Common Fixed Point Theorem
in Metric Spaces of Fisher and Sessa

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Abstract. In this paper it is shown that $T$ and $I$ have a unique common fixed point on a compact subset $C$ of a metric space $X$, where $T$ and $I$ are two self maps on $C$, $I$ is non-expansive and the pair $(T, I)$ is weakly commuting. In [3] Fisher and Sessa verified the same problem but with $C$ closed subset. Further we show this result by replacing compatibility with weakly commutativity of pair $(T, I)$ and continuity with non-expansiveness of $I$.

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1. Introduction

Many authors have written some papers in which two self maps on a closed convex set have a unique common fixed point for example [1], [3] and [9]. In 1986, Fisher and Sessa proved a fixed point theorem for two self maps on a subset of a Banach space which is closed convex(see [3]). Sessa in [9] generalized a result of Das and Naik [1]. They defined
two maps $T$ and $I$ on a metric space $(X,d)$ into itself to be weakly commuting iff
\[ d(TIx, ITx) \leq d(Ix, Tx), \]
for all $x$ in $X$.

A self map $I$ on a metric space $X$ is said to be non-expansive provided that
\[ d(Ix, Iy) \leq d(x, y), \]
for all $x, y$ in $X$. Two commuting maps clearly satisfy (1) but the converse is not generally true as is shown in the following example.

**Example 1.1.** Let $X = [0,1]$, and suppose $X$ is endowed with the Euclidean metric. Define $T$ and $I$ by $Tx = \frac{x}{x+4}$ and $Ix = \frac{x}{2}$ for any $x$ in $X$. Then
\[
d(TIx, ITx) = \frac{x}{x+8} - \frac{x}{2x+8} = \frac{x^2}{2(x+8)(x+4)} \leq \frac{x^2+2x}{2(x+4)} = \frac{x}{2} - \frac{x}{x+4} = d(Ix, Tx).
\]

But for any $x \neq 0$, $TIx = \frac{x}{x+8} > \frac{x}{2x+8} = ITx$.

Fisher and Sessa proved the following theorem.

**Theorem 1.2.** ([3]) Let $T$ and $I$ be two weakly commuting mappings from $C$ into itself satisfying the inequality
\[ d(T(x), T(y)) \leq ad(I(x), I(y)) + (1-a)\max\{d(T(x), I(x)), d(T(y), I(y))\} \]

for all $x, y$ in $C$ where $0 < a < 1$ and $C$ is a closed convex subset of a Banach space $X$. If $I$ is linear and non-expansive on $C$ and further $IC$ contains $TC$, then $T$ and $I$ have a unique common fixed point in $C$.

2. **Main Results**

Our aim is to modify of Theorem 1.2.
**Theorem 2.1.** Let $T$ and $I$ be two weakly commuting self maps on $C$ satisfying (2), where $C$ is a compact subset of the Banach space $X$. If $I$ is non-expansive on $C$ and $IC$ contains $TC$, then $T$ and $I$ have a unique common fixed point in $C$.

**Proof.** Let $x = x_0$ be an arbitrary point in $C$ and for any $n \in N$ choose $x_{n+1}$ such that $Tx_n = Ix_{n+1}$. Since $C$ is compact so $\{x_n\}$ has a convergence subsequence $\{y_k\}_{k=1}^{\infty}$ (converging to $x^*$ for some $x^* \in C$. In the following we show each $y_k$ with $y_k^*$ where it represent $k$’th member of $\{y_n\}$ and $n$’th element of $\{x_n\}$). Now we show

$$d(Tx^*, Ix^*) = 0.$$ 

$$d(Tx^*, Ix^*) \leq \limd(Tx^*, Ty^k_n) + \limd(Ty^k_n, Iy^k_n) + \limd(Iy^k_n, Ix^*)$$

$$\leq \limd(Ix^*, Iy^k_n) + \lim(1-a) \max\{d(Tx^*, Ix^*), d(Ty^k_n, Iy^k_n)\}$$

