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Common Fixed Point Theorem in Metric Spaces of Fisher and Sessa

A. R. Valipour Baboli
Karaj Branch, Islamic Azad University

M. B. Ghaemi*
Iran University of Science and Technology

Abstract. In this paper it is shown that T and I have a unique common fixed point on a compact subset C of a metric space X, where T and I are two self maps on C, I is non-expansive and the pair (T, I) is weakly commuting. In [3] Fisher and Sessa verified the same problem but with C closed subset. Further we show this result by replacing compatibility with weakly commutativity of pair (T, I) and continuity with non-expansiveness of I.

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1. Introduction

Many authors have written some papers in which two self maps on a closed convex set have a unique common fixed point for example [1], [3] and [9]. In 1986, Fisher and Sessa proved a fixed point theorem for two self maps on a subset of a Banach space which is closed convex(see [3]). Sessa in [9] generalized a result of Das and Naik [1]. They defined

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*Corresponding author

two maps T and I on a metric space (X, d) into itself to be weakly commuting iff

$$d(TIx, ITx) \leqslant d(Ix, Tx), \tag{1}$$

for all x in X.
A self map I on a metric space X is said to be non-expansive provided that

$$d(Ix, Iy) \leqslant d(x, y),$$

for all x, y in X. Two commuting maps clearly satisfy (1) but the converse is not generally true as is shown in the following example.

Example 1.1. Let $X = [0, 1]$, and suppose X is endowed with the Euclidean metric. Define T and I by $Tx = \frac{x}{x+4}$ and $Ix = \frac{x}{2}$ for any x in X. Then

$$\begin{aligned} d(TIx, ITx) &= \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)} \\ &\leqslant \frac{x^2+2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Ix, Tx). \end{aligned}$$

But for any $x \neq 0, TIx = \frac{x}{x+8} > \frac{x}{2x+8} = ITx.$

Fisher and Sessa proved the following theorem.

Theorem 1.2. ([3]) *Let T and I be two weakly commuting mappings from C into itself satisfying the inequality*

$$d(T(x), T(y)) \leqslant ad(I(x), I(y)) + (1-a)max\{d(T(x), I(x)), d(T(y), I(y))\} \tag{2}$$

for all x, y in C where $0 < a < 1$ and C is a closed convex subset of a Banach space X. If I is linear and non-expansive on C and further IC contains TC, then $T$and I have a unique common fixed point in C.

2. Main Results

Our aim is to modify of Theorem 1.2.

Theorem 2.1. *Let $T$and I be two weakly commuting self maps on C satisfying* (2)*, where C is a compact subset of the Banach space X. If I is non-expansive on C and IC contains TC, then T and I have a unique common fixed point in C.*

Proof. Let $x = x_0$ be an arbitrary point in C and for any $n \in N$ choose x_{n+1} such that $Tx_n = Ix_{n+1}$. Since C is compact so $\{x_n\}$ has a convergence subsequence $\{y_k\}_{k=1}^{\infty}$ (converging to x^* for some $x^* \in C$. In the following we show each y_k with y_n^k where it represent k'th member of $\{y_n\}$ and n'th element of $\{x_n\}$). Now we show

$$d(Tx^*, Ix^*) = 0.$$

$$\begin{aligned} d(Tx^*, Ix^*) &\leqslant \overline{lim} d(Tx^*, Ty_n^k) + \overline{lim} d(Ty_n^k, Iy_n^k) + \overline{lim} d(Iy_n^k, Ix^*) \\ &\leqslant \overline{lim} ad(Ix^*, Iy_n^k) + \overline{lim}(1-a) max\{d(Tx^*, Ix^*), d(Ty_n^k, Iy_n^k)\} \\ &+ \overline{lim} d(Ty_n^k, Iy_n^k) + \overline{lim} d(Iy_n^k, Ix^*). \end{aligned}$$

There are two cases if

$$\overline{lim} d(Tx^*, Ix^*) \geqslant \overline{lim} d(Ty_n^k, Iy_n^k),$$

then

$$\begin{aligned} ad(Tx^*, Ix^*) &\leqslant (a+1)\overline{lim} d(x^*, y_n^k) + \overline{lim} d(Ty_n^k, Iy_n^k) \\ &= \overline{lim} d(Ty_n^k, Iy_n^k) \\ &\leqslant \overline{lim} d(Ty_n^k, Ix_{n+1}) + \overline{lim} d(Ix_{n+1}, Iy_n^k) \\ &\leqslant \overline{lim} d(x_{n+1}, y_n^k) = 0, \end{aligned}$$

and $\quad \overline{lim} d(Ty_n^k, Iy_n^k) \geqslant d(Tx^*, Ix^*)$, then

$$\begin{aligned} d(Tx^*, Ix^*) &\leqslant (a+1)\overline{lim} d(x^*, y_n^k) + (2-a)\overline{lim} d(Ty_n^k, Iy_n^k) \\ &= (2-a)\overline{lim} d(Ty_n^k, Iy_n^k) \leqslant (2-a)(\overline{lim} d(Ty_n^k, Ix_{n+1}) + \overline{lim} d(Ix_{n+1}, Iy_n^k)) \\ &\leqslant (2-a)\overline{lim} d(x_{n+1}, y_n^k) = 0. \end{aligned}$$

