(x, \mu(x)) \mid x \in X\}$ by A and any FFS, $E = \{(\mu_E, \nu_E) \mid x \in X\}$ by E for simplify.

Definition 2.7. [3] An FS A of a Q-algebra $(X, *, 0_X)$ is a fuzzy Q-subalgebra (FQ_s) of X, if for any $x, y \in X, \mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ and in equality, it is referred to as strong.

3 On CLQ-algebras and ALQ-algebras.

In this section, we introduce the notions CLQ-algebras and ALQ-algebras via the commutative properties and associative properties on any given Q-algebra. Also two distinct tripes of powers of elements in Q-algebras are introduced and is established that CLQ-algebras are ALQ-algebras and two tripes of powers of elements in any ALQ-algebras are equal.

Let $n \in \mathbb{N}, (X, *, 0_X)$ be a Q-algebra and $x \in X$. Define $x^{\{2\}} = x * x, x^{\{3\}} = x^{\{2\}} * x$ and for any $n \geq 3$, $x^{\{n\}} = x^{\{n-1\}} * x$ and $x^{(2)} = x * x, x^{(3)} = x * x^{(2)}$ and for any $n \geq 3$, $x^{(n)} = x * x^{(n-1)}$.

Example 3.1. For any prime p, view the Q-algebra $(\mathbb{R}, *, \sqrt{p})$, that for any $x, y \in \mathbb{R}, x * y = x - y + \sqrt{p}$. Obviously, for any $k, n \in \mathbb{N}, x^{\{n\}} = (n-1)\sqrt{p} - (n-2)x, x^{(2k-1)} = \sqrt{p}$ and $x^{(2k)} = x$.

Let $x, y \in X$. We state that $(X, *, 0_X)$ is a CLQ-algebra, if is satisfied in the commutative law, i.e for any $x, y, z \in X, x * y = y * x$ and is an ALQ-algebra, if it is satisfied in the associative law, i.e for any $x, y, z \in X, x * (y * z) = (x * y) * z$.

Example 3.2. (i) Consider the Q-algebra $(Y, *, 0_Y)$ as shown in Table 2. One can see that $(X, *, 0_X)$ is not a CLQ-algebra and an ALQ-algebra, because of $a * b \neq b * a$ and $b * (b * b) \neq (b * b) * b$, respectively.

(ii) Let $n \in \mathbb{Z}$. Then $(\mathbb{Z}, -, 0)$ and $(n\mathbb{Z}, -, 0)$ are not CLQ-algebras and ALQ-algebras.

(iii) Klein four-group is an ALQ-algebra and a CLQ-algebras.

Proposition 3.3. *Let $(X, *, 0_X)$ be a Q-algebra and $x, y \in X$ and $n, m \in \mathbb{N}$. Then*

(i) *$(X, *, 0_X)$ is an ALQ-algebra, iff $(X, *, 0_X)$ is a CLQ-algebra.*

(ii) *if $(X, *, 0_X)$ is an ALQ-algebra, then $(x * y)^{\{n\}} = x^{\{n\}} * y^{\{n\}}$.*

(iii) *if $(X, *, 0_X)$ is an ALQ-algebra, then $(x^{\{n\}})^{\{m\}} = x^{\{mn\}}$.*

Proof. (i) Let $x, y, z \in X$. Since $(X, *, 0_X)$ is a CLQ-algebra,

$$(x * y) * z = (x * z) * y = (z * x) * y = (z * y) * x = x * (y * z)$$

and so $(X, *, 0_X)$ is an ALQ-algebra.

Conversely, let $(X, *, 0_X)$ be an ALQ-algebra. Then for any $x \in X$, $0_X * x = (x * x) * x = x * (x * x) = x * 0_X = x$ and so for any $x, y \in X$,

$$\begin{aligned} x * y &= 0_X * (x * y) = (y * y) * (x * y) = ((y * y) * x) * y = ((y * x) * y) * y \\ &= (y * x) * (y * y) = (y * x) * 0_X = y * x \end{aligned}$$

and so $(X, *, 0_X)$ is a CLQ-algebra.

(ii) Let $x, y \in X$. Since $(X, *, 0_X)$ is an ALQ-algebra

$$(x*y)^2 = (x*y)*(x*y) = (x*(x*y))*y = ((x*x)*y)*y = (x^2*y)*y$$
$$= x^2*(y*y) = x^2*y^2.$$

Now, by induction, we assume that $(x*y)^{\{n-1\}} = x^{\{n-1\}}*y^{\{n-1\}}$. Then

$$\begin{aligned}(x*y)^{\{n\}} &= (x*y)^{\{n-1\}}*(x*y)\\ &= (x^{\{n-1\}}*y^{\{n-1\}})*(x*y) = ((x^{\{n-1\}}*y^{\{n-1\}})*y)*x\\ &= (x^{\{n-1\}}*(y^{\{n-1\}}*y))*x = (x^{\{n-1\}}*y^{\{n\}})*x\\ &= (x^{\{n-1\}}*x)*y^{\{n\}} = x^{\{n\}}*y^{\{n\}}.\end{aligned}$$

(iii) Let $x \in X$ and $r, s \in \mathbb{N}$. Since $(X, *, 0_X)$ is an ALQ-algebra , $x^{\{r\}}*x^{\{s\}} = (\underbrace{x*\ldots*x}_{r-times})*(\underbrace{x*x*\ldots*x}_{s-times}) = (\underbrace{x*x*\ldots*x}_{(r+s)-times}) = x^{\{r+s\}}$.Clearly $(x^{\{n\}})^1 = x^{\{n\}} = x^{\{n1\}}$. Assume the statement holds for some positive integer $m = k$, i.e. $(x^{\{n\}})^{\{k\}} = x^{\{nk\}}$. For $m = k+1$, have

$(x^{\{n\}})^{\{k+1\}} = (x^{\{n\}})^{\{k\}}*x^{\{n\}} = x^{\{nk\}}*x^{\{n\}} = x^{\{nk+n\}} = x^{\{n(k+1)\}}$.

□

Corollary 3.4. *Let $(X, *, 0_X)$ be an ALQ-algebra, $x, y \in X$ and $m, n \in \mathbb{N}$. Then*

(i) $x^{\{n\}} = x^{(n)}$,

(ii) $(x*y)^{(n)} = x^{(n)}*y^{(n)}$.

(iii) $(x^{(n)})^{(m)} = x^{(mn)}$.

Based Corollary 3.4, from now on, will denote $x^{\{n\}} = x^{(n)}$ by x^n, for simplify.

Example 3.5. (i) Consider the Q-algebras $X, Y, \mathtt{Z}$ as Tables 1, 2 and 3. (1) In Q-algebra $(X, *, 0_X)$, for any $n \in \mathbb{N}, d^{\{2n\}} = d^{(2n)} = 0_X$ and $d^{\{2n-1\}} = d^{(2n-1)} = d$.

(2) In Q-algebra $(Y, *, 0_X)$, for any $n \geq 2, m \in \mathbb{N}, a^{\{n\}} = 0_X, a^{(2m-1)} = a$ and $a^{(2m)} = 0_X$.

(3) In Q-algebra $(Z, *, 0_X)$, for any $n \in \mathbb{N}, b^{\{n\}} = \begin{cases} a & n \equiv 0_X \pmod 3 \\ b & n \equiv 1 \pmod 3 \\ 0_X & n \equiv 2 \pmod 3 \end{cases}$,

$b^{(2n-1)} = b$ and $b^{(2n)} = 0_X$.

$*$	0_X	a	b	c	d
0_X	0_X	0_X	0_X	0_X	d
a	a	0_X	0_X	a	d
b	b	b	0_X	0_X	d
c	c	0_X	c	0_X	d
d	d	d	d	d	0_X

Table 1: $(X, *, 0_X)$

$*$	0_X	a	b	c
0_X	0_X	0_X	0_X	0_X
a	a	0_X	0_X	0_X
b	b	0_X	0_X	0_X
c	c	c	c	0_X

Table 2: $(Y, *, 0_Y)$

$*$	0_X	a	b
0_X	0_X	b	a
a	a	0_X	b
b	b	a	0_X

Table 3: $(Z, *, 0_Z)$

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4 On Fermatean fuzzy Q-subalgebras

In this section, we introduce the notion of Fermatean fuzzy Q-subalgebras and investigate the connection between Fermatean fuzzy Q-subalgebras and (anti)fuzzy Q-subalgebras.

From now on, we consider $(X, *, 0_X)$ as a Q-algebra and obtain the subsequent results. Let FS be a fuzzy sub-Q-algebra of X. We say FS is an anti fuzzy sub-Q-algebra of X, if at all $x, y \in X, \mu(x*y) \leq \mu(x) \wedge \mu(y)$.

In what follow, for any $k, l \in \mathbb{N}$, any $x \in X$ and any map $\mu : X \to \mathbb{I} = [0, 1]$, denote $E^{(k,l)} = \{(x, \mu_E^k(x), \nu_E^l(x)) \mid x \in X\}$. It is clear that $E^{(1,1)} = E$, for any $k \geq l \geq 2, \mu^k \subseteq \mu^l$ and for any $k \leq k', l \leq l'$, if $E^{(k,l)}$ is an FFS of X, then $E^{(k',l')}$ is an FFS of X.

Definition 4.1. Let E be an FFS of X. Then E is a Fermatean fuzzy Q-subalgebra (FFQ_s) of X, if for any $x, y \in X, \mu_E^3(x * y) \geq \min\{\mu_E^3(x), \mu_E^3(y)\}$ and $\nu_E^3(x * y) \leq \max\{\mu_E^3(x), \mu_E^3(y)\}$ and is an anti

FFQ_s of X, if for any $x, y \in X, \mu_E^3(x * y) \leq \max\{\mu_E^3(x), \mu_E^3(y)\}$ and $\nu_E^3(x * y) \geq \min\{\mu_E^3(x), \mu_E^3(y)\}$.