$$+ \limd(Ty^k_n, Iy^k_n) + \limd(Iy^k_n, Ix^*).$$

There are two cases if

$$\limd(Tx^*, Ix^*) \geq \limd(Ty^k_n, Iy^k_n),$$

then

$$ad(Tx^*, Ix^*) \leq (a+1)\limd(x^*, y^k_n) + \limd(Ty^k_n, Iy^k_n)$$

$$= \limd(Ty^k_n, Iy^k_n)$$

$$\leq \limd(Ty^k_n, Ix_{n+1}) + \limd(Ix_{n+1}, Iy^k_n)$$

$$\leq \limd(x_{n+1}, y^k_n) = 0,$$

and

$$\limd(Ty^k_n, Iy^k_n) \geq d(Tx^*, Ix^*),$$

then

$$d(Tx^*, Ix^*) \leq (a+1)\limd(x^*, y^k_n) + (2-a)\limd(Ty^k_n, Iy^k_n)$$

$$= (2-a)\limd(Ty^k_n, Iy^k_n)$$

$$\leq (2-a)(\limd(Ty^k_n, Ix_{n+1}) + \limd(Ix_{n+1}, Iy^k_n))$$

$$\leq (2-a)\limd(x_{n+1}, y^k_n) = 0.$$ 

So

$$d(Tx^*, Ix^*) = 0,$$
Set
\[ K_n = \{ x \in C : d(Tx, Ix) \leq \frac{1}{n} \} \text{ and } H_n = \{ x \in C : d(Tx, Ix) \leq \frac{a + 1}{a_n} \} . \]
Clearly for each \( n, K_n \neq \emptyset \) and \( K_1 \supseteq K_2 \supseteq \ldots \supseteq K_n \supseteq \ldots \). Thus each of the sets \( \overline{TK_n} \), where \( \overline{TK_n} \) denotes the closure of \( TK_n \), must be non-empty for \( n = 1, 2, \ldots \) and \( \overline{TK_1} \supseteq \overline{TK_2} \supseteq \ldots \supseteq \overline{TK_n} \supseteq \ldots \).

Further, for arbitrary \( x, y \in K_n \),
\[
d(Tx, Ty) \leq a[ d(Ix, Tx) + d(Tx, Ty) + d(Ty, Iy) ] + \frac{1 - a}{n} \leq \frac{(a + 1)}{n} + ad(Tx, Ty)
\]
and so
\[
d(Tx, Ty) \leq \frac{(a + 1)}{(1 - a)n},
\]
Thus
\[
\lim_{n \to \infty} diam(\overline{TK_n}) = \lim_{n \to \infty} diam(\overline{TK_n}) = 0.
\]
It follows, by a well known result of Cantor (see, e.g [2], p 156) the intersection \( \bigcap_{n=1}^{\infty} \overline{TK_n} \) contains exactly one point \( w \). Now let \( y \) be an arbitrary point in \( \overline{TK_1} \). Then for arbitrary \( \varepsilon > 0 \) there is a point \( y' \) in \( K_n \) such that
\[
d(Ty', y) < \varepsilon. \tag{3}
\]
Using the weak commutativity of \( T \) and \( I \) non-expansiveness of \( I \) and applying (1), (2) and (3) we have
\[
d(Ty, Iy) \leq d(Ty, TIy') + d(TIy', ITy') + d(ITy', Iy)
\leq ad(Iy, ITy') + (1 - a)\max\{d(Ty, Iy), d(TIy', I^2y')\}
+ d(TIy', ITy') + d(ITy', Iy)
\leq ad(y, Iy') + (1 - a)\max\{d(Ty, Iy), d(TIy', ITy') + d(ITy', I^2y')\}
\leq a[d(y, Ty') + d(Ty', Iy')] + (1 - a)\max\{d(Ty, Iy), d(Iy', Ty')
+ d(Ty', Iy')\} + \frac{1}{n} + \varepsilon
\leq a(\varepsilon + \frac{1}{n}) + (1 - a)\max\{d(Ty, Iy), \frac{1}{n} + \frac{1}{n}\} + \frac{1}{n} + \varepsilon
\leq (1 + a)(\varepsilon + \frac{1}{n}) + (1 - a)\max\{d(Ty, Iy), \frac{2}{n}\}.
\]
Since \( \epsilon \) is arbitrary it follows that
\[
d(Ty, Iy) \leq \frac{(a + 1)}{n} + (1 - a)\max\{d(Ty, Iy), \frac{2}{n}\}. \tag{4}
\]

There are two possible cases for the max-part in (4).
If \( d(Ty, Iy) \leq \frac{2}{n} \), then we have \( d(Ty, Iy) \leq \frac{2}{n} < \frac{(a + 1)}{an} \) directly.
But if \( d(Ty, Iy) > \frac{2}{n} \), (4) implies
\[
d(Ty, Iy) \leq \frac{(a + 1)}{n} + (1 - a)d(Ty, Iy)
\]
so
\[
d(Ty, Iy) \leq (a + 1)\frac{a}{a.n}.
\]

In both cases we see \( d(Ty, Iy) \leq \frac{a + 1}{a.n} \) and so \( y \) lies in \( H_n \).
Thus \( TK_n \subseteq H_n \) and so the point \( w \) must be in \( H_n \) for \( n = 1, 2, ... \).
It follows that
\[
d(Tw, Iw) \leq (a + 1)\frac{a}{a.n},
\]
for \( n = 1, 2, ... \) and so \( Tw = Iw \).
Since (1) holds, we also have \( ITw = TIw = T^2w \).
Thus
\[
d(T^2w, Tw) \leq ad(ITw, Iw) + (1 - a)\max\{d(T^2w, ITw), d(Tw, Iw)\} = ad(T^2w, Tw),
\]
so \( T^2w = Tw \) and \( Tw = w' \) is a fixed point of \( T \) for \( a < 1 \).
Further, \( Iw' = ITw = TIw = TTw = Tw' = w' \) and so \( w' \) is also a fixed point of \( I \). uniqueness, suppose \( w'' \) is a common fixed point too Then
\[
d(w', w'') = d(Tw', Tw'') 
\leq ad(Iw', Iw'') + (1 - a)\max\{d(Tw', Iw'), d(Tw'', Iw'')\}
\leq ad(w', w'')
\]
and the uniqueness of the common fixed point follows since \( a < 1 \). \( \square \)