So

$$d(Tx^*, Ix^*) = 0,$$

Set
$K_n = \{x \in C : d(Tx, Ix) \leqslant \frac{1}{n}\}$ and $H_n = \{x \in C : d(Tx, Ix) \leqslant \frac{a+1}{a.n}\}$.
Clearly for each $n, K_n \neq \emptyset$ and $K_1 \supseteq K_2 \supseteq ... \supseteq K_n \supseteq ....$ Thus each of the sets $\overline{TK_n}$, where $\overline{TK_n}$ denotes the closure of TK_n, must be non-empty for $n = 1, 2, ...$ and $\overline{TK_1} \supseteq \overline{TK_2} \supseteq ... \supseteq \overline{TK_n} \supseteq ....$
Further, for arbitrary $x, y \in K_n$,

$$d(Tx, Ty) \leqslant ad(Ix, Iy) + (1-a)max\{d(Tx, Ix), d(Ty, Iy)\}$$
$$\leqslant a[d(Ix, Tx) + d(Tx, Ty) + d(Ty, Iy)] + \frac{(1-a)}{n} \leqslant \frac{(a+1)}{n} + ad(Tx, Ty)$$

and so

$$d(Tx, Ty) \leqslant \frac{(a+1)}{(1-a)n},$$

Thus
$\lim_{n\longrightarrow\infty} diam(TK_n) = \lim_{n\longrightarrow\infty} diam(\overline{TK_n}) = 0.$
It follows, by a well known result of Cantor (see, e.g [2],p 156) the intersection $\bigcap_{n=1}^{\infty} \overline{TK_n}$ contains exactly one point w. Now let y be an arbitrary point in $\overline{TK_n}$. Then for arbitrary $\epsilon > 0$ there is a point y' in K_n such that

$$d(Ty', y) < \epsilon. \quad (3)$$

Using the weak commutativity of T and I non-expansiveness of I and applying (1),(2) and (3) we have

$$\begin{aligned} d(Ty, Iy) &\leqslant d(Ty, TIy') + d(TIy', ITy') + d(ITy', Iy) \\ &\leqslant ad(Iy, I^2y') + (1-a)max\{d(Ty, Iy), d(TIy', I^2y')\} \\ &+ d(TIy', ITy') + d(ITy', Iy) \\ &\leqslant ad(y, Iy') + (1-a)max\{d(Ty, Iy), d(TIy', ITy') + d(ITy', I^2y')\} \\ &\leqslant a[d(y, Ty') + d(Ty', Iy')] + (1-a)max\{d(Ty, Iy), d(Iy', Ty') \\ &+ d(Ty', Iy')\} + \frac{1}{n} + \epsilon \\ &\leqslant a(\epsilon + \frac{1}{n}) + (1-a)max\{d(Ty, Iy), \frac{1}{n} + \frac{1}{n}\} + \frac{1}{n} + \epsilon \\ &\leqslant (1+a)(\epsilon + \frac{1}{n}) + (1-a)max\{d(Ty, Iy), \frac{2}{n}\}. \end{aligned}$$

Since ϵ is arbitrary it follows that

$$d(Ty, Iy) \leqslant \frac{(a+1)}{n} + (1-a)max\{d(Ty, Iy), \frac{2}{n}\}. \tag{4}$$

There are two possible cases for the max-part in(4).
If $d(Ty, Iy) \leqslant \frac{2}{n}$,then we have $d(Ty, Iy) \leqslant \frac{2}{n} < \frac{(a+1)}{an}$ directly.
But if $d(Ty, Iy) > \frac{2}{n}$,(4) implies $d(Ty, Iy) \leqslant \frac{a+1}{n} + (1-a)d(Ty, Iy)$ so

$$d(Ty, Iy) \leqslant \frac{(a+1)}{a.n}.$$

In both cases we see $d(Ty, Iy) \leqslant \frac{a+1}{an}$ and so y lies in H_n.
Thus $\overline{TK_n} \subseteq H_n$ and so the point w must be in H_n for $n = 1, 2, ...$
It follows that

$$d(Tw, Iw) \leqslant \frac{(a+1)}{a.n},$$

for $n = 1, 2, ...$ and so $Tw = Iw$.
Since (1) holds, we also have $ITw = TIw = T^2w$.
Thus
$d(T^2w, Tw) \leqslant ad(ITw, Iw) + (1-a)max\{d(T^2w, ITw), d(Tw, Iw)\} = ad(T^2w, Tw)$, so $T^2w = Tw$ and $Tw = w'$ is a fixed point of T for $a < 1$.
Further, $Iw' = ITw = TIw = TTw = Tw' = w'$ and so w' is also a fixed point of I. uniqueness, suppose w'' is a common fixed point too Then

$$\begin{aligned} d(w', w'') &= d(Tw', Tw'') \\ &\leqslant ad(Iw', Iw'') + (1-a)max\{d(Tw', Iw'), d(Tw'', Iw'')\} \\ &\leqslant ad(w', w'') \end{aligned}$$

and the uniqueness of the common fixed point follows since $a < 1$. □

The following example satisfies Theorem 2.1.Notice that it does not satisfy conditions in Theorem 1.2 because C is non-convex.