Remark 4.2. Let E is an FFQ_s of X. Then

(i) for any $x \in X, \mu_E(0_X) \geq \mu_E^3(0_X) \geq \mu_E^3(x)$ and $\nu_E^3(0_X) \leq \nu_E^3(x) \leq \nu_E(x)$.

(ii) μ_E, ν_E are FQ_s and anti FQ_s of X, respectively.

The converse of Remark 5.2, is not true, necessarily based the Example 4.3.

Let $(X, *, 0_X)$ be a Q-algebra and $\emptyset \neq Y \subseteq X$. Will say Y is a subalgebra of X, if for any $x, t \in Y, x * t \in Y$ and will denote by $Y \leq_{sub} X$.

Example 4.3. (i) Let $(X, *, 0_X)$ be a Q-algebra and $Y \leq_{sub} X$. Then $E = \{(\chi_Y, \chi_Y^c)\}$ is an FFQ_s of X.

(ii) Consider the Q-algebra Z as Tables 3. Then A and B are FQ_s and anti FQ_s of Z, as Tables 4 and 5, respectively, while $E = (\mu, \nu)$ is not an FFQ_s of Z.

μ	0_X	a	b
0_X	0.95	0.8	0.8

Table 4: FS A

ν	0_X	a	b
0_X	0.85	0.95	0.95

Table 5: FS B

Proposition 4.4. *Let E be a FFQ_s of X and $n \geq 2$. Then for any $x \in X$,*

(i) $\mu(x^{\{n\}}) \geq \mu^3(x)$ *and* $\nu^3(x^{\{n\}}) \leq \nu(x)$.

(ii) $\mu(x^{(n)}) \geq \mu^3(x)$ *and* $\nu^3(x^{(n)}) \leq \nu(x)$.

Let E and F be FFQ_s of X. We discuss that $E \subseteq F$, if at all $x \in X, \mu_E(x) \leq \mu_F(x)$ and $\nu_E(x) \geq \nu_F(x)$. Obviously at all $n \geq n', E^{(n,n')} \subseteq E^{(n',n)}$ and at all $n \leq n', m \leq m', E^{(m',n)} \subseteq E^{(m,n')}$. Also, define

$$E \cap F = \{(x, \min\{\mu_E(x), \mu_F(x)\}, \max\{\nu_E(x), \nu_F(x)\})\} \text{ and}$$
$$E \cup F = \{(x, \max\{\mu_E(x), \mu_F(x)\}, \min\{\nu_E(x), \nu_F(x)\})\}.$$

Obviously, $E \cap F$ and $E \cup F$ are an FFQ_s of X.

Theorem 4.5. *Let A be an FQ_s of X. Then $[A, \frac{1}{3}, \frac{1}{3}] = \{(x, \sqrt[3]{\mu(x)}, \sqrt[3]{\mu^c(x)})\}$ is an FFQ_s of X.*

Proof. Let $x \in X$. Then $(\sqrt[3]{\mu(x)})^3 + (\sqrt[3]{\mu^c(x)})^3 = \mu(x) + 1 - \mu(x) = 1$ and so $[A, \frac{1}{3}, \frac{1}{3}] = \{(x, \sqrt[3]{\mu(x)}, \sqrt[3]{\mu^c(x)})\}$ is an FFS of X. Now, since A is an FQ_s of X,

$$\sqrt[3]{\mu(x*y))}^3 = \mu(x*y) \geq \mu(x) \wedge \mu(y) = \sqrt[3]{\mu(x)}^3 \wedge \sqrt[3]{\mu(y)}^3.$$

Also,

$$\begin{aligned}
(\sqrt[3]{\mu^c(x*y))}^3 &= 1 - \mu(x*y) \leq 1 - (\mu(x) \wedge \mu(y)) \leq (1-\mu(x)) \vee (1-\mu(y)) \\
&= \mu^c(x) \vee \mu^c(y) = \sqrt[3]{\mu^c(x)}^3 \vee \sqrt[3]{\mu^c(y)}^3.
\end{aligned}$$

Then $[A, \frac{1}{3}, \frac{1}{3}] = \{(x, \sqrt[3]{\mu(x)}, \sqrt[3]{\mu^c(x)})\}$ is an FFQ_s of X. $\square$

Theorem 4.6. *Let A be an FQ_s of X. Then $\langle A, \frac{1}{m}, \frac{1}{n}\rangle = \{(x, \sqrt[3]{\nu^c(x)}, \sqrt[3]{\nu(x)})\}$ is an FFQ_s of X.*

Proof. Let $x \in X$. Then $(\sqrt[3]{\nu(x)})^3 + (\sqrt[3]{\nu^c(x)})^3 = \nu(x) + 1 - \nu(x) = 1$ and so $\langle A, \frac{1}{3}, \frac{1}{3}\rangle = \{(x, \sqrt[3]{\nu^c(x)}, \sqrt[3]{\nu(x)})\}$ is an FFS of X. Now, being A is an FQ_s of X,

$$(\sqrt[3]{\nu^c(x*y))})^3 = \nu^c(x*y) \geq \nu(x) \wedge \nu(y) = (\sqrt[3]{\nu^c(x)})^3 \wedge (\sqrt[3]{\nu^c(y)})^3.$$

Also,

$$(\sqrt[3]{\nu(x*y)})^3 \quad = \quad \nu(x*y) \leq \nu(x) \vee \nu(y) = (\sqrt[n]{\nu^c(x)})^3 \vee (\sqrt[3]{\nu^c(y)})^3.$$

Then $\langle A, \frac{1}{3}, \frac{1}{3}\rangle = \{(x, \sqrt[3]{\nu^c(x)}, \sqrt[3]{\nu(x)})\}$ is an FFQ_s of X. $\square$

Let E be an FFS of X and $x \in X$. Define $E^c = \{(x, \nu_E(x), \mu_E(x)) \mid x \in X\}$, it is clear that E^c is an anti FFQ_s of X.

Corollary 4.7. *Let A be an FQ_s of X. Then*

(i) $[[A, \frac{1}{3}, \frac{1}{3}]] = \{(x, \sqrt[3]{\mu^c(x)}, \sqrt[3]{\mu(x)})\}$ *is an anti FFQ_s of X.*

(ii) $\langle\langle A, \frac{1}{3}, \frac{1}{3}\rangle\rangle = \{(x, \sqrt[3]{\nu(x)}, \sqrt[3]{\nu^c(x)})\}$ *is an anti FFQ_s of X.*

Definition 4.8. Let E be an FFS of X and $\alpha, \beta \in \mathbb{I}$. Define $E^{(\alpha,\beta)} = \{x \in X \mid \mu_E^3(x) \geq \alpha, \nu_E^3(x) \leq \beta\}$ is known as Fermatean fuzzy level subset of E.

Theorem 4.9. *Let E be a FFS of X and $\alpha, \beta \in \mathbb{I}$. Then $E^{(\alpha,\beta)} \leq_{sub} X$ iff E is a FFQ_s of X.*

Proof. Let $x, y \in E^{(\alpha,\beta)}$. Then $\mu_E^3(x) \geq \alpha$ and $\nu_E^3(y)) \leq \beta$. This implies that $\mu_E^3(x * y) \geq \mu_E^3(x) \wedge \mu_E^3(y) \geq \alpha \wedge \alpha = \alpha$ and $\nu_E^3(x * y) \leq \nu_E^3(x) \vee \nu_E^3(y) \leq \beta \vee \beta = \beta$ and so $x * y \in E^{(\alpha,\beta)}$.

Conversely, let $\mu_E^3(x) \wedge \mu_E^3(y) = \alpha$ and $\nu_E^3(x) \vee \nu_E^3(y) = \beta$. Then $\mu_E^3(x) \geq \alpha, \mu_E^3(y) \geq \alpha$ and $\nu_E^3(x) \leq \beta, \nu_E^3(y) \leq \beta$. This implies that

$x, y \in E^{(\alpha,\beta)}$ and so $x * y \in E^{(\alpha,\beta)}$, because $E^{(\alpha,\beta)}$ is a Q-subalgebra of X. Therefore, $\mu_E^3(x) \geq \alpha = \mu_E^3(x) \wedge \mu_E^3(y)$ and $\nu_E^3(x) \leq \beta = \nu_E^3(x) \vee \nu_E^3(y)$, so E is an FFQ_s of X. □

Definition 4.10. Let E be an FFS of ALQ-algebra X. Define $\sqrt[3]{E} = (\mu_{\sqrt[3]{E}}, \nu_{\sqrt[3]{E}})$ known as Fermatean fuzzy nil radical of E, which $\mu_{\sqrt[3]{E}}(x) = \bigwedge_{n \geq 1} \mu_E(x^n)$ and $\nu_{\sqrt[3]{E}} = \bigvee_{n \geq 1} \nu_E(x^n)$

Example 4.11. Consider the ALQ-algebra $(X, *, 0)$ in the following Table 6, where $X = \{0, a, b\}$.