The following example satisfies Theorem 2.1. Notice that it does not satisfy conditions in Theorem 1.2 because \( C \) is non-convex.
Example 2.2. Choosing \( C = [0, \frac{1}{2}] \cup \{1\}, Ix = \frac{x}{2} \) and \( Tx = \frac{x}{x + 4} \) then \( TC = [0, \frac{1}{9}] \cup \{\frac{1}{5}\} \subseteq [0, \frac{1}{4}] \cup \{\frac{1}{2}\} = IC \) \( I \) is non-expansive and the pair \((I, T)\) is weakly commuting, where both of them are self maps. Further \( I \) and \( T \) have a unique common fixed point which we know it is 0. The following corollary is a trivial conclusion of Theorem 2.1.

**Corollary 2.3.** Let \( T \) be a mapping from \( C \) into itself satisfying the inequality

\[
d(T(x), T(y)) \leq ad(I(x), I(y)) + (1 - a)\max\{d(T(x), I(x)), d(T(y), I(y))\},
\]

for all \( x, y \in C \), where \( C \) is a compact sub set of the Banach space \( X \), \( I \) is the identity map on \( C \), and \( 0 < a < 1 \). Then \( T \) has a unique fixed point.

We note that the weak commutativity in Theorem 2.1 is a necessary condition. It suffices to consider the following example.

**Example 2.4.** Let \( X = R \) and let \( C = [0, 1] \). Define \( T \) and \( I \) by \( Tx = \frac{1}{3}, Ix = \frac{x}{2} \) for any \( x \in C \), it is clear that all the conditions of Theorem 2.1 are satisfied except weak commutativity since for \( x = \frac{1}{2} \), \( d(TI(\frac{1}{2}), IT(\frac{1}{2})) = \frac{1}{6} > \frac{1}{12} = d(T(\frac{1}{2}), I(\frac{1}{2})) \). However \( T \) and \( I \) do not have a common fixed point.

In 1990, G. Jungck extended a fixed point theorem of Fisher and Sessa by replacing the requirements of weak commutativity and non-expansiveness by compatibility and continuity respectively.

G. Jungck[7] defined two self maps to be compatible iff whenever \((x_n)\) is a sequence in \( X \) such that \( Tx_n, Ix_n \to t \) for some \( t \in X \), then \( d(ITx_n, TIx_n) \to 0 \). Clearly, commuting maps are weakly commuting, and weakly commuting maps are compatible. Also non-expansiveness requires continuity of a map.

**Lemma 2.5.** (Proposition 2.2, [7]). Let \( f, g : (X, d) \to (X, d) \) be compatible.
1. If \( f(t) = g(t) \), then \( fg(t) = gf(t) \).
2. Suppose that \( \lim_n f(x_n) = \lim_n g(x_n) = t \) for some \( t \) in \( X \).

(a) If \( f \) is continuous at \( t \), \( \lim_n gf(x_n) = f(t) \).
(b) If \( f \) and \( g \) are continuous at \( t \), then \( f(t) = g(t) \) and \( fg(t) = gf(t) \).

**Lemma 2.6.** ([6]). Let \( T \) and \( I \) be compatible self maps of a metric space \( (X, d) \) where \( I \) is continuous. Suppose there exist real number \( r > 0 \) and \( a \in (0, 1) \) such that for all \( x, y \in X \),

\[
d(Tx, Ty) \leq rd(Ix, Iy) + a \max\{d(Tx, Ix), d(Ty, Iy)\}
\]  

Then \( Tw = Iw \) for some \( w \in X \) iff \( A = \bigcap\{cl(T(K_n)) : n \in N\} \neq \emptyset \),

where \( k_n = \{x \in X : d(Tx, Ix) \leq \frac{1}{n}\} \).

Using Lemmas 2.5 and 2.6 we have the following corollary

**Corollary 2.7.** Let \( T \) and \( I \) be two compatible self maps on a compact subset \( C \) of a complete metric space \( X \) Suppose that \( I \) is continuous linear and \( TC \subseteq IC \) If there exists \( a \in (0, 1) \) such that \( T \) and \( I \) satisfy the following inequality

\[
d(T(x), T(y)) \leq ad(I(x), I(y)) + (1 - a)\max\{d(T(x), I(x)), d(T(y), I(y))\},
\]

for all \( x, y \in C \). Then \( T \) and \( I \) have a unique common fixed point in \( C \).

**Example 2.8.** Let \( X = [0, 1] \) and \( C = [0, 1] \) with the Euclidean metric and define \( I \) and \( T \) by \( Ix = \frac{x}{2}, Tx = \frac{x}{x + 3} \) for any \( x \in C \) Now \( C \) is compact and \( I, T : C \rightarrow C, TC = [0, \frac{1}{4}] \subseteq [0, \frac{1}{2}] = IC \) and \( I \) is linear and continuous. Clearly \( I \) and \( T \) are compatible on \( C \) and so satisfy in inequality(2) Then \( x = 0 \) is a unique common fixed point in \( C \).

**References**


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