Example 2.2. Choosing $C = [0, \frac{1}{2}] \bigcup \{1\}, Ix = \frac{x}{2}$ and $Tx = \frac{x}{x+4}$ then $TC = [0, \frac{1}{9}] \bigcup \{\frac{1}{5}\} \subseteq [0, \frac{1}{4}] \bigcup \{\frac{1}{2}\} = IC$ I is non-expansive and the pair (I, T) is weakly commuting, where both of them are self maps. Further I and T have a unique common fixed point which we know it is 0. The following corollary is a trivial conclusion of Theorem 2.1.

Corollary 2.3. *Let T be a mapping from C into itself satisfying the inequality*

$$d(T(x), T(y)) \leqslant ad(I(x), I(y)) + (1-a)max\{d(T(x), I(x)), d(T(y), I(y))\},$$

for all $x, y \in C$, where C is a compact sub set of the Banach space X , I is the identity map on C , and $0 < a < 1$. Then T has a unique fixed point.

We note that the weak commutativity in Theorem 2.1 is a necessary condition. It suffices to consider the following example.

Example 2.4. Let $X = R$ and let $C = [0, 1]$. Define T and I by $Tx = \frac{1}{3}, Ix = \frac{x}{2}$ for any $x \in C$, it is clear that all the conditions of Theorem 2.1 are satisfied except weak commutativity since for $x = \frac{1}{2}, d(TI(\frac{1}{2}), IT(\frac{1}{2})) = \frac{1}{6} > \frac{1}{12} = d(T(\frac{1}{2}), I(\frac{1}{2}))$. However T and I do not have a common fixed point.

In 1990, G. Jungck extended a fixed point theorem of Fisher and Sessa by replacing the requirements of weak commutativity and non-expansiveness by compatibility and continuity respectively.

G.Jungck[7] defined two self maps to be compatible iff whenever (x_n) is a sequence in X such that $Tx_n, Ix_n \longrightarrow t$ for some $t \in X$, then $d(ITx_n, TIx_n) \longrightarrow 0$. Clearly, commuting maps are weakly commuting, and weakly commuting maps are compatible. Also non-expansiveness requires continuity of a map.

Lemma 2.5. (Proposition 2.2, [7]). *Let $f, g : (X, d) \longrightarrow (X, d)$ be compatible.*
1. If $f(t) = g(t)$, then $fg(t) = gf(t)$.

2. suppose that $lim_n f(x_n) = lim_n g(x_n) = t$ *for some* t *in* X.
(a) *If* f *is continuous at* t , $lim_n gf(x_n) = f(t)$.
(b) *If* f *and* g *are continuous at* t, *then* $f(t) = g(t)$ *and* $fg(t) = gf(t)$.

Lemma 2.6. ([6]). *Let* T *and* I *be compatible self maps of a metric space* (X, d) *where* I *is continuous. Suppose there exist real number* $r > 0$ *and* $a \in (0, 1)$ *such that for all* $x, y \in X$,

$$d(Tx, Ty) \leqslant rd(Ix, Iy) + amax\{d(Tx, Ix), d(Ty, Iy)\} \tag{5}$$

Then $Tw = Iw$ *for some* $w \in X$ *iff* $A = \bigcap\{cl(T(K_n)) : n \in N\} \neq \emptyset$, *where* $k_n = \{x \in X : d(Tx, Ix) \leqslant \frac{1}{n}\}$.

Using Lemmas 2.5 and 2.6 we have the following corollary

Corollary 2.7. *Let* T *and* I *be two compatible self maps on a compact subset* C *of a complete metric space* X *Suppose that* I *is continuous linear and* $TC \subseteq IC$ *If there exists* $a \in (0, 1)$ *such that* T *and* I *satisfy the following inequality*

$$d(T(x), T(y)) \leqslant ad(I(x), I(y)) + (1 - a)max\{d(T(x), I(x)), d(T(y), I(y))\},$$

for all $x, y \in C$. *Then* T*and* I *have a unique common fixed point in* C.

Example 2.8. Let $X = [0, 1]$ and $C = [0, 1]$ with the Euclidean metric and define I and T by $Ix = \frac{x}{2}, Tx = \frac{x}{x+3}$ for any $x \in C$ Now C is compact and $I, T : C \longrightarrow C$, $TC = [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = IC$ and I is linear and continuous. Clearly I and T are compatible on C and so satisfy in inequality(2) Then $x = 0$ is a unique common fixed point in C.

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Alireza Valipour Baboli
Department of Mathematics
College of Basic Sciences
Ph.D Student of Mathematics
Karaj Branch, Islamic Azad Univercity
Alborz, Iran
E-mail: a.valipour@umz.ac.ir

Mohammad Bagher Ghaemi
Department of Mathematics
Science and Technology
Associate Professor of Mathematics
Iran Univercity
Tehran, Iran
E-mail: mghaemi@iust.ac.ir