*	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

Table 6: ALQ-algebra $(X, *, 0)$

and $E = (\mu_E, \nu_E)$ by $\mu_E(0) = 0.9, \mu_E(a) = 0.6, \mu_E(b) = 0.3, \nu_E(0) = 0.2, \nu_E(a) = 0.5, \nu_E(b) = 0.7$. Thus

$$\sqrt[3]{E} = (\mu_{\sqrt[3]{E}}, \nu_{\sqrt[3]{E}}) = \begin{cases} (0.9, 0.2) & \text{for } x = 0, \\ (0.6, 0.5) & \text{for } x = a, \\ (0.3, 0.7) & \text{for } x = b, \end{cases}.$$

Theorem 4.12. *Let $(X, *, 0_X)$ be an ALQ-algebra and E be a FFQ_s of X. Then $\sqrt[3]{E}$ is a FFQ_s of X.*

Proof. Let $x \in X$ and $n \geq 1$. Since E is an FFS of X, $0 \leq \mu_E^3(x^n) + \nu_E^3(x^n)) \leq 1$ and so

$$\mu^3_{\sqrt[3]{E}}(x) + \nu^3_{\sqrt[3]{E}}(x) = \bigwedge_{n\geq 1} \mu_E^3(x^n) + \bigvee_{n\geq 1} \nu_E^3(x^n) \leq 1$$

and so $\sqrt[3]{E}$ is an FFS of X. In addition,

$$\mu^3_{\sqrt[3]{E}}(x * y) = \bigwedge_{n\geq 1} \mu_E^3((x*y)^n) = \bigwedge_{n\geq 1} \mu_E^3(x^n * y^n) \geq \bigwedge_{n\geq 1} (\mu_E^3(x^n) \wedge \mu_E^3(y^n))$$
$$\geq \bigwedge_{n\geq 1} \mu_E^3(x^n) \wedge \bigwedge_{n\geq 1} \mu_E^3(y^n) = \mu^3_{\sqrt[3]{E}}(x) \wedge \mu^3_{\sqrt[3]{E}}(y).$$

Also,

$$\nu^3_{\sqrt[3]{E}}(x * y) = \bigvee_{n\geq 1} \nu_E^3((x*y)^n) = \bigvee_{n\leq 1} \nu_E^3(x^n * y^n) \leq \bigvee_{n\geq 1} (\nu_E^3(x^n) \vee \nu_E^3(y^n))$$
$$\leq \bigvee_{n\geq 1} \nu_E^3(x^n) \vee \bigvee_{n\geq 1} \nu_E^3(y^n) = \nu^3_{\sqrt[3]{E}}(x) \vee \nu^3_{\sqrt[3]{E}}(y).$$

Therefore, $\sqrt[3]{E}$ is a FFQ_s of X. $\square$

Theorem 4.13. *Let $(X, *, 0_X)$ be a Q-algebra and E, F be FFQ_s of X. Then*

(i) $\sqrt[3]{E} \subseteq E$,

(ii) if $E \subseteq F$, *then* $\sqrt[3]{E} \subseteq \sqrt[3]{F}$,

(iii) if X *is an ALQ-algebra, then* $\sqrt[3]{\sqrt[3]{E}} = \sqrt[3]{E}$.

(iv) if X *is an ALQ-algebra, then* $(\sqrt[3]{E})^{(3,3)} = E^{(3,3)}$.

Proof. *(i)* Let $x \in X$. Then $\mu_{\sqrt[3]{E}}(x) = \bigwedge_{n\geq 1} \mu_E(x^n) = \mu_E(x) \wedge (\bigwedge_{n\geq 2} \mu_E(x^n)) \leq \mu_E(x)$ and so $\mu_E \supseteq \mu_{\sqrt[3]{E}}$. In addition, $\nu_{\sqrt[3]{E}}(x) =$

$\bigvee_{n\geq 1} \nu_E(x^n) = \nu_E(x) \vee (\bigvee_{n\geq 2} \nu_E(x^n)) \geq \nu_E(x)$ and $\nu_{\sqrt[3]{E}} \supseteq \nu_E$ and so $E \supseteq \sqrt[3]{E}$.

(ii) Let $x \in X$. Then $\mu_E(x) \leq \mu_F(x)$ and $\nu_E(x) \geq \nu_F(x)$, so

$$\mu_{\sqrt[3]{E}}(x) = \bigwedge_{n\geq 1} \mu_E(x^n) \leq \bigwedge_{n\geq 1} \mu_F(x^n) = \mu_{\sqrt[3]{F}}(x)$$

and $\nu_{\sqrt[3]{E}}(x) = \bigvee_{n\geq 1} \nu_E(x^n) \geq \bigvee_{n\geq 1} \nu_F(x^n) = \nu_{\sqrt[3]{F}}(x)$. Hence $\sqrt[3]{E} \subseteq \sqrt[3]{F}$.

(iii) By (i), $\sqrt[3]{E} \subseteq \sqrt[3]{\sqrt[3]{E}}$. Let $x \in X$. Then by definition and Lemma 3.3,

$$\mu_{\sqrt[3]{\sqrt[3]{E}}}(x) = \bigwedge_{n\geq 1} \mu_{\sqrt[3]{E}}(x^n) = \bigwedge_{n\geq 1} \bigwedge_{m\geq 1} \mu_E((x^n)^m) = \bigwedge_{n\geq 1} \bigwedge_{m\geq 1} \mu_E(x^{mn})$$
$$= \bigwedge_{k\geq 1} \mu_E(x^k) = \mu_{\sqrt[3]{E}}(x).$$

In similar to $\nu_{\sqrt[3]{\sqrt[3]{E}}}(x) = \nu_{\sqrt[3]{E}}(x)$ and so $\sqrt[3]{\sqrt[3]{E}} = \sqrt[3]{E}$.

(iv) Let $x \in X$. Then

$$(\mu^3_{\sqrt[3]{E}})(x) = \bigwedge_{n\geq 1} \mu^3_E(x^n) = \mu^3_E(x) \wedge \mu^3_E(0_X) = \mu^3_E(x) \textit{ and}$$
$$(\nu^3_{\sqrt[3]{E}})(x) = \bigvee_{n\geq 1} \nu^3_E(x^n) = \nu^3_E(x) \vee \nu^3_E(0_X) = \nu^3_E(x).$$

Hence, $(\sqrt[3]{E})^{(3,3)} = \{(x, \mu^3_E(x), \nu^3_E(x)) \mid x \in X\}$. □

Corollary 4.14. *Let $(X, *, 0_X)$ be an ALQ-algebra. Then $\sqrt[3]{E} = E$.*

Example 4.15, shows that the converse of Corollary 4.14, is not true necessarily.

Example 4.15. Let A be any FQ_s of Z as Table 3. Then

- (i) by Theorem 4.5, $\sqrt[3]{[A, \frac{1}{3}, \frac{1}{3}]} = [A, \frac{1}{3}, \frac{1}{3}]$, while Z is not ALQ-algebra.

(*i*) by Theorem 4.6, $\sqrt[3]{\langle A, \frac{1}{3}, \frac{1}{3}\rangle} = \langle A, \frac{1}{3}, \frac{1}{3}\rangle$, while Z is not ALQ-algebra.

Definition 4.16. Let E and F be FFS of X. Define $E \circ F = \{(x, (\mu_{E\circ F})(x), (\nu_{E\circ F})(x) \mid x \in X\}$, *which* $(\mu_{E\circ F})(x) = \bigvee\limits_{a*b=x} (\mu_E(a) \wedge \mu_F(b))$ *and* $(\nu_{E\circ F})(x) = \bigwedge\limits_{a*b=x} (\nu_E(a) \vee \nu_F(b))$.

Theorem 4.17. *Let E be an FFS of X. Then the subsequent are equivalent:*

(*i*) *E is FFQ_s of X,*

(*ii*) $(E \circ E)^{(3,3)} \subseteq E^{(3,3)}$.

Proof. Let $x, y \in X$. Then there are $c, d, c', d' \in X$ in a way that

$$\begin{aligned}
\mu^3_{E\circ E}(x) + \nu^3_{E\circ E}(x) &= \Big(\bigvee_{a*b=x} (\mu_E(a) \wedge \mu_E(b))\Big)^3 + \Big(\bigwedge_{a*b=x} (\nu_E(a) \vee \nu_E(b))\Big)^3 \\
&= \Big(\bigvee_{a*b=x} (\mu^3_E(a) \wedge \mu^3_E(b))\Big) + \Big(\bigwedge_{a*b=x} (\nu^3_E(a) \vee \nu^3_E(b))\Big) \\
&= (\mu^3_E(c) \wedge \mu^3_E(d)) + (\nu^3_E(c') \vee \nu^3_E(d')) \leq 1.
\end{aligned}$$

Hence, $E \circ E$ is an FFS of X and so $(E \circ E)^{(3,3)}$ is an FFS of X.

Since E is an FFQ_s of X,

$$\mu^3_{E\circ E}(x) = \Big(\bigvee_{a*b=x} (\mu_E(a) \wedge \mu_E(b))\Big)^3 = \bigvee_{a*b=x} (\mu^3_E(a) \wedge \mu^3_E(b)) \leq \bigvee_{a*b=x} \mu^3_E(a*b) = \mu^3_E(x),$$
$$\nu^3_{E\circ E}(x) = \Big(\bigwedge_{a*b=x} (\nu_E(a) \vee \nu_E(b))\Big)^3 = \bigwedge_{a*b=x} (\nu^3_E(a) \vee \nu^3_E(b)) \geq \bigwedge_{a*b=x} \nu^3_E(a*b) = \nu^3_E(x).$$

Hence $(E \circ E)^{(3,3)} \subseteq E^{(3,3)}$.

Conversely, let $(E \circ E)^{(3,3)} \subseteq E^{(3,3)}$. Then with choose, $x = a$ and

$y = b$, have

$$\mu^3_{E\circ E}(x*y) = \Big(\bigvee_{a*b=x*y}(\mu_E(a)\wedge\mu_E(b))\Big)^3 = \bigvee_{a*b=x*y}(\mu^3_E(a)\wedge\mu^3_E(b)) \geq \mu^3_E(x)\wedge\mu^3_E(y)$$
$$\geq \mu^3_{E\circ E}(x)\wedge\mu^3_{E\circ E}(y) \text{ and}$$
$$\nu^3_{E\circ E}(x*y) = \Big(\bigwedge_{a*b=x*y}(\nu_E(a)\vee\nu_E(b))\Big)^3 = \bigwedge_{a*b=x*y}(\nu^3_E(a)\vee\nu^3_E(b)) \leq \nu^3_E(x)\vee\nu^3_E(y)$$
$$\leq \nu^3_{E\circ E}(x)\vee\nu^3_{E\circ E}(y).$$

Hence $(E\circ E)^{(3,3)}$ is an FFQ_s of X. $\square$

Theorem 4.18. *Let E and F be FFQ_s of X. If X is an ALQ-algebra, then the subsequent are equivalent:*

(i) $(E\circ F)^{(3,3)}$ *is an FFQ_s of X,*

(ii) $(E\circ F)^{(3,3)} = (F\circ E)^{(3,3)}$.

Proof. Let $x, y\in X$ and $(E\circ F)^{(3,3)} = (F\circ E)^{(3,3)}$. Then

$$\begin{aligned}(\mu^3_E\circ\mu^3_F)\circ(\mu^3_E\circ\mu^3_F) &= \mu^3_E\circ(\mu^3_F\circ\mu^3_E)\circ\mu^3_F\\ &= \mu^3_E\circ(\mu^3_E\circ\mu^3_F)\circ\mu^3_F\\ &= (\mu^3_E\circ\mu^3_E)\circ(\mu^3_F\circ\mu^3_F)\subseteq\mu^3_E\circ\mu^3_F\end{aligned}$$

and

$$\begin{aligned}(\nu^3_E\circ\nu^3_F)\circ(\nu^3_E\circ\nu^3_F) &= \nu^3_E\circ(\nu^3_F\circ\nu^3_E)\circ\nu^3_F\\ &= \nu^3_E\circ(\nu^3_E\circ\nu^3_F)\circ\nu^3_F\\ &= (\nu^3_E\circ\nu^3_E)\circ(\nu^3_F\circ\nu^3_F)\supseteq\nu^3_E\circ\nu^3_F.\end{aligned}$$

Hence $((E\circ F)\circ(E\circ F))^{(3,3)}\subseteq(E\circ F)^{(3,3)}$ and by Theorem 4.17, $(E\circ F)^{(3,3)}$ is an FFQ_s of X.

Conversely, let $(E\circ F)^{(3,3)}$ be an FFQ_s of X. Since X is an ALQ-algebra,

$$\begin{aligned}(\mu^3_{E\circ F})(x) &= \bigvee_{a*b=x}\mu^3_E(a)\wedge\mu^3_F(b)\\ &= \bigvee_{b*a=x}\mu^3_E(b)\wedge\mu^3_F(a) = (\mu^3_{F\circ E})(x).\end{aligned}$$

In similar to, $(\nu^3_{E\circ F})(x) = (\nu^3_{F\circ E})(x)$. Hence, $(E \circ F)^{(3,3)} = (F \circ E)^{(3,3)}$.
□

Theorem 4.19. *Let E and F be FFQ_s of X. If $E \subseteq F$ and $\nu_E(0_X) = \nu^3_E(0_X)$, then $E^{(3,1)} \subseteq E \circ F$.*

Proof. Let $x, y \in X$. Since $x * 0_X = x$,

$$\begin{aligned}(\mu_{E\circ F})(x) \quad &= \quad \bigvee_{a*b=x} (\mu_E(a) \wedge \mu_F(b)) \geq \mu_E(x) \wedge \mu_F(0_X) \geq \mu^3_E(x) \wedge \mu^3_F(x) = \mu^3_E(x) \\ \text{and} \quad (\nu_{E\circ F})(x) &= \bigwedge_{a*b=x} (\nu_E(a) \vee \nu_F(b)) \leq \nu_E(x) \vee \nu_F(0_X) \leq \nu_E(x) \vee \nu_E(0_X) \\ &= \quad \nu_E(x) \vee \nu^3_E(0_X) \leq \nu_E(x) \vee \nu^3_E(x) = \nu_E(x).\end{aligned}$$

This implies that $\mu^3_E \subseteq \mu_{E\circ F}$ and $\nu_{E\circ F} \subseteq \nu_E$ and so $E^{(3,1)} \subseteq E \circ F$.
□

Corollary 4.20. *Let E and F be FFQ_s of X.*

(i) If $\nu_E(0_X) = \nu^3_E(0_X)$, then $E^{(3,1)} \subseteq E \circ E$.

(ii) $E \subseteq E \circ E$.

(iii) If $E \subseteq F$, then $E \subseteq E \circ F$.

Corollary 4.21. *Let A be an FQ_s of X. Then $[A, \frac{1}{3}, \frac{1}{3}] \subseteq [A, \frac{1}{3}, \frac{1}{3}] \circ [A, \frac{1}{3}, \frac{1}{3}]$.*

Theorem 4.22. *Let E and F be FFQ_sas of X. Then*

(i) $\sqrt[3]{E} \cap \sqrt[3]{F} = \sqrt[3]{E \cap F}$, and $\nu^3_{\sqrt[3]{E} \cup \sqrt[3]{F}} \supseteq \nu^3_{\sqrt[3]{E\circ F}}$,

(ii) If $E \subseteq F$, then $\sqrt[3]{E^{(3,3)}} \subseteq \sqrt[3]{(E \circ F)^{(3,3)}}$. $\nu_E(0_X) = \nu^3_E(0_X)$, then $E \circ F \subseteq E^{(3,1)}$.

Proof. (i) Let $x, y \in X$. Using Theorem 4.13, $\sqrt[3]{E \cap F} \subseteq \sqrt[3]{E} \cap \sqrt[3]{F}$.

$$
\begin{aligned}
(\mu_{\sqrt[3]{E}} \cap \mu_{\sqrt[3]{F}})(x) &= \mu_{\sqrt[3]{E}}(x) \wedge \mu_{\sqrt[3]{F}}(x) = \bigwedge_{n\geq 1} \mu_E(x^n) \wedge \bigwedge_{n\geq 1} \mu_F(x^n) \\
&= \bigwedge_{n\geq 1} (\mu_F(x^n) \wedge \mu_E(x^n)) \\
&\quad = \bigwedge_{n\geq 1} (\mu_F \cap \mu_E)(x^n) = \bigwedge_{n\geq 1} \mu_{(E\cap F)}(x^n) = (\mu_{\sqrt[3]{F\cap E}})(x) \text{ and} \\
&\quad (\nu_{\sqrt[3]{E}} \cup \nu_{\sqrt[3]{F}})(x) = \nu_{\sqrt[3]{E}}(x) \vee \nu_{\sqrt[3]{F}}(x) = \bigvee_{n\geq 1} \nu_E(x^n) \vee \bigvee_{n\geq 1} \nu_F(x^n) \\
&= \bigvee_{n\geq 1} (\nu_F(x^n) \vee \nu_E(x^n)) \\
&\quad = \bigvee_{n\geq 1} (\nu_F \cup \nu_E)(x^n) = \bigvee_{n\geq 1} \nu_{(E\cup F)}(x^n) = (\nu_{\sqrt[3]{F\cup E}})(x).
\end{aligned}
$$

(ii) Let $x \in X$. Then

$$
\begin{aligned}
\mu^3_{\sqrt[3]{E\circ F}}(x) &= \bigwedge_{n\geq 1} \mu^3_{E\circ F}(x^n) = \bigwedge_{n\geq 1} \bigvee_{a*b=x^n} (\mu^3_F(a) \wedge \mu^3_E(b)) \geq \bigwedge_{n\geq 1} (\mu^3_F(x^n) \wedge \mu^3_E(0_X)) \\
&\geq \bigwedge_{n\geq 1} (\mu^3_F(x^n) \wedge \mu^3_E(x^n)) = (\bigwedge_{n\geq 1} (\mu^3_F(x^n)) \wedge (\bigwedge_{n\geq 1} \mu^3_E(x^n)) \\
&= \mu^3_{\sqrt[3]{E}}(x) \wedge \mu^3_{\sqrt[3]{F}}(x) = (\mu^3_{\sqrt[3]{E}} \cap \mu^3_{\sqrt[3]{F}})(x).
\end{aligned}
$$

Hence $\mu^3_{\sqrt[3]{E}} \cap \mu^3_{\sqrt[3]{F}} \subseteq \mu^3_{\sqrt[3]{E\circ F}}$. Also

$$
\begin{aligned}
\nu^3_{\sqrt[3]{E\circ F}}(x) &= \bigvee_{n\geq 1} \nu^3_{E\circ F}(x^n) = \bigvee_{n\geq 1} \bigwedge_{a*b=x^n} (\nu^3_F(a) \vee \nu^3_E(b)) \\
&\quad \leq \bigvee_{n\geq 1} (\nu^3_F(x^n) \vee \nu^3_E(0_X)) \\
&\leq \bigvee_{n\geq 1} (\nu^3_F(x^n) \vee \nu^3_E(x^n)) = (\bigvee_{n\geq 1} (\nu^3_F(x^n)) \vee (\bigvee_{n\geq 1} \nu^3_E(x^n)) \\
&= \nu^3_{\sqrt[3]{E}}(x) \vee \nu^3_{\sqrt[3]{F}}(x) = (\nu^3_{\sqrt[3]{E}} \cup \nu^3_{\sqrt[3]{F}})(x).
\end{aligned}
$$

Hence $\nu^3_{\sqrt[3]{E\circ F}} \subseteq (\nu^3_{\sqrt[3]{E}} \cup \nu^3_{\sqrt[3]{F}})$. Since $E \subseteq F$, we get $\mu^3_{\sqrt[3]{E}} \cap \mu^3_{\sqrt[3]{F}} = \mu^3_{\sqrt[3]{E}}$ and $\nu^3_{\sqrt[3]{E}} \cup \nu^3_{\sqrt[3]{F}} = \nu^3_{\sqrt[3]{E}}$. This implies that $\sqrt[3]{E^{(3,3)}} \subseteq \sqrt[3]{(E\circ F)^{(3,3)}}$.
□

Theorem 4.23. *Let E be an FFQ_s of X. Then* $\sqrt[3]{E^{(3,3)}} = \sqrt[3]{(E \circ E)^{(3,3)}}$.

Proof. Let $x \in X$. Then

$$\mu^3_{E\circ E}(x) = \bigwedge_{n\geq 1} \mu^3_E(a) \wedge \mu^3_E(b) \leq \mu^3_E(a * b) = \mu^3_E(x) \ and$$
$$\nu^3_{E\circ E}(x) = \bigvee_{n\geq 1} \nu^3_E(a) \vee \nu^3_E(b) \geq \nu^3_E(a * b) = \nu^3_E(x).$$

Thus $\mu^3_{E\circ E} \subseteq \mu^3_E$ and $\nu^3_E \subseteq \nu^3_{E\circ E}$ and so $(E \circ E)^{(3,3)} \subseteq E^{(3,3)}$. Therefore, $\sqrt[3]{(E \circ E)^{(3,3)}} \subseteq \sqrt[3]{E^{(3,3)}}$ and by Theorem 4.22, $\sqrt[3]{E^{(3,3)}} = \sqrt[3]{(E \circ E)^{(3,3)}}$. □

Let $(X, *, 0_X)$ and $(X', *', 0_{X'})$ be Q-algebras. Then a map $f : X \to X'$ is referred to as a homomorphism, if for any $x, y \in X, f(x * y) = f(x) *' f(y)$. If f is onto, then it is referred to as an epimorphism. For any FS, $A = \{(x, \mu(x)) \mid x \in X\}$ of X and FS, $B = \{(x, \nu(x)) \mid x \in X'\}$ of X', recall that $(\mu_f)(y) = f(\mu)(y) = \begin{cases} \bigvee_{y=f(x)} \mu(x) & f^{-1}(y) \neq \emptyset \\ 0 & f^{-1}(y) = \emptyset \end{cases}$ and $(\nu_{f^{-1}})(x) = f^{-1}(\nu)(x) = \nu(f(x))$.

Theorem 4.24. *Let F be an FFQ_s of X'. Then for homomorphism $f : X \to X'$,*

(i) $f^{-1}(F)$ *is an* FFQ_s *of* X.

(ii) $\sqrt[3]{f^{-1}(F)} = f^{-1}(\sqrt[3]{F})$.

(iii) $f^{-1}(F) \subseteq f^{-1}(\sqrt[3]{F})$.

Theorem 4.25. *Let E be an FFQ_s of X and $f : X \to X'$ be an isomorphism. Then*

(i) $f(E)$ *is an* FFQ_s *of* X'.

(ii) $\sqrt[3]{f(E)} \subseteq \sqrt[3]{f(\sqrt[3]{E})}$.

(iii) $\sqrt[3]{f(E)} = f(\sqrt[3]{E})$.

Proof. (i) Let $x', y' \in X'$. Then there is $c, d \in X$ in a way that $f^3(\mu_E)(y)+f^3(\nu_E)(y) = \bigvee_{y=f(x)} \mu_E^3(x)+ \bigvee_{y=f(x)} \nu_E^3(x) = \mu_E^3(c)+\mu_E^3(d) \leq 1$ and so $f(E)$ is an FSS of X'. Since f is onto, there are $x, y \in X$ in a way that $x' = f(x), y' = f(y)$ and so

$$\begin{aligned}
((\mu_f^3)_E)(x' *' y') &= f^3(\mu_E)(x' *' y') = \bigvee_{x'*'y'=f(a)} \mu_F^3(a) \geq \bigvee_{f(x*y)=f(a)} \mu_F^3(x*y) \\
&\geq \bigvee_{f(x*y)=f(a)} (\mu_F^3(x) \wedge \mu_F^3(x)) = (\bigvee_{f(x)=x'} \mu_F^3(x)) \wedge (\bigvee_{f(y)=y'} \mu_F^3(y)) \\
&= f^3(\mu_E)(x') \wedge f^3(\mu_E)(y'), \text{ and since f is one to one} \\
&\quad ((\nu_f^3)_E)(x' *' y') = f^3(\nu_E)(x' *' y') = \bigvee_{x'*'y'=f(a)} \nu_F^3(a) \\
&\quad = \bigvee_{f(x*y)=f(a)} \nu_F^3(x*y) \\
&\leq \bigvee_{f(x*y)=f(a)} (\nu_F^3(x) \wedge \nu_F^3(x)) = (\bigvee_{f(x)=x'} \nu_F^3(x)) \wedge (\bigvee_{f(y)=y'} \nu_F^3(y)) \\
&= f^3(\nu_E)(x') \wedge f^3(\nu_E)(y').
\end{aligned}$$

Hence $f(E)$ is an FFQ_s of X'.

(ii) Let $y \in X'$. Since f is onto, then $f^{-1}(y) \neq \emptyset$, so

$$\mu_{f(\sqrt[3]{E})}(y) = \bigwedge_{n\geq 1} \mu_{f(E)}(y^n) = \bigwedge_{n\geq 1} \bigvee_{f(x)=y^n} \mu_E(x) \geq \bigvee_{f(x)=y} \mu_E(x) = \mu_{f(E)}(y) \textit{ and}$$
$$\nu_{f(\sqrt[3]{E})}(y) = \bigvee_{n\geq 1} \nu_{f(E)}(y^n) = \bigvee_{n\geq 1} \bigvee_{f(x)=y^n} \nu_E(x) \leq \bigvee_{f(x)=y} \nu_E(x) = \nu_{f(E)}(y).$$

Hence $\mu_{f(E)} \subseteq \mu_{f(\sqrt[3]{E})}, \nu_{f(E)}(y) \supseteq \nu_{f(\sqrt[3]{E})}$ and so $f(E) \subseteq f(\sqrt[3]{E})$.

(iii) Let $y \in X'$. Since f is onto, then $f^{-1}(y) \neq \emptyset$, so for any $0 < \epsilon$,

$$\begin{aligned}
& \nu_{\sqrt[3]{f(E)}}(y) - \epsilon \leq (\nu_{f(E)})(y^m) = (\nu_{f(E)})(f(x^m)) \\
= \quad & (\nu_{f^{-1}(f(E))})(x^m) = \nu_E(x^m) \leq (\nu_{\sqrt[3]{E}})(x) \leq \bigvee_{f(z)=x} \nu_{f(\sqrt[3]{E})}(z) = \nu_{f(\sqrt[3]{E})}(y) \textit{ and} \\
& \mu_{\sqrt[3]{f(E)}}(y) + \epsilon \geq (\mu_{f(E)})(y^m) = (\mu_{f(E)})(f(x^m)) \\
= \quad & (\mu_{f^{-1}(f(E))})(x^m) = \mu_E(x^m) \geq (\mu_{\sqrt[3]{E}})(x) \geq \bigwedge_{f(z)=x} \mu_{f(\sqrt[3]{E})}(z) = \mu_{f(\sqrt[3]{E})}(y)
\end{aligned}$$

Hence, $\mu_{\sqrt[3]{f(E)}} \supseteq \mu_{f(\sqrt[3]{E})}, \nu_{\sqrt[3]{f(E)}} \subseteq \nu_{f(\sqrt[3]{E})}$ and so $(\sqrt[3]{f(E)}) \supseteq f(\sqrt[3]{E})$.

On the other hand, f is onto, then by definition of supremum, there is $x \in X$ in a way that

$$\begin{aligned}
\nu_{f(\sqrt[3]{E})}(y) - \epsilon \leq \nu_{\sqrt[3]{E}}(x) = \bigvee_{n\geq 1} \nu_E(x^n) \textit{ and} \\
\mu_{f(\sqrt[3]{E})}(y) + \epsilon \geq \mu_{\sqrt[3]{E}}(x) = \bigwedge_{n\geq 1} \mu_E(x^n)
\end{aligned}$$

In addition, since

$$\begin{aligned}
\nu_{f(\sqrt[3]{E})}(y) = \bigvee_{f(x)=y} \nu_{\sqrt[3]{E}}(x) = \bigvee_{f(x)=y} \bigvee_{n\geq 1} \nu_E(x^n) = \bigvee_{n\geq 1} \bigvee_{f(x)=y} \nu_E(x^n) \textit{ and} \\
\mu_{f(\sqrt[3]{E})}(y) = \bigvee_{f(x)=y} \mu_{\sqrt[3]{E}}(x) = \bigvee_{f(x)=y} \bigwedge_{n\geq 1} \mu_E(x^n) = \bigwedge_{n\geq 1} \bigvee_{f(x)=y} \mu_E(x^n),
\end{aligned}$$

there is $m \in \mathbb{N}$ in a way that

$$\begin{aligned}
(\nu_{f(\sqrt[3]{E})})(y) - \epsilon \leq \bigvee_{f(z)=y^m} \nu_E(z) = (\nu_{f(E)})(y^m) \leq \bigvee_{n\geq 1} (\nu_{f(E)})(y^n) = (\nu_{\sqrt[3]{f(E)}})(y) \textit{ and} \\
(\mu_{f(\sqrt[3]{E})})(y) + \epsilon \geq \bigwedge_{f(w)=y^m} \nu_E(w) = (\mu_{f(E)})(y^m) \geq \bigwedge_{n\geq 1} (\mu_{f(E)})(y^n) = (\mu_{\sqrt[3]{f(E)}})(y).
\end{aligned}$$

Thus $\mu_{f(\sqrt[3]{E})} \supseteq \mu_{\sqrt[3]{f(E)}}$ and $\nu_{f(\sqrt[3]{E})} \subseteq \nu_{\sqrt[3]{f(E)}}$. This implies that $f(\sqrt[3]{E}) \supseteq \sqrt[3]{f(E)}$ and so $f(\sqrt[3]{E}) = \sqrt[3]{f(E)}$. $\square$

Example 4.26. Consider the Q-algebras $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ as Tables 7 and 8, that $X = \{0_X, 1, 2, 3\}$ and $Y = \{0_Y, b, c, d\}$.

$*_X$	0_X	1	2	3
0_X	0_X	3	2	1
1	1	0_X	3	2
2	2	1	0_X	3
3	3	2	1	0_X

Table 7: $(X, *, 0_X)$

$*_Y$	0_X	b	c	d
0_Y	0_Y	b	d	c
b	b	0_Y	c	d
c	c	d	0_Y	b
d	d	c	b	0_Y

Table 8: $(Y, *, 0_Y)$

Now, define a bijection $\varphi : X \to Y$ by $\varphi = \{(0_X, 0_Y), (1, c), (2, b), (3, d)\}$, which is an isomorphism. Define a Fermatean fuzzy set $E = (\mu_E, \nu_E)$ on X by

x	0_X	1	2	3
$\mu_E(x)$	0.9	0.6	0.6	0.6
$\nu_E(x)$	0.4	0.7	0.7	0.7

Then $\varphi(E) = (\mu_{\varphi(E)}, \nu_{\varphi(E)})$ on Y is

y	0_Y	b	c	d
$\mu_{f(E)}(y)$	0.9	0.6	0.6	0.6
$\nu_{f(E)}(y)$	0.4	0.7	0.7	0.7

.

One can see that $\sqrt[3]{\varphi(E)} = \varphi(E) = \varphi(\sqrt[3]{E})$.

5 On Fermatean fuzzy Q-Ideal

In this section, we present the Fermatean fuzzy Q-ideals(FFI) based the fuzzy Q-ideals and investigate their properties. We present the notion 3-regular FS and make a correspondence between of an FFQ_s and FFQ_i.

Let X be a Q-algebra and A be an FS. Then A is referred to as a fuzzy Q-ideal (FQ_i) of X, if for any $x, y \in X, \mu(0_X) \geq \mu(x), \mu(x) \geq \min\{\mu(x * y), \mu(y)\}$ and it is referred to as an anti FQ_i of X, if for any $x, y \in X, \mu(0_X) \leq \mu(x), \mu(x) \leq \max\{\mu(x * y), \mu(y)\}$. Obviously, if A is an FQ_i of X, then A^c is an anti FQ_i of X, which $\mu^c(x) = 1 - \mu(x)$.

Definition 5.1. Let E be an $FFS = (\mu_E, \nu_E)$ of X. Then E is a Fermatean fuzzy Q-ideal (FFI) of X, if for any $x, y \in X, (1), \mu_E^3(0_X) \geq \mu_E^3(x), \nu_E^3(0_X) \leq \nu_E^3(x), (2), \mu_E^3(x) \geq \min\{\mu_E^3(x * y), \mu_E^3(y)\}$ and (3) : $\nu_E^3(x) \leq \max\{\nu_E^3(x * y), \mu_E^3(y)\}$.

Remark 5.2. Let E is an FFQ_i of X. Then μ_E, ν_E are FQ_i and anti FQ_i of X, respectively.

The converse of Remark 5.2, is not true, necessarily based the Example 4.3.

Example 5.3. (i) Let I be an ideal of X. Then $E = (\chi_I, \chi_I^c)$ is an FFQ_i of X.

(ii) Let A be an FQ_i of X. Then $[A, \frac{1}{3}, \frac{1}{3}]$ is an FFQ_i of X.

(iii) Let A be an FQ_i of X. Then $\langle A, \frac{1}{3}, \frac{1}{3}\rangle$ is an FFQ_i of X.

Proposition 5.4. *Let $E = (\mu_E, \nu_E)$ be an FFQ_i of X. Then μ_E and ν_E are order-reserving and order-preserving, respectively.*

Example 5.5. Consider the Q-algebra $(X, *, 0)$ in Table 9)

$*_X$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Table 9: $(X, *, 0)$

and the FFQ_i $E = (\mu_E, \nu_E)$ defined by

x	0	a	b	c
$\mu_E(x)$	0.75	0.75	0.75	0.75
$\nu_E(x)$	0.1	0.3	0.5	0.75

Proposition 5.6. *Any FFQ_i of X is a FFQ_s of X.*

The opposite of Proposition 5.6, may not be true.

Example 5.7. Consider the Q-algebra Y as Tables 2. Then $E = (\mu_E, \nu_E)$ is an FFQ_s of Y as Tables 10 and 11, while is not an FFQ_i of Y.

μ_E	0_X	a	b	c
0_X	0.5	0.4	0.3	0.2

Table 10: FS μ_E

ν_E	0_X	a	b	c
0_X	0.3	0.4	0.5	0.6

Table 11: FS ν_E

Let $m \in \mathbb{N}, E$ be an FFS of X and $x, y, z \in X$. We state that E is m-regular, if $x * y \leq z$, then $\mu_E^m(x) \geq \min\{\mu_E^m(y), \mu_E^m(z)\}$ and $\nu_E^m(x) \leq \max\{\nu_E^m(y), \nu_E^m(z)\}$.

Theorem 5.8. *Let X be a Q-algebra. Then*

(i) *Any FFQ_i of X, is 3-regular.*

(ii) *Any 3-regular FFQ_s of X is an FFQ_i of X.*

Proof. (i) Let $x, y, z \in X, x * y \leq z$ and $E = (\mu_E, \nu_E)$ be an FFQ_i of X. Then $(x * y) * z = 0_X$ and so

$$\begin{aligned} \mu^3(x) &\geq \min\{\mu^3(x * y), \mu^3(y)\} \geq \min\{\min\{\mu^3((x * y) * z), \mu^3(z)\}, \mu^3(y)\} \\ &= \min\{\min\{\mu^3(0_X), \mu^3(z)\}, \mu^3(y)\} = \min\{\mu^3(z), \mu^3(y)\}. \end{aligned}$$

In similar to, $\nu_E^3(x) \leq \max\{\nu_E^3(y), \nu_E^3(z)\}$ and so E is 3-regular.

(ii) Let $x, y \in X$ and E be a 3-regular FFQ_s of X. Then by definition, $\mu_E(0_X) \geq \mu_E^3(x)$ and $\nu_E^3(0_X) \leq \nu_E(x)$. Since, for any $x, y \in X, x * (x * y) \leq_X y$, It implies that by hypothesis that

$$\mu_E(x) \geq \min\{\mu_E(x * y), \mu_E(y)\} \textit{ and } \nu_E(x) \leq \min\{\nu_E(x * y), \nu_E(y)\}.$$

Hence, E is an FFQ_i of X. □

Theorem 5.9. *Let E and F be 3-regular FFQ_i of X, X', respectively and $f : X \to X'$ be an isomorphism. Then*

(i) *$f(E)$ is a 3-regular an FFQ_s of X'.*

(ii) *$f^{-1}(F)$ is a 3-regular an FFQ_s of X.*

Proof. *(i) Using Theorem 4.25, $f(E)$ is a an FFQ_s of X'. Now, for $x', y', z' \in X'$, if $x' *' y \leq_{X'} z'$, then there are $x, y, z \in X$ in a way that $f(x * y) \leq_{X'} f(z)$ and so $x * y \leq_X z$ because, f is a bijection. Since F is a 3-regular an FFQ_s of X,*

$$\begin{aligned} f^3(\mu_E)(x') &= \bigvee_{f(x)=x'} \mu_E^3(x) \geq \bigvee_{\substack{f(y)=y' \\ f(z)=z'}} \min\{\mu_E^3(y), \mu_E^3(z)\} \\ &\geq \min\{\bigvee_{f(y)=y'} \mu_E^3(y), \bigvee_{f(z)=z'} \mu_E^3(z)\} \\ &= \min\{f^3(\mu_E)(y'), f^3(\mu_E)(z')\}. \end{aligned}$$

In similar to, $f^3(\nu_E)(x') \leq \min\{f^3(\nu_E)(y'), f^3(\nu_E)(z')\}$ and so $f(E)$ is a 3-regular an FFQ_s of X'.

(ii) By Theorem 4.24, and similar to (i), the proof is obtained. □

Corollary 5.10. *Let E and F be FFQ_i of X, X', respectively and $f : X \to X'$ be an isomorphism. Then*

(i) $f(E)$ is an FFQ_i of X'.

(ii) $f^{-1}(F)$ is an FFQ_i of X.

Example 5.11. Consider the isomorphic Q-algebras $(X, *_X, 0_X)$ and $(Y, *_Y, 0_Y)$ Tables 7 and 8. Define $E = (\mu_E, \nu_E)$ on X by

x	0_X	1	2	3
$\mu_E(x)$	0.9	0.5	0.5	0.5
$\nu_E(x)$	0.2	0.7	0.7	0.7

One can check that E is an FFQ_i of X, hence it is 3-regular. Now, $\varphi(E)$ on Y is given by

y	0_Y	b	c	d
$\mu_{\varphi(E)}(y)$	0.9	0.5	0.5	0.5
$\nu_{\varphi(E)}(y)$	0.2	0.7	0.7	0.7

and clearly $\varphi(E)$ is a 3-regular FFQ_s of Y.

Next, define $F = (\mu_F, \nu_F)$ on Y by

y	0_Y	b	c	d
$\mu_F(y)$	0.8	0.4	0.4	0.4
$\nu_F(y)$	0.3	0.8	0.8	0.8

One can check that F is an FFQ_i (hence 3-regular) of Y. Then

$$\varphi^{-1}(F) = (\mu_{\varphi^{-1}(F)}, \nu_{\varphi^{-1}(F)})$$

on X is

x	0_X	1	2	3
$\mu_{\varphi^{-1}(F)}(x)$	0.8	0.4	0.4	0.4
$\nu_{\varphi^{-1}(F)}(x)$	0.3	0.8	0.8	0.8

which is a 3-regular FFQ_s of X.

Theorem 5.12. *Let $E = (\mu_E, \nu_E)$ be an FFS of X. Then*

(i) E is an FFQ_i of X iff μ_E^3 and ν^{c3}_E are FQ_i of X.

(ii) E is an FFQ_i of X iff $F = (\mu_E, \mu_E^c)$ and $G = (\nu_E^c, \nu_E)$ are FFQ_i of X.

Proof. Let $x, y \in X$.

(i) If E is an FFQ_i of X, clearly μ_E is an FQ_i of X. In addition,

$$\begin{aligned} \nu^{c3}_E(0_X) &= 1 - \nu_E^3(0_X) \geq 1 - \nu_E^3(x) = \nu^{c3}_E(x) \text{ and} \\ & \nu^{c3}_E(x) = 1 - \nu_E^3(x) \geq 1 - \max\{\nu_E^3(x * y), \nu_E^3(x)\} \\ & = \min\{1 - \nu_E^3(x * y), 1 - \nu_E^3(y)\} \\ &= \min\{\nu^{c3}_E(x * y), \nu^{c3}_E(y)\}. \end{aligned}$$

Hence, ν_E is an FQ_i of X.

Conversely, assume that μ_E^3 and ν^{c3}_E are FQ_i of X. Then for each $x, y \in X, \mu_E^3(0_X) \geq \mu_E^3(x), \nu_E^3(0_X) = 1 - \nu^{c3}_E(0_X) \leq 1 - \nu^{c3}_E(x) = \nu_E^3(x)$ and

$$\begin{aligned} 1 - \nu_E^3(x) &= \nu^{c3}_E(x) \geq \min\{\nu^{c3}_E(x * y), \nu^{c3}_E(y)\} \\ & = \min\{1 - \nu_E^3(x * y), 1 - \nu_E^3(y)\} \\ &= 1 - \max\{\nu_E^3(x * y), \nu_E^3(y)\}, \end{aligned}$$

that is $\nu_E^3(x) \leq \max\{\nu_E^3(x*y), \nu_E^3(y)\}$. Moreover, μ_E^3 is an FQ_i of X, so E is an FFQ_i of X.

(ii) Let E be an FFQ_i of X. Then by item $(i), \mu_E^3$ and ${\nu^3}^c_E$ are FQ_i of X and so ${\mu^3}^c_E$ and ${\nu^3}_E$ are anti FQ_i of X. It implies that $F = (\mu_E, \mu_E^c)$ and $G = (\nu_E^c, \nu_E)$ are FFQ_i of X.

Conversely, let $F = (\mu_E, \mu_E^c)$ and $G = (\nu_E^c, \nu_E)$ are FFQ_i of X. By item (i), μ_E^3 and ${\nu^3}^c_E$ are FQ_i of X and so $E = (\mu_E, \nu_E)$ is an FFQ_i of X. $\square$

Theorem 5.13. *Let E, F be* 3*-regular FFS of X. Then*

(i) $\sqrt[3]{E}$ *is a* 3*-regular FFS of X.*

(ii) $E \circ F$ *is a* 3*-regular FFS of X.*

Proof. Let $x, y, z \in X$ and $x * y \leq_X z$.

(i) Since E is a 3-regular FFS of X,

$$\mu^3_{\sqrt[3]{E}}(x) = \bigwedge_{n\geq 1} \mu_E^3(x^n) \geq \min\{\bigwedge_{n\geq 1} \mu_E^3(y^n), \bigwedge_{n\geq 1} \mu_E^3(z^n)\} = \min\{\mu^3_{\sqrt[3]{E}}(y), \mu^3_{\sqrt[3]{E}}(z)\}$$

and

$$\nu^3_{\sqrt[3]{E}}(x) = \bigvee_{n\geq 1} \nu_E^3(x^n) \leq \max\{\bigvee_{n\geq 1} \nu_E^3(y^n), \bigvee_{n\geq 1} \nu_E^3(z^n)\} = \max\{\nu^3_{\sqrt[3]{E}}(y), \nu^3_{\sqrt[3]{E}}(z)\}.$$

Hence, $\sqrt[3]{E}$ is a 3-regular FFS of X. (ii) Since E, F are 3-regular FFS of X and $x * 0_X = x, y * 0_X = y, z * 0_X = z$,

$$\begin{aligned}
(\mu^3_{E\circ F})(x) &= \bigvee_{a*b=x} (\mu_E^3(a) \wedge \mu_F^3(b)) \leq (\mu_E^3(x) \wedge \mu_F^3(0_X)) \\
&\geq (\mu_E^3(y) \wedge \mu_E^3(z) \wedge \mu_F^3(0_X)) = (\nu_E^3(y) \wedge \mu_F^3(0_X) \wedge \mu_E^3(z) \wedge \mu_F^3(0_X)) \\
&\geq \bigvee_{c*d=y} (\mu_E^3(c) \wedge \mu_F^3(d)) \wedge \bigvee_{p*q=z} (\mu_E^3(p) \wedge \mu_F^3(q)) = (\mu^3_{E\circ F})(y) \wedge (\mu^3_{E\circ F})(z).
\end{aligned}$$

and

$$\begin{aligned}
(\nu^3_{E\circ F})(x) &= \bigwedge_{a*b=x} (\nu_E^3(a) \vee \nu_F^3(b)) \leq (\nu_E^3(x) \vee \nu_F^3(0_X)) \\
&\leq (\nu_E^3(\vee \nu_E^3(z) \vee \nu_F^3(0_X)) = (\nu_E^3(y) \vee \mu_F^3(0_X \vee \nu_E^3(z) \vee \nu_F^3(0_X)) \\
&\leq \bigwedge_{c*d=y} (\nu_E^3(c) \vee \nu_F^3(d \vee \bigwedge_{p*q=z} (\nu_E^3(p) \vee \nu_F^3(q)) = (\nu^3_{E\circ F})(y) \vee (\nu^3_{E\circ F})(z).
\end{aligned}$$

Hence, $E \circ F$ is a 3-regular FFS of X $\square$

Corollary 5.14. *Let $(X, *, 0_X)$ be an ALQ-algebra and E be an FFQ_i of X. Then*

(i) $\sqrt[3]{E}$ *is an* FFQ_i *of* X.

(ii) If $(E \circ F)^{(3,3)} = (F \circ E)^{(3,3)}$, *then* $F \circ E$ *is an* FFQ_i *of* X.

Let E be an FS of X and $\alpha, \beta \in \mathbb{I}$. Define $U(E, \alpha) = \{x \in X \mid \mu_E^3(x) \geq \alpha\}$ and $L(E, \beta) = \{x \in X \mid \nu_E^3(x) \leq \beta\}$ are call as upper α-level cut of A and lower α-level cut of A, respectively.

Theorem 5.15. *An FFS E is an FFQ_i of X iff for any $\alpha, \beta \in \mathbb{I}, \emptyset \neq U(E, \alpha)$ and $\emptyset \neq L(E, \beta)$ are FQ_i of X.*

Proof. Let $\alpha, \beta \in \mathbb{I}$ and E be an FFQ_i of X. Firstly, $\mu_E^3(x) \geq \alpha$ and $\nu_E^3(x) \leq \beta$, so $0_X \in U(E, \alpha) \cap L(E, \beta)$. Assume that $x * y, y \in U(E, \alpha)$. Then $\mu_E^3(x * y) \geq \alpha, \mu_E^3(y) \geq \alpha$ and $\mu_E^3(x) \geq \min\{\mu_E^3(x * y), \mu_E^3(y)\} \geq \min\{\alpha, \alpha\}$ and so $y \in U(E, \alpha)$. It implies that $U(E, \alpha)$ is an FQ_i of X and in similar to $L(E, \beta)$ is an FQ_i of X.

Conversely, let $U(E, \alpha)$ and $L(E, \beta)$ are FQ_i of X. For any $x \in X, \mu_E^3(x) = \alpha, \nu_E^3(x) = \beta$. So $\{x, 0_X\} \subseteq U(E, \alpha) \cap L(E, \beta)$, because $U(E, \alpha)$ and $L(E, \beta)$ are FQ_i of X. Hence $\mu_E^3(0_X) \geq \alpha$ and $\nu_E^3(0_X) \leq \beta$. Let for some $x, y \in X, \mu_E^3(x) < \min\{\mu_E^3(x * y), \mu_E^3(y)\}$, then by taking $\gamma = \dfrac{1}{4}(\mu_E^3(x) + \min\{\mu_E^3(x * y), \mu_E^3(y)\})$, we have $\mu_E^3(x) < \gamma < \min\{\mu_E^3(x * y), \mu_E^3(y)\}$. Hence $x \in$ *not* $\in U(E, \gamma)$, while $x * y \in U(E, \gamma)$ and $y \in U(E, \gamma)$, which is a contradiction, because $U(E, \gamma)$ is an ideal of X. Moreover, if there is $x, y \in X$ in a way that $\nu_E^3(x) > \max\{\nu_E^3(x * y), \nu_E^3(y)\}$, by taking $* == \dfrac{1}{2}(\mu_E^3(x) + \min\{\mu_E^3(x * y), \mu_E^3(y)\})$, we get that

$\max\{\mu_E^3(x * y), \mu_E^3(y)\} < \gamma < \mu_E^3(x)$. Therefore, $x * y, y \in L(E, *)$, but $x \notin L(E, *)$ and is a contradiction, because $L(E, *)$ is an ideal of X. Hence E is an FFQ_i of X. $\square$

Corollary 5.16. *Let $\{E_i\}_{i \in I}$ be a collection of FFQ_i of X. Then*

(i) $\bigcap_{i \in I} E_i$ *is an* FFQ_i *of* X.

(ii) If $\{E_i\}_{i\in I}$ be a chain, then $\bigcup\limits_{i\in I} E_i$ is an FFQ_i of X.

Let $J = [0, 0.75] \subseteq [0, 1], \mathcal{J} = \{J_\lambda \mid \lambda \in \mathbb{J}\}$ be a collection of ideals of X and $\lambda, \lambda' \in \mathbb{J}$. Define $\lambda \leq \lambda'$ iff $J_{\lambda'} \subseteq J_\lambda$ and say $(\mathcal{J}, \subseteq)$ is a $\mathbb{J}$-poset. Also we say that X is a $\mathbb{J}$-coverd, if $X = \bigcup\limits_{\lambda\in\mathbb{J}} J_\lambda$ and $(\mathcal{J}, \subseteq)$ is a $\mathbb{J}$-poset. So based the Theorem 5.15 and Corollary 5.16, have the subsequent results.

Theorem 5.17. *Let X be a $\mathbb{J}$-coverd Q-algebra. Then $E = \{(x, \sup\{\lambda \in \mathbb{J} \mid x \in J_\lambda\}, \inf\{\lambda \in \mathbb{J} \mid x \in J_\lambda\})\}$ is an FFQ_i of X.*

Proof. Let $x, y \in X$. Then there is $\lambda \in \mathbb{J}$ in a way that $x \in I_\lambda$ and

$$\begin{aligned}&(\sup\{\lambda \in \mathbb{J} \mid x \in J_\lambda\})^3 + (\inf\{\lambda \in \mathbb{J} \mid x \in I_\lambda\})^3\\ &= \sup\{\lambda^3 \in \mathbb{J} \mid x \in J_\lambda\} + \inf\{\lambda^3 \in \mathbb{J} \mid x \in I_\lambda\} \leq 1\end{aligned}$$

and so E is an FFS of X. Let $x \in X$ and $\alpha \in \mathbb{I}$. Then there is $\lambda \in \mathbb{J}$ in a way that $x \in I_\lambda$. If $\alpha = \bigvee\limits_{\lambda<\alpha} \lambda$, then $U(E, \alpha) = \bigcap\limits_{\lambda<\alpha} I_\lambda$, because of, $\forall \lambda < \alpha, x \in U(E, \alpha) \Leftrightarrow x \in \bigcap\limits_{\lambda<\alpha} I_\lambda$. Thus $U(E, \alpha)$ is an FQ_i of X. If $\alpha \neq \bigvee\limits_{\lambda<\alpha} \lambda$, then there is $* > 0$ in a way that $(\alpha - *, \alpha) \cap \mathbb{J} = \emptyset$. Assume that $x \in \bigcup\limits_{\lambda\geq\alpha} I_\lambda$, then there is $\lambda \geq \alpha$ in a way that $x \in I_\lambda$ and so $\mu_E(x)\lambda \geq \alpha$. Hence $\bigcup\limits_{\lambda\geq\alpha} I_\lambda \subseteq U(E, \alpha)$. Suppose $x \notin \bigcup\limits_{\lambda\geq\alpha} I_\lambda$, then $\forall \lambda \geq \alpha, x \notin I_\lambda$ and so for all $\lambda > \alpha - *, x \notin I_\lambda$. Thus $\mu_E(x) \leq \alpha - * < \alpha$ and so $x \notin U(E, \alpha)$. Therefore, $\bigcup\limits_{\lambda\geq\alpha} I_\lambda = U(E, \alpha)$ and so in any cases, $U(E, \alpha)$, is an FQ_i of X. Now, let $\beta = \bigwedge\limits_{\beta<\lambda} \lambda$. Then $L(E, \beta) = \bigcap\limits_{\beta<\lambda} I_\lambda$, because of $\forall \beta < \lambda, x \in I_\lambda \Leftrightarrow x \in \bigcap\limits_{\beta<\lambda} I_\lambda$. Hence $L(E, \beta)$ is an FQ_i of X. If $\beta \neq \bigwedge\limits_{\beta<\lambda} \lambda$, then there is $* > 0$ in a way that $(\beta, \beta + *) \cap J = \emptyset$.

Let $x \in \bigcup_{\lambda \leq \beta} I_\lambda$, then there is $\lambda \leq \beta$ in a way that $x \in I_\lambda$. Then $\mu_E(x) \leq \lambda < \beta$ so that $x \in L(E, \beta)$. Hence $\bigcup_{\lambda \leq \beta} I_\lambda \subseteq L(E, \beta)$. If $x \notin \bigcup_{\lambda \leq \beta} I_\lambda$, then $\forall\ \lambda \leq \beta, x \notin I_\lambda$, special $x \notin I_\lambda$, which $\lambda < \beta + *$. It implies that $\nu_E(x) \geq \beta + * > \beta$ and so $x \notin L(E, \beta)$. Therefore, $L(E, \beta) \subseteq \bigcup_{\lambda \leq \beta} I_\lambda$ an in any cases, $L(E, \beta) \subseteq \bigcup_{\lambda \leq \beta} I_\lambda$, which implies that $L(E, \beta)$ is an FQ_i of X. $\square$

Example 5.18. Let $X = \{0, a, b, c\}$ be the Q-algebra given by Table 9. Take $\mathbb{J} = [0,\, 0.75] \subseteq [0, 1]$ and define a descending chain of ideals:

$$J_{0.1} = \{0\}, \quad J_{0.3} = \{0, a\}, \quad J_{0.5} = \{0, a, b\}, \quad J_{0.75} = \{0, a, b, c\} = X.$$

Clearly $J_{0.75} \subseteq J_{0.5} \subseteq J_{0.3} \subseteq J_{0.1}$ corresponds to $\lambda \leq \lambda' \iff J_{\lambda'} \subseteq J_\lambda$ and $X = \bigcup_{\lambda \in \mathbb{J}} J_\lambda$, so X is $\mathbb{J}$-covered.

Now define $E = (\mu_E, \nu_E)$ as in the theorem, by

$$\mu_E(x) = \sup\{\lambda \in \mathbb{J} \mid x \in J_\lambda\}, \qquad \nu_E(x) = \inf\{\lambda \in \mathbb{J} \mid x \in J_\lambda\}$$

and so

x	0	a	b	c
$\mu_E(x)$	0.75	0.75	0.75	0.75
$\nu_E(x)$	0.1	0.3	0.5	0.75.

One can see that E is indeed an FFQ_i of X.

6 Conclusion

Fermatean fuzzy subsets offer a stronger and more adaptable approach to managing uncertainty in complex decision-making scenarios when compared to traditional fuzzy sets, intuitionistic fuzzy sets, and Pythagorean fuzzy sets. Their less restrictive constraints enable a more accurate representation of real-world ambiguities, making them particularly effective across a wide range of fields, including medicine, engineering, and risk management. In this paper, we orient the idea of Fermatean fuzzy

Q-subalgebras as fuzzy Q-subalgebras and investigate their properties. Also

(i) The notions of ALQ-algebra and CLQ-algebra are introduced and is proved that, ALQ-algebra and CLQ-algebra are equal.

(ii) It has been proven that any Fermatean fuzzy Q-subalgebra is a generalization of fuzzy Q-subalgebra and anti fuzzy Q-subalgebra.

(iii) Fermatean fuzzy level subset of Fermatean fuzzy Q-subalgebra is defined and is shown that Fermatean fuzzy subset is a Fermatean fuzzy Q-subalgebra iff its Fermatean fuzzy level subset is a Q-algebra.

(iv) The notation of Fermatean fuzzy nil radical is introduced and is shown that Fermatean fuzzy nil radical of any Fermatean fuzzy Q-subalgebra is a Fermatean fuzzy Q-subalgebra.

(v) It is shown that in any ALQ-algebra, the fuzzy nil radical of a Fermatean fuzzy Q-subalgebra is equal to Fermatean fuzzy Q-subalgebra.

(vi) Using the isomorphisms, it is transformed the Fermatean fuzzy Q-subalgebras between two Q-algebras.

(vii) The notion of Fermatean fuzzy Q-ideals is presented and is made a link between Fermatean fuzzy Q-subalgebras and Fermatean fuzzy Q-ideals.

6.1 Suggestions for future researches

We hope to obtain more results regarding Q-algebras such as:

(i) orient the (anti) (m, n)-fuzzy Q-subalgebras,

(ii) introduce the notion of (m, n)-fuzzy Q-hypersubalgebra,

(iii) present the (m, n)-fuzzy graphs regard to (m, n)-fuzzy Q-subalgebras,

(iv) present the (m, n)-fuzzy hypergraphs regard to (m, n)-fuzzy Q-hypersubalgebra,

(v) compute the domination number of (m,n)-fuzzy graphs of (m,n)-fuzzy graphs,

(vi) compute domination number of (m,n)-fuzzy hypergraphs of (m,n)-fuzzy Q-hypersubalgebra.

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